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SUMMING SUBSEQUENCES OF RANDOM VARIABLES

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ABSTRACT. Given an increasing sequence N of positive integers and $k \geqq 1$, call any one to one correspondence $\tau: N \to \mathbf{N}^k$ an ordering (or numbering) of N onto $\mathbf{N}^k$. Let (X_n) be a sequence of random variables satisfying $\sup_n \mathrm{E}\,|X_n|\,(\log^+|X_n|)^{k-1} < \infty$. Then there exists a subsequence $N_0 = (i_n)$ such that, for any further subsequence $N_1 = (i_{j_n})$ and any ordering τ satisfying $|\tau(i_{j_n})| \leqq j_n$ for all $n \geqq 1$, we have $(X_{\tau^{-1}(s)})$ converges Cesàro a.s. for $s \in \mathbf{N}^k$.

1. Introduction and notation. The theorem of Komlós [2] is a generalized strong law of large numbers. If (X_n) is an L_1-bounded sequence of random variables, then there exists a subsequence such that every further subsequence converges Cesàro a.s., to the same limit. In this paper, the following Komlós-type property is considered. Given a sequence (X_n) satisfying a certain moment condition, there exists a subsequence (X_n^0) such that any ordering, to a degree, of any subsequence of (X_n^0) into $\mathbf{N}^k$ converges Cesàro a.s. The limit is independent of the particular subsequence of (X_n^0), and of the ordering. As a corollary (taking $k = 1$), to a large degree, permutations of the Komlós subsequences converge Cesàro a.s.

This latter result cannot be obtained from Komlós's proof, which uses martingale difference sequences. The method used here is patterned after Etemadi's [1] proof of the strong law of large numbers for pairwise independent, identically distributed random variables. Despite the fact that we begin with a sequence (X_n) rather than an array, the moment condition must be stronger than L_1-bounded to obtain the result; we suppose $\sup_n \mathrm{E}|X_n|(\log^+|X_n|)^{k-1} < \infty$. This condition is not always necessary, but Smythe [4] has shown that if $\mathrm{E}|X_n|(\log^+|X_n|)^{k-1} = \infty$, then the strong law of large numbers fails to hold for a k-dimensional array of i.i.d. random variables. Consequently, a multiparameter Komlós-type theorem cannot hold in general if (X_n) is only L_1-bounded.

In the following, let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $k \geqq 1$, we consider $\mathbf{N}^k$ with the coordinatewise partial ordering $\leqq$. For $s = (s_1, \ldots, s_k) \in \mathbf{N}^k$, denote $|s| = s_1 \cdot \ldots \cdot s_k$. If $j \geqq 1$, let $d_j = \mathrm{card}\{s \in \mathbf{N}^k: |s| = j\}$, the number of ways of writing

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j as a product of k positive integers. Let N be an increasing sequence of positive integers; any one to one correspondence $\tau: N \to \mathbf{N}^k$ will be called an ordering (or numbering) of N onto $\mathbf{N}^k$. Given such an ordering τ, denote $X_s = X_{\tau^{-1}(s)}$ for $s \in \mathbf{N}^k$. For a random variable X, let $F_a(X) = X \cdot I_{|X| \leqq a}$ be truncation at the value $a \geqq 0$. Finally, constants appear in the arguments, e.g., c, K,which are basically unimportant and may differ at each appearance.

2. The Results. We begin with some preliminary results, similar to those in **[2, 3]**.

LEMMA 1. *Let (X_n) be a sequence of random variables. Then there exists a subsequence $N_0 = (i_n)$ and a sequence of nonnegative scalars (M_j) such that, for any further subsequence $N_1 = (i_{j_n})$ and ordering τ: $N_1 \to \mathbf{N}^k$ satisfying $|\tau(i_{j_n})| \leqq j_n$, $n \geqq 1$, we have*

$$\frac{M_j}{2} \leqq \int_{j-1<|X_s| \leqq j} |X_s| \, dP \leqq M_j + \frac{1}{j^2}, \qquad s \in \mathbf{N}^k, 1 \leqq j \leqq |s|^2. \tag{1}$$

PROOF. For each $j \geqq 1$, there exists a subsequence $I_j \subset I_{j-1}$ (taking $I_0 = \mathbf{N}$) and a scalar $M_j \geqq 0$ such that, for all $n \in I_j$,

$$\frac{M_j}{2} \leqq \int_{j-1<|X_n| \leqq j} |X_n| \, dP \leqq M_j + \frac{1}{j^2}.$$

Let i_n be the n^{th} element of I_{n^2}, and denote $N_0 = (i_n)$. With this construction it is easy to see that (1) holds.

LEMMA 2. *Let (X_n) be a sequence of random variables. Then there exists a subsequence $N_0 = (i_n)$ and a sequence of (bounded) random variables (β_j) such that, for any further subsequence $N_1 = (i_{j_n})$ and ordering τ: $N_1 \to \mathbf{N}^k$ satisfying $|\tau(i_{j_n})| \leqq j_n$, $n \geqq 1$, we have*

$$|\mathrm{E}(F_j(X_s)\beta_j - \beta_j^2)| \leqq 1, \qquad s \in \mathbf{N}^k, 1 \leqq j \leqq |s|, \tag{2}$$

$$|\mathrm{E}(F_p(X_r) - \beta_p)(F_q(X_s) - \beta_q)| \leqq 1/2^{|s|}, \textit{for} \tag{3}$$
$$1 \leqq p \leqq |r|, 1 \leqq q \leqq |s|, 1 \leqq |r| \leqq |s|, r \neq s \in \mathbf{N}^k.$$

PROOF. For each $j \geqq 1$, $(F_j(X_n))$ is uniformly integrable. So there exists a subsequence $(X_{j,n})$ of $(X_{j-1,n})$ and a random variable β_j, $|\beta_j| \leqq j$ a.s., such that $F_j(X_{j,n}) \to \beta_j$ weakly in L_1. By diagonalizing, we can suppose that $F_j(X_n) \to \beta_j$ weakly in L_1 for each $j \geqq 1$.

For $j = 1$, there exists a subsequence $I_1 \subset \mathbf{N}$ such that, for all $n \in I_1$, $|\mathrm{E}(F_1(X_n)\beta_1 - \beta_1^2)| \leqq 1$. Let i_1 be the first element of I_1.

For $j > 1$, suppose $I_j \subset I_{j-1}$ is a subsequence from which an index i_j has been chosen. We wish to determine i_{j+1}.

Since $F_{j+1}(X_n) - \beta_{j+1} \to 0$ weakly in L_1, there exists a subsequence $I_{j+1} \subset I_j$ such that, for all $n \in I_{j+1}$,

$$|\mathrm{E}(F_{j+1}(X_n)\beta_{j+1} - \beta_{j+1}^2)| \leqq 1,$$
$$|\mathrm{E}(F_p(X_{i_m}) - \beta_p)\,(F_q(X_n) - \beta_q)| \leqq 1/2^{j+1},$$
$$\textit{for } 1 \leqq p \leqq m \leqq j,\ 1 \leqq q \leqq j+1.$$

Choose i_{j+1} to be the $j+1^{\text{st}}$ element of I_{j+1}. The subsequence $N_0 = (i_n)$ is now completely determined and (2), (3) can be verified.

Lemmas 1 and 2, which apply to any sequence (X_n) whatsoever, provide estimates used in establishing the following results.

LEMMA 3. *Suppose* $\sup_n \mathrm{E}|X_n|(\log^+|X_n|)^{k-1} < \infty$. *Then there exists a subsequence* $N_0 = (i_n)$ *such that for any further subsequence* $N_1 = (i_{j_n})$ *and ordering* $\tau\colon N_1 \to \mathbf{N}^k$ *satisfying* $|\tau(i_{j_n})| \leqq j_n$, $n \geqq 1$, *we have*

$$\sum_{s \in \mathbf{N}^k} \frac{\mathrm{E}(F_{|s|}(X_s))^2}{|s|^2} < \infty, \tag{4}$$

$$\sum_{s \in \mathbf{N}^k} P(|X_s| > |s|) < \infty. \tag{5}$$

PROOF. We take N_0 to be the subsequence given by Lemma 1. Let N_1 be a further subsequence and τ an ordering, $|\tau(i_{j_n})| \leqq j_n$, $n \geqq 1$. From (1) and the hypothesis on (X_n), we get $\sum_{j=1}^\infty (\log j)^{k-1} M_j < \infty$. Using this and the fact that $\sum_{i=1}^j d_i \leqq cj(\log j)^{k-1}$, we can obtain (4) and (5).

LEMMA 4. Suppose $\sup_n \mathrm{E}|X_n|(\log^+|X_n|)^{k-1} < \infty$. Then there exists a subsequence $N_0 = (i_n)$ and a sequence of (bounded) random variables (β_n) such that, for any further subsequence $N_1 = (i_{j_n})$ and ordering τ: $N_1 \to \mathbf{N}^k$, $|\tau(i_{j_n})| \leqq j_n$, $n \geqq 1$, we have

$$\sum_s \frac{\mathrm{E}(Z_s)^2}{|s|^2} < \infty, \tag{6}$$

$$\sum_{s \neq t} |\mathrm{E}(Z_s Z_t)| < \infty, \tag{7}$$

(8) there exists $X \in L_1$ such that $\beta_n \to X$ a.s. (and in L_1),

where $Z_s = F_{|s|}(X_s) - \beta_{|s|}$, $s \in \mathbf{N}^k$.

PROOF. Relations (6) and (7) may be readily shown using the previous estimates. For (8), we suppose, without loss of generality, that $F_j(X_n) \to \beta_j$ weakly in L_1 for each $j \geqq 1$. From (1)

$$\sum_{j=1}^\infty \mathrm{E}|\beta_j - \beta_{j-1}| \leqq \sum_{j=1}^\infty \underset{n}{\underline{\lim}}\, \mathrm{E}|F_j(X_n) - F_{j-1}(X_n)|$$
$$\leqq \sum_{j=1}^\infty \left(M_j + \frac{1}{j^2}\right) < \infty.$$

Thus, $\beta_j \to X$ a.s. (and in L_1) for some $X \in L_1$.

The limit in (8) (= the Cesàro limit in Theorem 5, following) can be identified as follows.

For $k > 1$, the hypothesis $\sup_n \mathrm{E}|X_n|(\log^+|X_n|)^{k-1} < \infty$ implies that (X_n) is uniformly integrable. So there exists a subsequence converging weakly in L_1 to a random variable X. By starting with this subsequence, it is easy to show $\lim_{n\to\infty} \beta_n = X$ a.s. (and in L_1). In this case, X is the only possible limit in (8) and Theorem 5. Conversely, given the limit X in (8), there exists a subsequence (X_{j_n}) such that $X_{j_n} \to X$ weakly in L_1 (assuming $k > 1$).

If $k = 1$, a well-known "subsequence splitting" lemma asserts that there exists a subsequence (X_{j_n}) which is equivalent (in the sense of Khintchin) to a sequence (Y_n) which converges weakly in L_1. By starting with (X_{j_n}), X may be identified as the weak limit of (Y_n). Conversely, as above, if X is given in (8), then there exists a subsequence of (X_n) which is equivalent to a sequence covering weakly in L_1 to X.

THEOREM 5. *If* $\sup_n \mathrm{E}|X_n|(\log^+|X_n|)^{k-1} < \infty$, *then there exists a subsequence* $N_0 = (i_n)$ *and* $X \in L_1$ *such that for each further subsequence* $N_1 = (i_{j_n})$ *and ordering* $\tau\colon N_1 \to \mathbf{N}^k$ *satisfying* $|\tau(i_{j_n})| \leqq j_n$, $n \geqq 1$, *we have*

$$\lim_s \frac{1}{|s|} \sum_{r \leqq s} X_r = X \text{ a.s.}$$

PROOF. Without loss of generality, we can assume $X_n \geqq 0$, $n \geqq 1$. We take N_0 and (β_j) to satisfy (5)–(8); let N_1 be a subsequence and τ an appropriate ordering. Define $S_u = \sum_{r\leqq u} X_r$, $S_u^* = \sum_{r\leqq u} F_{|r|}(X_r)$ and $T_u = \sum_{r\leqq u}\beta_{|r|}$. For $\alpha > 1$ and $s = (s_1, \ldots, s_k) \in \mathbf{N}^k$, denote $m(s) = ([\alpha^{s_1}], \ldots, [\alpha^{s_k}]) \in \mathbf{N}^k$ and let $\varepsilon > 0$. Now, from (6) and (7),

$$\begin{aligned}
&\sum_s P(|S^*_{m(s)} - T_{m(s)}| > |m(s)|\,\varepsilon)\\
&\qquad \leqq c \sum_s \frac{1}{|m(s)|^2} \mathrm{E}(S^*_{m(s)} - T_{m(s)})^2\\
&\qquad \leqq c \sum_s \frac{1}{|m(s)|^2} \sum_{t\leqq m(s)} \mathrm{E}(Z_t^2) + c \sum_s \frac{1}{|m(s)|^2} \sum_{t\neq u} |\mathrm{E}(Z_t Z_u)|\\
&\qquad \leqq c \sum_t \frac{\mathrm{E}(Z_t)^2}{|t|^2} + K < \infty.
\end{aligned}$$

Hence, $1/m(s)(S^*_{m(s)} - T_{m(s)}) \to 0$ a.s. By (5) and (8), $1/m(s) S_{m(s)} \to X$ a.s., for some $X \in L_1$. By monotonicity of the partial sums,

$$\frac{1}{\alpha^k} X \leqq \varliminf_s \frac{S_s}{|s|} \leqq \varlimsup_s \frac{S_s}{|s|} \leqq \alpha^k X \text{ a.s.}$$

Since this holds for all $\alpha > 1$, we conclude

$$\frac{S_s}{|s|} \to X \text{ a.s.}$$

REMARK. For $k > 1$, (X_n) is uniformly integrable; so we have L_1 convergence as well.

By applying the theorem to the one-dimensional case, we get a corollary showing that the subsequences in the Komlós theorem can be permuted to a large degree.

COROLLARY 6. Suppose $\sup_n \mathrm{E}|X_n| < \infty$. Then there exists a subsequence (X_{i_n}) and $X \in L_1$ such that for any further subsequence $(X_{i_{j_n}})$ and any permutation $\pi\colon \mathbf{N} \to \mathbf{N}$ satisfying $\pi(n) \leqq j_n$, $n \geqq 1$, we have

$$\lim_n \frac{1}{n} \sum_{m=1}^{n} X'_m = X \text{ a.s.},$$

where

$$X'_m = X_{i_{j_{\pi^{-1}(m)}}}, \ m \geqq 1.$$

The degree of permutation is governed by the sparsity of the subsequence $(X_{i_{j_n}})$ in (X_{i_n}) the thinner the subsequence, the more freedom to permute. This result does not follow from Komlós' proof in [2], nor from the maximal inequality in [3]. In these, (X_{i_n}) is constructed in a way so that X_{i_n} is nearly independent of the "past." Permutations cause the need to be independent of the "future" as well, and this is where the breakdown occurs. The success of the Etemadi-type approach appears to be in replacing "nearly independent" with "nearly pairwise uncorrelated."

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