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ON THE NUMBER OF MINIMAL PRIME IDEALS IN THE COMPLETION OF A LOCAL DOMAIN

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Let R be a local Noetherian domain. It is well-known that the number of minimal prime ideals in the completion of R is greater than or equal to the number of maximal ideals in the integral closure of R. An (unproved) exercise in [2] states that the reverse inequality holds if R is one-dimensional. The purpose of this note is to show how this latter fact can be generalized to local domains of dimension greater than one. Specifically, let $x_1, \ldots, x_d$ be a system of parameters for R and set

$$T = R\left[\frac{x_2}{x_1}, \cdots, \frac{x_d}{x_1}\right]_{MR[\frac{x_2}{x_1}, \cdots, \frac{x_d}{x_1}]}$$

(M is the maximal ideal of R). We will show that if R is quasi-unmixed, then the number of maximal ideals in the integral closure of T is greater than or equal to the number of minimal prime ideals in the completion of R. As a corollary we deduce a criterion for local domains to be analytically irreducible and we close with a bound for the number of minimal prime ideals in the completion of R in the non-quasi-unmixed case.

NOTATION. Throughout, (R, M) will denote a local Noetherian ring with maximal ideal M. We will use "—" to denote integral closure—both for rings and ideals. Recall that for an ideal $I \subseteq R$, $\bar{I}$, the integral closure of I, is the set of elements $x \in R$ satisfying an equation of the form

$$x^n + i_1 x^{n-1} + \cdots + i_n = 0, \quad i_k \in I^k, \ 1 \leqq k \leqq n.$$

It is well-known that $\bar{I}$ is an ideal of R contained in the radical of I. We will use "*" to denote the completion of a local ring. Recall that a local ring R is quasi-unmixed in case $\dim R^*/p^* = \dim R$, for all minimal primes $p^* \subseteq R^*$. Any other standard facts or terminology from local ring theory appear here as they do in [2].

REMARK. Lemmas 1 and 2 below are more or less well-known, but we have included their easy proofs for the sake of exposition.

LEMMA 1. (c.f. [6, p. 354]): *Let R be a Noetherian domain and $I \subseteq R$*

an ideal. Write $I = (x_1, \ldots, x_d)$ *and set* $S_i = R[x_1/x_i, \ldots, x_d/x_i]$. *Then for all* $n \geqq 1$ $\overline{I^n} = \bigcap_{i=1}^{d} [\overline{I^n S_i} \cap R]$.

PROOF. Let $V \supseteq R$ be a valuation domain. There exists an i such that $S_i \subseteq V$. Since $I^n V$ is principal and V is integrally closed, $\overline{I^n V} = I^n V$. Therefore $\overline{I^n S_i} \subseteq I^n V$. Since $\overline{I^n} = \bigcap [I^n V \cap R]$, the intersection ranging over all valuation domains $V \supseteq R$ [**6**], the result follows.

LEMMA 2. *Let R be a local ring and* $I \subseteq R$ *an ideal. Then*

$$\bigcap_{n \geqq 1} \overline{I^n} = \text{nil rad } (R).$$

PROOF. Clearly nil rad $(R) \subseteq \bigcap_{n \geqq 1} \overline{I^n}$. To show $\bigcap_{n \geqq 1} \overline{I^n} \subseteq$ nil rad (R), observe that an element $x \in R$ belongs to $\overline{I^n}$ if and only if the image of x in R/p belongs to $(\overline{I^n + p/p})$ for all minimal primes $p \subseteq R$. Consequently we may assume that R is a domain and show $\bigcap_{n \geqq 1} \overline{I^n} = 0$. Since R is Noetherian, a theorem of Chevally implies that there exists a DVR V containing R with $IV \neq V$. Since V is integrally closed and $I^n V$ is principal, $\overline{I^n V} = I^n V$. Therefore $\bigcap_{n \geqq 1} \overline{I^n} \subseteq \bigcap_{n \geqq 1} I^n V = 0$.

PROPOSITION 3. *Let* (R, M) *be a quasi-unmixed local domain. Let* $x_1, \ldots, x_d$ *be a system of parameters, set* $I = (x_1, \ldots, x_d)$ *and*

$$T = R\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right]_{MR[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}]}.$$

Suppose that $M_1, \ldots, M_k$ *are the maximal ideals in* $\bar{T}$. *Then, for all* $n \geqq 1$, $\overline{I^n} = \bigcap_{i=1}^{k} (I^n \bar{T}_{M_i} \cap R)$.

PROOF. Let $n \geqq 1$. Since $I^n T$ is principal, $\overline{I^n T} = I^n \bar{T} \cap T$. Moreover, since $\bar{T}$ is a Krull domain (in fact a Dedekind domain, since T is one-dimensional) $\bigcap_{i=1}^{k} (I^n \bar{T}_{M_i} \cap \bar{T}) = I^n \bar{T}$, as the M_i are precisely the prime divisors of $I^n \bar{T}$. Therefore $\bigcap_{i=1}^{k} (I^n \bar{T}_{M_i} \cap T) = \overline{I^n T}$. Thus, by Lemma 1, it suffices to show that $\overline{I^n T} \cap S_i = \overline{I^n S_i}$ for all $i = 1, \ldots, d$, where $S_i = R[x_1/x_i, \ldots, x_d/x_i]$. Since $x_1, \ldots, x_d$ are analytically independent, each $x_j/x_i \notin MS_i$. So $S_{iMS_i} = T$. As localization commutes with integral closure, we will be done if we show that $\overline{I^n S_i}$ is MS_i-primary. Suppose Q is a prime divisor of $\overline{I^n S_i}$. Since R is quasi-unmixed, S is locally quasi-unmixed [**3**, 2.5]. Therefore, by [**4**, Theorem 2.12], height $Q = 1$ ($I^n S_i$ is principal). Since MS_i is the unique height one prime in S_i containing $I^n S_i$ (this is well-known), we have $Q = MS_i$. Thus $\overline{I^n S_i}$ is MS_i-primary and the proposition is proved.

THEOREM 4. *Let* (R, M) *be a quasi-unmixed local domain. Let* $x_1, \ldots, x_d$ *be a system of parameters and set*

$$T = R\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right]_{MR[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}]}.$$

Let r and k respectively denote the number of maximal ideals in $\bar{R}$ and $\bar{T}$. Then $r \leqq$ number of minimal prime ideals in $R^* \leqq k$.

PROOF. The first inequality follows from the proof of (33.10) in [**2**]. For the second inequality, let $M_1, \ldots, M_k$ be the maximal ideals in $\bar{T}$ and set $T_i = \bar{T}_{M_i}$. Then for each $i = 1, \ldots, k$ there is a natural map of M-adic completions $\varphi_i: R^* \to T_i^*$. In fact, if we set $I = (x_1, \ldots, x_d)$ then, since I is M-primary in R and M_iT_i-primary in each T_i, we may view these completions as being with respect to the I-adic topology. We will show $\bigcap_{i=1}^k \ker \varphi_i \subseteq \text{nil rad}\,(R^*)$. The theorem will readily follow from this. Indeed, it suffices to note that since each T_i^* is a domain (in fact, a DVR) each $\ker \varphi_i$ is prime. Of course any collection of primes whose intersection is contained in nil rad (R^*) must include the minimal primes of R^*.

Now suppose $x^* \in \bigcap_{i=1}^k \ker \varphi_i$. We may select a sequence of elements $\{x_n\}$ in R such that $x^* - x_n \in I^n R^*$. The choice of x^* implies that for each $i = 1, \ldots, k$ $\{x_n\}$ is a null sequence in T_i with respect to the IT_i-adic topology. Therefore, after suitably refining the sequence $\{x_n\}$ we may further assume that $x_n \in I^nT_i$ for all n and all i. By Proposition 3, $x_n \in \overline{I^n}$ for all n. Therefore $x^* = x_n + x^* - x_n \in \overline{I^n}R^* + I^nR^* \subseteq \overline{I^nR^*}$ for all n. By Lemma 2, x^* is nilpotent.

COROLLARY 5. (cf. [**2**; Exercise 1, page 122]). *Let R be a one-dimensional local domain. Then the number of minimal prime ideals in R^* is equal to the number of maximal ideals in $\bar{R}$.*

PROOF. Since a one-dimensional local domain is quasi-unmixed, and T in Theorem 4 is just R, the result follows.

The next corollary is a criterion for a local domain to be analytically irreducible. It is an immediate consequence of Theorem 4 and the definitions.

COROLLARY 6. *Let R and T be as in* Theorem 4. *Assume further that R is analytically unramified. If $\bar{T}$ is local, then R is analytically irreducible.*

REMARK. If R is a localization of a finitely generated algebra over a field or the integers, then R is quasi-unmixed and analytically unramified. Hence Corollary 6 applies to most of the local rings from geometry.

Our final proposition uses Rees rings, rather than overrings to bound the number of minimal prime ideals in R^*.

PROPOSITION 7. *Let (R, M) be a local domain and $I \subseteq R$ an M-primary ideal. Write $\mathscr{R} = R[\mathrm{It}, t^{-1}]$, t an indeterminate, for the Rees ring of R with respect to I. Then the number of minimal prime ideals in R^* is less than* *or equal to the number of prime divisors of $t^{-1}\bar{\mathscr{R}}$. In particular, if R is analytically unramified and $t^{-1}\bar{\mathscr{R}}$ is primary, then R is analytically irreducible.*

PROOF. Let $\mathscr{P}_1, \ldots, \mathscr{P}_k$ be the prime divisors of $t^{-1}\bar{\mathscr{R}}$ and set $V_i = \bar{\mathscr{R}}_{\mathscr{P}_i}$ (since $\bar{\mathscr{R}}$ is a Krull domain there are finitely many $\mathscr{P}_i$ and each V_i is a DVR). Since $t^{-n}\bar{\mathscr{R}} \cap R = \overline{I^n}$ it follows that $\bigcap_{i=1}^{k} t^{-n}V_i \cap R = \overline{I^n}$ for all n. Let ψ_i: $R^* \to V_i^*$ be the natural map (where R^* is viewed as the I-adic completion of R and V_i^* as the t^{-1}-adic completion of V_i). Then just as in the proof of Theorem 4 one shows $\bigcap_{i=1}^{k} \ker \psi_i \subseteq$ nil rad(R^*) and the proposition follows.

REMARK. In case R is quasi-unmixed and I is generated by a system of parameters, it can be shown that there is a one-to-one correspondence between the prime divisors of $t^{-1}\bar{\mathscr{R}}$ and the maximal ideals of $\bar{T}$ (for T as in Theorem 4). Thus Theorem 4 can be recovered from Proposition 7.

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