ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 2, Spring 1988

ON THE COLLAPSE OF THE RESONANCE STRUCTURE IN A THREE-PARAMETER FAMILY OF COUPLED OSCILLATORS

HANS G. OTHMER AND MASAJI WATANABE

1. Introduction. In many biological systems it is necessary to scynchronize or otherwise organize the temporal activity of a population of cells, and this is usually achieved through stimulation or 'forcing' by a pacemaker, or by mutual coupling within the population. While the pacemaker is sometimes external to the organism, as in the case of circadian rhythms, many interesting examples in physiology involve endogenous pacemakers. Examples include the oscillatory networks of neurons (the central pattern generators) that underlie a variety of periodic behavior [**1, 2**] and the SA node in the mammalian heart. In the SA node individual cells generate periodic outputs, and the problem is to understand how the output is synchronized in the population [**3**]. In central pattern generators the periodic output is often a network property, in that individual cells do not burst periodically in isolation [**1**], and the problem is to understand the patterns of interaction that can generate the observed periodic behavior. However, synchronization is not always desirable, as is illustrated by the fact that synchronized bursting of large numbers of neurons underlies epileptic seizures [**4**]. Many other examples can be given to underscore the fact that knowledge of how coupling affects the collective behavior of aggregates of cells is important for understanding both normal and pathological processes in numerous biological and physiological systems [**5**].

From a mathematical standpoint the simplest system that is relevant in this context is a single periodically-forced system, and for such systems much is known about the dependence of solutions on the period and amplitude of the forcing function [**2**]. However, much less is known about the dependence of solutions for a system of coupled oscillators on biologically-relevant parameters, such as the intrinsic frequency of the oscillators and the coupling strength. Some results can be gotten by asymptotic methods in the limit of very weak or very

Supported in part by NIH Grant GM 29123.
Received by the editors on May 24, 1986.

strong coupling [**6, 7, 8, 9, 10**], but previous work on directly-coupled oscillators shows that much of the interesting, and perhaps biologically-important behavior arises at intermediate values of the parameters, where there are subtle balances between competing processes [**6, 8, 9, 10**]. For example, it is shown in [**10**] that, for moderate values of the coupling strength, a system of two identical coupled oscillators may simultaneously have both stable synchronized solutions and stable de-synchronized solutions. In addition, there are sequences of period-doubling bifurcations and chaos as the coupling strength is varied, but none of this behavior exists at very weak or very strong coupling [**8, 9,** and **10**]. This work has demonstrated the value of choosing a simple model that can be more-or-less completely understood by a combination of analytical and numerical techniques. Our objectives in this paper are to develop a perturbation technique that is applicable to other than directly-coupled oscillators, examples of which are given in [**11**] and [**12**], and to extend the results in [**8, 9**, and **10**] to nonsingular linear coupling.

In the following section we develop a procedure for studying periodic solutions of perturbed systems for which the unperturbed system has an N-parameter family of periodic solutions. This approach, which is related to earlier work by Urabe [**13**], does not rely on the existence of a periodic surface for the perturbed system, and thus is applicable in degenerate cases where other approaches such as averaging fail. When $N = 1$ a straight-forward application of the implicit function theory yields a local branch of periodic solutions [**14**], but when $N > 1$ the appropriate linear problem is singular and some form of reduction procedure is required. In §2 we study a system of two directly-coupled oscillators and show how to obtain the bifurcation equations for general coupling functions. This reduction procedure also enables us to prove that there is no subharmonic bifurcation, and to obtain persistence of a smooth invariant torus via a center manifold construction.

In §3 we illustrate the reduction procedure for the system studied in [**10**] and extend the results in that paper to more general types of coupling. In particular, we recover the perturbation results in [**10**] directly, without appeal to results on persistence of invariant manifolds and without construction of the leading-order terms in a perturbation expansion of the perturbed manifold. In §4 we show how the resonance structure found in [**10**] collapses as the coupling matrix approaches a multiple of the identity. The results show how the infinite family of resonance zones found in [**10**] arise from a codimension-four singularity that exists when the coupling matrix is the identity. In §5 we prove in a limiting case that a certain bifurcation which was found numerically in [**10**] occurs, and we are able to construct a global branch of periodic solutions that result from this bifurcation. This branch varies between a branch on which the oscillators are π radians out of phase and one on which the oscillators are in phase, as the coupling parameter varies, and at zero coupling one of the oscillators is at rest.

2. Perturbation of oscillators via weak coupling. To introduce the general reduction of equations in a neighborhood of a periodic orbit, we consider the parameterized autonomous system

$$\frac{dx}{dt} = F(x, \delta) \tag{1}$$

where $F : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$. Here and hereafter we assumed that all vector fields are smooth (i.e., C^r for $r \geq 2$) unless stated otherwise. Let $\phi(t, x_0, \delta)$ be the solution of (1) with $\phi(0, x_0, \delta) = x_0$, and let $\phi_0(t)$ denote a nonconstant periodic solution of the unperturbed system

$$\frac{dx}{dt} = F(x, 0) \equiv F_0(x) \tag{2}$$

with least period $T > 0$. Suppose that $\Psi(s)$ is an $n \times (n-1)$ matrix, each entry of which is a smooth function of s, with the properties

$$\begin{aligned} \Psi(s+T) &= \Psi(s) \\ \Psi(s)^T \Psi(s) &= I_{(n-1)\times(n-1)} \\ F_0(\phi_0(s))^T \Psi(s) &= O_{1\times(n-1)} \end{aligned} \tag{3}$$

for all $s \in \mathbf{R}$. We shall say that $\Psi(\cdot)$ is admissable for a given pair $(F_0, \phi_0(\cdot))$ if $\Psi(\cdot)$ satisfies (3). If P_s is a local section at $\phi_0(s)$ defined by restricting the range of $\Psi(s)$ to a sufficiently small neighborhood of the orbit at $\phi_0(s)$, then for each $x \in P_s$ there is a $y \in \mathbf{R}^{n-1}$ such that

$$x = \phi_0(s) + \Psi(s)y.$$

The y's are local coordinates in a hyperplane normal to the orbit at $\phi(s)$. If x_0 is sufficiently close to the orbit of ϕ_0 and δ is sufficiently small, then $\phi(t, x_0, \delta) \in P_s$ for some t, and it can be shown that there are smooth functions $y(s)$ and $t(s)$ such that

$$\phi(t(s), x_0, \delta) = \phi_0(s) + \Psi(s)y(s)$$

[**15, 13**]. The functions t and y satisfy the periodic system

$$\frac{dt}{ds} = \tau(s, y, \delta) \tag{4}$$

$$\frac{dy}{ds} = Y(s, y, \delta), \tag{5}$$

where

$$\tau(s, y, \delta) = \frac{F_0(\phi_0(s))^T[F_0(\phi_0(s)) + \Psi'(s)y]}{F_0(\phi_0(s))^T F(\phi_0(s) + \Psi(s)y, \delta)} = 1 + \mathcal{T}(s, y, \delta),$$

and

$$Y(s, y, \delta) = \Psi(s)^T[\tau(s, y, \delta)F(\phi_0(s) + \Psi(s)y, \delta) - \Psi'(s)y]. \tag{6}$$

By the definition of s, $t(s) - t(0)$ is the time it takes for the solution through $x_0 \in P_0$ to reach P_s, and $y(s)$ measures the distance between $\phi(t(s), x_0, \delta)$ and $\phi_0(s)$. Evidently $\tau(s, 0, 0) = 1$, and therefore $\mathcal{T}(s, 0, 0) = 0$.

Equations (4) and (5) are equivalent to equation (1) in the following sense. If $x_0 \in P_0, y_0 = \Psi(0)^T[x_0 - \phi_0(0)]$, and $t(s)$ satisfies (4) with $t(0) = 0$, then

$$y(s, y_0, \delta) = \Psi(s)^T[\phi(t(s), x_0, \delta) - \phi_0(s)],$$

where $y(s, y_0, \delta)$ denotes the solution of (5) with the property that $y(0, y_0, \delta) = y_0$. Conversely, if

$$\tau(s, y(s, y_0, \delta), \delta) > 0$$

for all $s \in R$, and if $x_0 = \phi_0(0) + \Psi(0)y_0$, then

$$\phi(t, x_0, \delta) = \phi_0(s(t)) + \Psi(s(t))y(s(t), y_0, \delta),$$

where $s(\cdot)$ denotes the inverse of $t(\cdot)$. In particular, if $\phi(t, x_0, \delta)$ is a periodic solution of (1), then $y(s, y_0, \delta)$ is a periodic solution of (5), and conversely, if $y(s, y_0, \delta)$ is a kT-periodic solutions of (5) for some positive integer k, then $\phi(t, x_0, \delta)$ is a periodic solution of (1) with period $t(kT)$. Thus, for small δ, (1) has a periodic solution near ϕ_0 if and only if (5) has a small periodic solution.

The function $Y(s, y, \delta)$ in (6) has the form

$$Y(s, y, \delta) = P(s)y + Q(s, y) + \delta G(s, y, \delta)$$

where

$$P(s) = \Psi(s)^T[DF_0(\phi_0(s))\Psi(s) - \Psi'(s)], \tag{7}$$

$Q(s, y)$ and $G(s, y, \delta)$ are smooth functions of (s, y, δ), periodic in s of period T, and

$$Q(s, y) \sim O(|y|^2)$$

uniformly for $s \in R$. The variational equation of (2) with respect to $\phi_0(s)$ is

$$\frac{d\xi}{ds} = DF(\phi_0(s))\xi, \tag{8}$$

and $\xi = \phi_0'$ gives rise to 1 as a Floquet multiplier of (8). The remaining $n - 1$ Floquet multipliers of (8) are the Floquet multipliers of

$$\frac{dy}{ds} = P(s)y.$$

Any solution of (8) has the representation $\xi = \alpha\phi_0' + \Psi y$ for some function $\alpha(\cdot)$.

Next we shall apply this reduction to a system of two identical coupled oscillators. The generalization to a system of N coupled oscillators is straightforward, although it is advantageous to take account of any special structure in the geometry of the coupling function in this case [**12**]. Let $f : \mathbf{R}^m \to \mathbf{R}^m$, and suppose that $\eta(t)$ is a nonconstant hyperbolic periodic solution of

$$\frac{dz}{dt} = f(z)$$

with least period $T > 0$. Let $z_1, z_2 \in \mathbf{R}^m$, denote $x \in \mathbf{R}^n$ by

$$x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and define $F : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ by

$$F(x, \delta) = \begin{bmatrix} f(z_1) + \delta g_1(z_1, z_2, \delta) \\ f(z_2) + \delta g_2(z_1, z_2, \delta) \end{bmatrix}$$

where $g_1, g_2 : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^m$. The unperturbed system (2) has a one-parameter family of periodic solution

$$\phi_0(t, \theta) = \begin{bmatrix} \eta(t) \\ \eta(t + \theta) \end{bmatrix} \tag{9}$$

parameterized by $\theta \in [0, T)$. The family $\{\phi_0(t, \theta) | \theta \in [0, T), t \in \mathbf{R}\}$ defines a two-dimensional torus T_0^2 that is invariant under the flow when $\delta = 0$. Since we do not require that $g_1 \equiv g_2$, the formulation includes the case of weak coupling of two oscillators whose frequency differs by a term that is $O(\delta)$. By modifying the formulation slightly, it can also be used when the ratio of the frequencies differs from any integer (not necessarily one) by a term that is $O(\delta)$. Thus phase-locking of non-identical oscillators can be studied in this framework.

When $N = 2$, a given$\Phi(\cdot)$ that is admissable for $(f, \eta(\cdot))$ determines $m - 1$ normal directions to each orbit, and the remaining one of the $2m - 1 (= n - 1)$ normal directions is uniquely determined. Thus let $\Psi(s, \theta)$ denote the $n \times (n - 1)$ matrix given by

$$\Psi(s, \theta) = \begin{bmatrix} a(s, \theta) & \Phi(s) & O \\ b(s, \theta) & O & \Phi(s + \theta) \end{bmatrix},$$

where

$$a(s, \theta) = -\frac{|f(\eta(s + \theta))|}{|f(\eta(s))| \sqrt{|f(\eta(s))|^2 + |f(\eta(s + \theta))|^2}} f(\eta(s)),$$

$$b(s, \theta) = \frac{|f(\eta(s))|}{|f(\eta(s + \theta))| \sqrt{|f(\eta(s))|^2 + |f(\eta + \theta))|^2}} f(\eta(s + \theta)).$$

Clearly $\Psi(s,\theta)$ is a smooth function of (s,θ), and it follows that $\Psi(\cdot,\theta)$ is admissible for $(F,\phi(\cdot,\theta))$. Thus the perturbed system (1) has a periodic solution in a neighborhood of the orbit of $\phi(\cdot,\theta_0)$ for some fixed θ_0 if and only if, for some θ near θ_0, the equation

$$\frac{dy}{ds} = Y(s,y,\theta,\delta) \tag{10}$$

has a periodic solution. The function Y is given by

$$Y(s,y,\theta,\delta) = \Psi(s,\theta)^T[\tau(s,y,\theta,\delta)F(\phi_0(s,\theta)+\Psi(s,\theta)y,\delta) - \Psi'(s,\theta)y],$$

where

$$\tau(s,y,\theta,\delta) = \frac{F_0(\phi_0(s,\theta))^T[F_0(\phi_0(s,\theta))+\Psi'(s,\theta)y]}{F_0(\phi_0(s,\theta))^T F(\phi_0(s,\theta)+\Psi(s,\theta)y,\delta)} = 1+\mathcal{T}(s,y,\theta,\delta),$$

as before.

Equation (10) can be written in the form

$$\frac{dy}{ds} = P(s,\theta)y + Q(s,y,\theta) + \delta G(s,y,\theta,\delta) \tag{11}$$

where the terms on the right-hand side are defined as follows:

$$P(s,\theta) = \begin{bmatrix} P_{11}(s,\theta) & P_{12}(s,\theta) \\ P_{21}(s,\theta) & P_{22}(s,\theta) \end{bmatrix}$$

$$P_{11}(s,\theta) = \frac{d}{ds}\left(\ln\sqrt{\frac{|f(\eta(s))|^2\cdot|f(\eta(s+\theta))|^2}{|f(\eta(s))|^2+|f(\eta(s+\theta))|^2}}\right)$$

$$P_{12}(s,\theta) = \left(a(s,\theta)^T\Lambda(s), b(s,\theta)^T\Lambda(s+\theta)\right), \quad P_{21}(s,\theta)\equiv 0,$$

$$P_{22}(s,\theta) = \begin{bmatrix} \Phi(s)^T\Lambda(s) & O \\ O & \Phi(s+\theta)^T\Lambda(s+\theta) \end{bmatrix}$$

and

$$\Lambda(s) = Df(\eta(s))\Phi(s) - \Phi'(s).$$

Furthermore,

$$Q(s,y,\theta) \sim O(|y|^2)$$

uniformly for $s, \theta \in \mathbf{R}$.

The fundamental matrix $\Omega(s,\theta)$ of the variational system

$$\frac{dy}{ds} = P(s,\theta)y$$

with $\Omega(0,\theta) = I$ has the form

$$\Omega(s,\theta) = \begin{bmatrix} \Omega_{11}(s,\theta) & \Omega_{12}(s,\theta) \\ O & \Omega_{22}(s,\theta) \end{bmatrix},$$

where

$$\Omega_{11}(s,\theta) = \sqrt{\frac{|f(\eta(0))|^2 + |f(\eta(\theta))|^2}{|f(\eta(0))|^2 \cdot |f(\eta(\theta))|^2}} \sqrt{\frac{|f(\eta(s))|^2 \cdot |f(\eta(s+\theta))|^2}{|f(\eta(s))|^2 + |f(\eta(s+\theta))|^2}}$$

$$\Omega_{12}(s,\theta) = \int_0^s \Omega_{11}(s,\theta)\Omega_{11}^{-1}(u,\theta)P_{12}(u,\theta)\Omega_{22}(u,\theta)du$$

$$\Omega_{22}(s,\theta) = \begin{bmatrix} V(s) & O \\ O & V(s+\theta)V^{-1}(\theta) \end{bmatrix}$$

and $V(s)$ is the fundamental matrix solution of

$$\frac{dv}{ds} = \Phi(s)^T \Lambda(s) v \tag{12}$$

with $V(0) = I$. Clearly $\Omega_{11}(kT,\theta) = 1$, and we assume that $\Omega_{12}(kT,\theta) = 0$, for we can always choose a coordinate system in which this is true, and it is easy to see that this choice does not alter the fact that $P_{21}(s,\theta) \equiv 0$. Moreover, since $P_{22}(s,\theta)$ is the direct sum of the matrices $P(s)$ associated with each m-dimensional subsystem, the fact that the orbit $\eta(t)$ is hyperbolic implies that $\Omega_{22}(kT,\theta)$ has no eigenvalues of modulus one.

In the coupled system the first coordinate of y is a 'phase-like' coordinate, in that it measures distance orthogonal to the orbit $\phi_0(s,\theta)$ in the tangent space to T_0^2. (When there are N oscillators there are $N-1$ such coordinates.) The remaining $2(m-1)$ coordinates are normal coordinates, and it is advantageous to split y into 'phase-like' and normal coordinates. Therefore we write

$$y = \begin{bmatrix} \varphi \\ r \end{bmatrix}$$

where φ is the first component and r is the vector consisting of the last $n-2$ components of y. Similarly let

$$Q(s,y,\theta) = \begin{bmatrix} Q_1(s,\varphi,r,\theta) \\ Q_2(s,\varphi,r,\theta) \end{bmatrix},$$

$$G(s,y,\theta,\delta) = \begin{bmatrix} G_1(s,\varphi,r,\theta,\delta) \\ G_2(s,\varphi,r,\theta,\delta) \end{bmatrix}.$$

Then (11) becomes

$$\frac{d\varphi}{ds} = P_{11}(s,\theta)\varphi + P_{12}(s,\theta)r + Q_1(s,\varphi,r,\theta) + \delta G_1(s,\varphi,r,\theta,\delta) \tag{13}$$

$$\frac{dr}{ds} = P_{22}(s,\theta)r + Q_2(s,\varphi,r,\theta) + \delta G_2(s,\varphi,r,\theta,\delta). \tag{14}$$

The solution of this system that satisfies the initial condition

$$\varphi(0,r_0,\theta,\delta) = 0, \quad r(0,r_0,\theta,\delta) = r_0.$$

will be denoted

$$y(s,r_0,\theta,\delta) = \begin{bmatrix} \varphi(s,r_0,\theta,\delta) \\ r(s,r_0,\theta,\delta) \end{bmatrix}.$$

To determine whether any of the one-parameter family of solutions (9) that exists at $\delta = 0$ can be continued for $\delta \neq 0$, we look for solutions of the equations

$$\varphi(kT,r_0,\theta,\delta) = 0, \quad r(kT,r_0,\theta,\delta) - r_0 = 0 \tag{15}$$

for some positive integer k. Since $y \equiv 0$ is a solution of (11) when $\delta = 0, (r_0,\delta) = (0,0)$ satisfies (15) for any $\theta \in [0,T)$. Moreover it is easily shown that

$$\frac{\partial}{\partial r_0}[r(kT,r_0,\theta,\delta) - r_0] = \Omega_{22}(kT,\theta) - I.$$

This matrix is invertible in light of the remarks following (12), and the implicit function theorem implies that there is a smooth $R_k(\theta,\delta)$ defined for all $\theta \in [0,T)$ and small δ, with the properties that

$$r(kT,R_k(\theta,\delta),\theta,\delta) - R_k(\theta,\delta) = 0$$

and $R_k(\theta,\delta) \to 0$ as $\delta \to 0$. It follows that the equation

$$\varphi(k,T,R_k(\theta,\delta),\theta,\delta) = 0 \tag{16}$$

is satisfied at $\delta = 0$ for any $\theta \in [0,T)$. Thus there is a C^{r-1} function $h_k(\theta,\delta)$ such that

$$\varphi(kT,R_k(\theta,\delta),\theta,\delta) = \delta h_k(\theta,\delta),$$

and, for $\delta \neq 0$, (16) is equivalent to the equation

$$h_k(\theta,\delta) = 0.$$

If $h_k(\theta,0)$ has a simple zero at some θ_0 i.e., if $(\partial h_k(\theta,0)/\partial\theta)_{\theta_0} \neq 0$, then the original equation (10) has a periodic solution near $\phi_0(\cdot,\theta_0)$ whose period is close to kT for all small δ.

It can be shown that

$$\begin{aligned} h_k(\theta,0) = \int_0^{kT} & \Omega_{11}^{-1}(u,\theta)[P_{12}(u,\theta) \\ & \cdot \int_0^u \Omega_{22}(u,\theta)\Omega_{22}^{-1}(v,\theta)G_2(v,0,\theta,0)dv + G_1(u,0,\theta,0)]du \end{aligned}$$

and therefore

$$h_k(\theta,0) = ch_1(\theta,0).$$

In other words, $h_k(\theta,0)$ and $h_1(\theta,0)$ have the same zeros (if there are any) and they give rise to the same periodic solutions for small δ. Since h_k is a smooth function of δ these zeros are smooth functions of δ. We summarize these results in the following proposition.

PROPOSITION 1. *Suppose that the periodic orbit $\eta(t)$ of the uncoupled system is hyperbolic and that the bifurcation equation*

(17)

$$\begin{aligned} h_1(\theta,0) = \int_0^{T} & \Omega_{11}^{-1}(u,\theta)[P_{12}(u,\theta) \\ & \cdot \int_0^u \Omega_{22}(u,\theta)\Omega_{22}^{-1}(v,\theta)G_2(v,0,\theta,0)dv + G_1(u,0,\theta,0)]du \end{aligned}$$

has a simple zero at θ_0. Then, given an arbitrary neighborhood $\mathcal{N}$ of the orbit $\phi_0(\cdot, \theta_0)$, there is $\delta_0 > 0$ such that, for all $\delta \in (-\delta_0, \delta_0)$ (1) has a periodic solution ϕ_δ that is a smooth function of δ, whose orbit is contained in $\mathcal{N}$, and whose period is close to T. Furthermore, for sufficiently small δ there is a neighborhood of the orbit of $\phi_0(\cdot, \theta_0)$ in which there is no periodic solution of (1) whose least period is close to kT for any $k > 1$.

REMARK 1. Under the conditions in the proposition, bifurcation from the continuum of solutions that exists at $\delta = 0$ is transcritical, which means that the bifurcating solutions exist on both sides of $\delta = 0$. If one does not require that the zero be simple, then the dependence of θ on δ need not be smooth, and bifurcation may be one-sided.

REMARK 2. The stability of the bifurcating branches can be determined by a perturbation analysis of the critical multiplier or exponent. A branch is asymptotically stable for $\delta > 0$ and unstable for $\delta < 0$ if the $O(\delta)$ term in the critical exponent is negative. It should be noted that there is no exchange of stability at $\delta = 0$, even though there is a change in the stability of the bifurcating branch at the bifurcation point.

REMARK 3. As we mentioned in the Introduction, this method for treating the continuation of periodic solutions separately from the continuation of an invariant surface can be used in cases where it is not possible to prove that the surface persists under perturbation. However, it is known that the invariant torus perturbs smoothly in the problem just analyzed [**10**], and in the remainder of this remark we indicate how this can be proven within the present framework. We can write the integrated form of equations (13) and (14) as

$$\begin{aligned} \varphi_1 &\equiv \varphi(T) = \varphi_0 + \mathcal{F}_1(r_0, \varphi_0, \delta) \\ r_1 &\equiv (T) = \Omega_{22}(T, \theta) r_0 + \mathcal{F}_2(r_0, \varphi_0, \delta), \end{aligned}$$

and to these equations we append the equation $\delta_1 = \delta_0$ for the parameter. These can be written in the form

$$x_1 = G(x_0, \theta),$$

where $x \equiv (\varphi, \delta, r)^T$ and $G : \mathbf{R}^{2m} \times [0,T] \to \mathbf{R}^{2m}$. Since the spectrum of $\Omega_{22}(T,\theta)$ lies strictly within the unit disk, the spectrum of $DG(0,\theta)$ has $2(m-1)$ points within the unit disk and two points on the unit disk. Thus the center manifold theorem for maps in the form given in [**16**] can be used to prove the existence of a center manifold, whose representation is

$$r = h(\varphi, \theta, \delta)$$

where h is T-periodic in θ. The $\varphi = 0$ section of this generates a closed curve on the section $s = 0$, and the perturbed torus has the representation

$$x = \phi_0(s,\theta) + \Psi(s,\theta) \begin{bmatrix} \varphi(s, h(0,\theta,\delta), \theta, \delta) \\ r(s, h(0,\theta,\delta), \theta, \delta) \end{bmatrix}.$$

3. Preliminaries for a three-parameter analysis of coupled planar oscillators. In order to obtain an analytically-tractable problem for intermediate coupling strengths, one must choose a simple vector field having a periodic solution in the uncoupled state, and simple coupling functions. It was shown in [**8, 9,** and **10**] that a great deal can be done analytically when two planar systems described by the vector field

$$f(x,y) = \begin{bmatrix} ax + \beta y - x(x^2+y^2) \\ -\beta x + ay - y(x^2+y^2) \end{bmatrix}, \quad \alpha, \beta > 0,$$

are coupled linearly. In the coordinates

$$z_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad z_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix},$$

the governing equations analyzed in [**8, 9**, and **10**] are

$$\frac{dz_1}{dt} = f(z_1) + \delta D(z_2 - z_1), \quad \frac{dz_2}{dt} = f(z_2) + \delta D(z_1 - z_2), \tag{18}$$

where D is the 2×2 matrix with all entries equal to one. Since the vector field f is invariant under rotations, the vector field for the coupled system is equivalent under an orthogonal transformation to one in which the coupling matrix is given by $D = \operatorname{diag}(D_1, D_2) = \operatorname{diag}(2,0)$. Our purpose here is to determine how the structure of the bifurcation set changes when the coupling matrix is made nonsingular. When D_2 is small an elementary perturbation argument shows that the structure given in [**8, 9** and **10**] persists, but we shall analyze the changes that occur as D ranges from diag $(2,0)$ to diag $(2,2)$. However, we first rederive the perturbation results given in [**10**] to illustrate how simple the reduction procedure is for this system. Without loss of generality we may set $\alpha = 1$ and fix D_1, and in order to compare our results with those in [**8, 9** and **10**], we set $D_1 = 2$. Furthermore, we write $D_2 = 2(1-2\varepsilon)$ where $\varepsilon \in [0, 1/2]$. In component form (18) becomes

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_1 + \beta y_1 - x_1(x_1^2 + y_1^2) + 2\delta(x_1 - x_1) \\
\frac{dy_1}{dt} &= -\beta x_1 + y_1 - y_1(x_1^2 + y_1^2) + 2\delta(1-2\varepsilon)(y_2 - y_1) \\
\frac{dx_2}{dt} &= x_2 + \beta y_2 - x_2(x_2^2 + y_2^2) + 2\delta(x_1 - x_2) \\
\frac{dy_2}{dt} &= -\beta x_2 + y_2 - y_2(x_2^2 + y_2^2) + 2\delta(1-2\varepsilon)(y_1 - y_2).
\end{aligned} \tag{19}
$$

Thus $\varepsilon = 0$ corresponds to a coupling matrix that is a multiple of the identity, and $\varepsilon = 1/2$ corresponds to the problem studied in [**8, 9** and **10**].

Each two-dimensional subsystem of the uncoupled system has a unique periodic solution, whose period is $T = 2\pi/\beta$, given by

$$\eta(t) = \begin{bmatrix} \cos\beta t \\ -\sin\beta t \end{bmatrix}. \tag{20}$$

An admissible $\Phi(\cdot)$ is

$$\Phi(s) = \begin{bmatrix} \cos\beta s \\ -\sin\beta s \end{bmatrix},$$

and for this choice of $\Phi(\cdot), a(s,\theta)$ and $b(s,\theta)$ are given by

$$a(s,\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sin\beta s \\ \cos\beta s \end{bmatrix}, \quad b(s,\theta) = -\frac{1}{\sqrt{2}}\begin{bmatrix} \sin\beta(s+\theta) \\ \cos\beta(s+\theta) \end{bmatrix}.$$

Therefore $P(s,\theta)$ is given by

$$P(s,\theta) \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

and it follows that $\Omega_{11}(s,\theta) \equiv 1, \Omega_{12}(s,\theta) \equiv (0,0)$, and

$$\Omega_{22}(s,\theta) = \begin{bmatrix} e^{-2s} & 0 \\ 0 & e^{-2s} \end{bmatrix}.$$

Clearly $\Omega_{22}(kT,\theta) - I$ is nonsingular for any positive integer k. Thus the equation for y has the form assumed in (11) without applying a coordinate change.

One finds that

$$G_1(u,0,\theta,0) = (-4(1-\varepsilon)\sin\beta\theta - 2\varepsilon\sin 2\beta u + 2\varepsilon\sin 2\beta(u+\theta))/\sqrt{2}$$

and according to (17), $h_1(\theta,0)$ is given by

$$h_1(\theta,0) = \int_0^{2\pi/\beta} G_1(u,0,\theta,0)du = -\frac{4\sqrt{2}(1-\varepsilon)\pi}{\beta}\sin\beta\theta.$$

Therefore Proposition 1 applies, and the simple zeros of h_1 are $\theta_0 = 0$ and $\theta_0 = \pi/\beta$ for all k. It is easy to show that the former is asymptotically stable for $\delta > 0$ and unstable for $\delta < 0$, and that the latter has the opposite stability properties. This leads to the bifurcation diagram shown in Figure 1.

In the following we call the orbits of periodic solution which bifurcate from $\theta = 0$ the in-phase orbits and those that bifurcate from $\theta = \pi/\beta$ the out-of-phase orbits, and we denote them by ω_0 and ω_π, respectively. The orbit of ω_0 lies in the linear subspace $\mathcal{O} \subset \mathbf{R}^4$ defined as

$$\mathcal{O} = \{(z_1,z_2) \mid z_1 = z_2\},$$

and because $g_i(z,z) = 0, \omega_0$ exists for all $\delta \in \mathbf{R}$. The orbit of ω_π lies in the subspace

$$\prod = \{(z_1,z_2) \mid z_1 = -z_2\}.$$

The variational equation of (19) with respect to either ω_0 or ω_π splits into the systems

$$(21) \qquad \frac{d\xi_1}{dt} = K(t)\xi_1, \quad \frac{d\xi_2}{dt} = [K(t) - 2\delta D]\xi_2$$

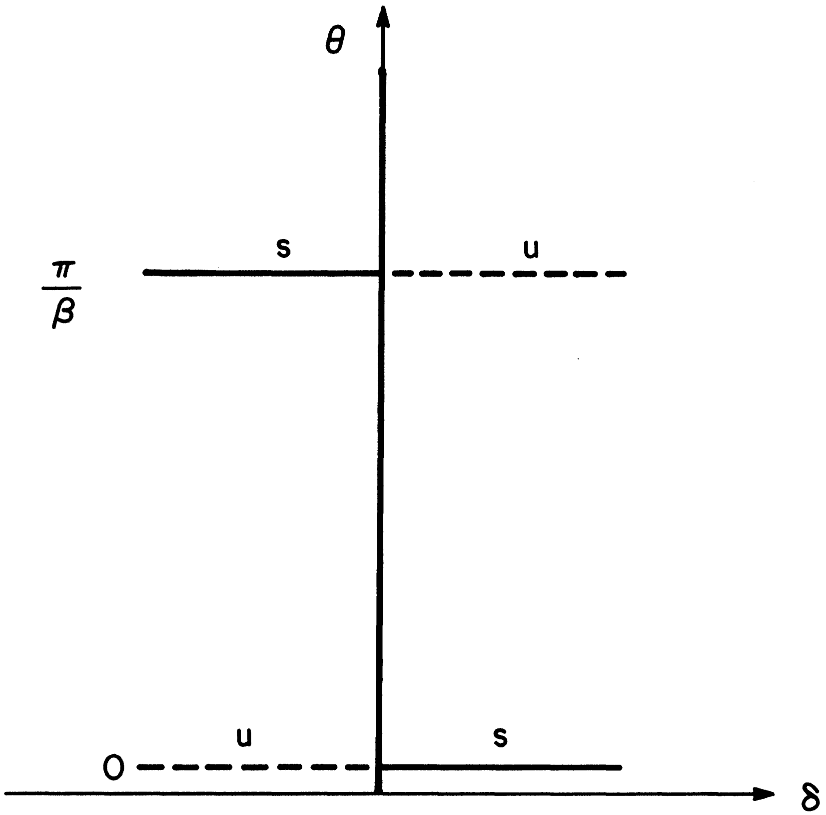

Figure 1. The bifurcation diagram for bifurcation from the continuum of solutions at $\delta = 0$. The solution labelled $0(\pi/\beta)$ corresponds to ω_0 (respectively, ω_π).

where $K(t)$ is the Jacobian of f along the orbit. Therefore the fundamental matrix solution of (21) has the form

$$W(t) = \begin{bmatrix} W_1(t) & 0 \\ 0 & W_2(t) \end{bmatrix}$$

where W_1 and W_2 satisfy the first and the second equation in (21) respectively, and the initial conditions $W_1(0) = I$ and $W_2(0) = I$.

To simplify the description of the changes in the stability properties of periodic orbits, we define the Floquet signature of an orbit $\gamma \in \mathbf{R}^4$

as follows. We associate a '+', a '−', or a '0' with each multiplier of the periodic variational system for the orbit, according as the modulus of the multiplier is greater than one, less than one, or equal to one, respectively. The Floquet signature of the orbit is then defined as $\sigma(\gamma) = (\sigma_1, \sigma_2, \sigma_3, \sigma_3)$, where σ_i is $+, -$, or 0. When dealing with orbits that lie in $\mathcal{O}$ or in $\prod$, the first (second) pair of entries of $\sigma(\cdot)$ will refer to the multipliers associated with the first (respectively, second) equation in (21).

It is quite easy to show that ω_0 is asymptotically stable for all $\delta > 0$, and unstable for $\delta < 0$ and either sufficiently large or sufficiently small in magnitude. A sketch of the proof of this fact goes as follows. In the coordinates

$$w_1 = \frac{z_1 + z_2}{2}, \quad w_2 = \frac{z_1 - z_2}{2}$$

adapted to $\mathcal{O}$ and $\prod$, (18) becomes

$$\begin{aligned} \frac{dw_1}{dt} &= \frac{1}{2}[f(w_1 + w_2) + f(w_1 - w_2)] \\ \frac{dw_2}{dt} &= \frac{1}{2}[f(w_1 + w_2) - f(w_1 - w_2] - 2\delta D w_2 \end{aligned} \tag{22}$$

and ω_0 is given by

$$w_1(t) = \begin{bmatrix} \cos\beta t \\ -\sin\beta t \end{bmatrix}, \quad w_2(t) \equiv 0.$$

For this orbit the matrix $K(t)$ is given as

$$K(t) = \begin{bmatrix} -(1 + \cos 2\beta t) & \beta + \sin 2\beta t \\ -\beta + \sin 2\beta t & -(1 - \cos 2\beta t) \end{bmatrix}.$$

Let $\{\lambda_1, \lambda_2\}$ and $\{\lambda_3 \lambda_4\}$ be the eigenvalues of $W_1(T)$ and $W_2(T)$ respectively. Since

$$W_1(t) = \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix},$$

$\lambda_1 = 1$ and $\lambda_2 = e^{-2T}$. By an argument identical to that used in proposition 2 of [**10**], it can be shown that the pair $\{\lambda_3, \lambda_4\}$ lies inside the unit circle when $\delta > 0$. Similarly one can show that at least one of the pair lies outside the unit circle when $\delta < 0$ and either $|\ \delta\ |$ is sufficiently small or $\delta < -1/4(1-\varepsilon)$. Thus the Floquet signature of ω_0 is $(0,-,-,-)$ for $\delta > 0$, and $(0,-,\pm,+)$ for $\delta < 0$ and either sufficiently large or sufficiently small in magnitude. It is not known whether or not ω_0 is stable on a subset of the excluded set of negative δ values.

Next we determine the region of existence of ω_π in $(\beta,\delta,\varepsilon)$ space. Because f is odd, $w_1 \equiv 0$ is the first component of a solution of (22) provided that w_2 satisfies

$$\frac{dw_2}{dt} = f(w_2) - 2\delta D w_2.$$

The components (u,v) of w_2 satisfy the system

$$\begin{aligned} \frac{du}{dt} &= (1-4\delta)u + \beta v - u(u^2+v^2) \\ \frac{dv}{dt} &= -\beta u + (1-4\delta(1-2\varepsilon))v - u(u^2+v^2). \end{aligned} \tag{23}$$

For $-\beta/4\varepsilon < \delta < \min(1/4(1-\varepsilon), \beta/4\varepsilon)$ we introduce the coordinate transformation

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} \mathcal{P}\cos\varsigma \\ \mathcal{P}\sin\varsigma \end{pmatrix} = AR_\phi \begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$A = \begin{pmatrix} \sqrt{1+k} & 0 \\ 0 & \sqrt{1-k} \end{pmatrix}, \quad R_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

and

$$k = \frac{4\delta\varepsilon}{\sqrt{[1-4\delta(1-\varepsilon)]^2+\beta^2}}, \quad \phi = \frac{1}{2}\arctan\frac{-\beta}{1-4\delta(1-\varepsilon)}. \tag{24}$$

Then (23) becomes

$$\begin{aligned} \frac{d\mathcal{P}}{dt} &= (1-k\cos 2\varsigma)\left\{[1-4\delta(1-\varepsilon)]\mathcal{P} - \frac{\mathcal{P}^3}{1-k^2}\right\} \\ \frac{d\varsigma}{dt} &= -\beta\sqrt{1-k^2} + [1-4\delta(1-\varepsilon)]k\sin 2\varsigma, \end{aligned}$$

and by (24), $| k |< 1$ and thus $\frac{d\varsigma}{dt} < 0$. Therefore $P = \sqrt{(1-k^2)(1-4\delta(1-\varepsilon))}$ is the orbit of a limit cylce in the (U,V)-plane whose period is

$$T_\varepsilon(\delta,\beta) = \int_0^{2\pi} \frac{d\varsigma}{\beta\sqrt{1-k^2} - [1-4\delta(1-\varepsilon)]k\sin 2\varsigma} = \frac{2\pi}{\sqrt{\beta^2-(4\delta\varepsilon)^2}}.$$

In the polar coordinates $u = \rho\cos\theta, v = \rho\sin\theta$, (23) becomes

$$(25)\qquad \begin{aligned} \frac{d\rho}{dt} &= \rho - \rho^3 - 4\delta\rho(1-2\varepsilon\sin^2\theta) \\ \frac{d\theta}{dt} &= -\beta + 4\varepsilon\delta\sin 2\theta, \end{aligned}$$

and in these coordinates the limit cycle is given by

$$\rho^2(t) = \frac{[1-4\delta(1-4\varepsilon)](1-k^2)}{1+k\cos 2(\theta(t)-\phi)}.$$

where $\theta(t)$ satisfies (25). In w coordinates ω_π is given by $(0,0,\rho(t)\cos\theta(t),\rho(t)\sin\theta(t))^T$.

The region of existence of ω_π and its stability properties relative to $\prod$ are readily deduced from the preceding results. For convenience we summarize the information about ω_0 and ω_π in the following proposition (see also Figure 2).

PROPOSITION 2.

(a) *The periodic solution ω_0 exists for all $\delta \in \mathbf{R}$ and $\varepsilon \in [0,1/2]$. It is asymptotically stable for all $\delta \in \mathbf{R}^+$, and there is a $\delta_0(\varepsilon) > 0$ such that ω_0 is unstable for all $\delta \in (-\infty,-1/4(1-\varepsilon)) \cup (-\delta_0,0)$.*

(b) *The periodic solution ω_π exists for $-\beta/4\varepsilon < \delta < \min\{1/4(1-\varepsilon),\beta/4\varepsilon\}$. When $\beta > \varepsilon/(1-\varepsilon)$ and $\delta > 0$, ω_π disappears via a Hopf bifurcation at the origin as $\delta \to 1/4(1-\varepsilon)$. When $\delta > 0$ and $\beta < \varepsilon(1-\varepsilon)$, or when $\delta < 0$, the period $T_\varepsilon(\delta,\beta) \to \infty$ as $\delta \to \pm\beta/4\varepsilon$, and a pair of fixed points appear on ω_π.*

4. Changes in the resonance structure for $\varepsilon \in [0,1/2]$. In this section we analyze how the stability of ω_π varies with β,δ, and ε. Since most of the results are direct extensions of those in **[10]** to the

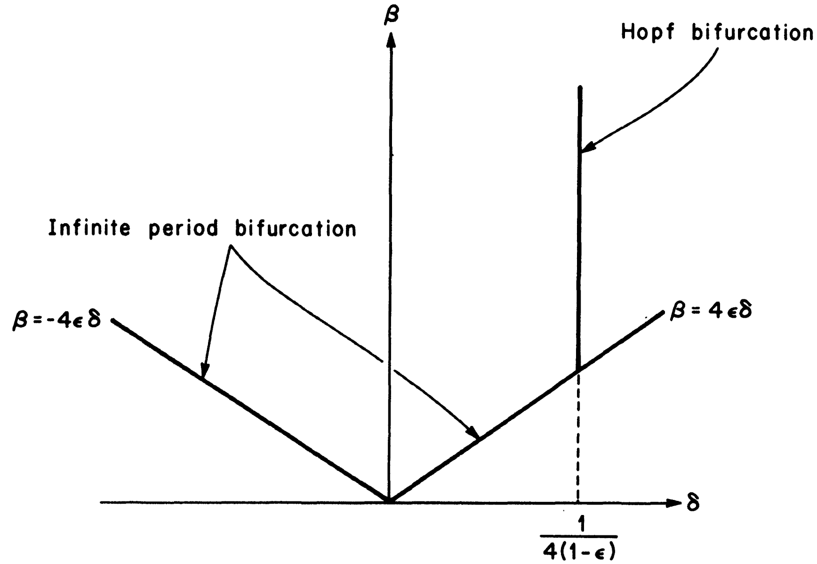

Figure 2. The region of existence of ω_π in the (δ,β)-plane for fixed $\varepsilon \in [0,1/2]$.

case $\varepsilon \neq 1/2$, we refer the reader to that paper for background. The variational equation of (18) with respect to ω_π has the form in (21) with

(26)

$$K(t) = \begin{bmatrix} 1-2\rho^2(t)-\rho^2(t)\cos 2\theta(t) & \beta-\rho^2(t)\sin 2\theta(t) \\ -\beta-\rho^2(t)\sin 2\theta(t) & 1-2\rho^2(t)+\rho^2(t)\cos 2\theta(t) \end{bmatrix}.$$

It follows from Proposition 2 that $(\sigma_3,\sigma_4) = (0,-)$ in the Floquet signature of ω_π, and therefore we only have to determine how $\{\lambda_1,\lambda_2\}$ vary with β,δ and ε.

According to Liouville's formula,

$$\lambda_1\lambda_2 = \exp\left(\int_0^{T\varepsilon(\delta,\beta)} trK(s)ds\right).$$

This integral can be evaluated explicitly, and the result is that

$$\lambda_1\lambda_2 = \exp\left\{\frac{4\pi[8(1-\varepsilon)\delta - 1]}{\sqrt{\beta^2 - (4\varepsilon\delta)^2}}\right\}.$$

Consequently

$$\lambda_1\lambda_2 \begin{cases} < 1 & \text{if } \delta < 1/8(1-\varepsilon) \\ = 1 & \text{if } \delta = 1/8(1-\varepsilon), \\ > 1 & \text{if } \delta > 1/8(1-\varepsilon) \end{cases}$$

which implies that ω_π is unstable for $\delta > 1/8(1-\varepsilon)$ and is asymptotically stable whenever $\delta < 1/8(1-\varepsilon)$ and $|\lambda_1| = |\lambda_2|$.

In order to analyze the variational equation

$$\frac{d\xi_1}{dt} = K(t)\xi_1 \tag{27}$$

with $K(t)$ as given in (26) we introduce the polar coordinates

$$\xi_1 = \begin{bmatrix} R\cos\Psi/2 \\ R\sin\Psi/2 \end{bmatrix}$$

and double angle $\Theta = 2\theta$. Then (27) and equation (25) define the autonomous system

$$\begin{aligned} \frac{d\Theta}{dt} &= -2(\beta - 4\varepsilon\delta\sin\Theta) \\ \frac{d\Psi}{dt} &= -[\beta - S(\Theta)\sin(\Psi - \Theta)] \\ \frac{dR}{dt} &= R[1 - 2S(\Theta) - S(\Theta)\cos(\Psi - \Theta)], \end{aligned} \tag{28}$$

where

$$S(\Theta) = \frac{[1 - 4(1-\varepsilon)\delta](1-k^2)}{1 + k\cos(\Theta - \Phi)}, \quad \Phi = \arctan\frac{-\beta}{1 - 4(1-\varepsilon)\delta}.$$

When $\beta > 4\varepsilon\delta$ we can use Θ as the independent variable in (28) and obtain the system

$$\begin{aligned} \frac{d\Psi}{d\Theta} &= \frac{\beta - S(\Theta)\sin(\Psi - \Theta)}{\beta - 4\varepsilon\delta\sin\Theta} \\ \frac{dR}{d\Theta} &= -\frac{R[1 - 2S(\Theta) - S(\Theta)\cos(\Psi - \Theta)]}{2(\beta - 4\varepsilon\delta\sin\Theta)}. \end{aligned} \tag{29}$$

Let $\Psi(\Theta, \Theta_0, \Psi_0, \delta, \beta)$ be the solution of (29) which satisfies $\Psi(\Theta_0, \Theta_0, \Psi_0, \delta, \beta) = \Psi_0$. Since the right hand side of (29) is 2π-periodic in Θ and Ψ, the flow of this equation defines a circle map C of the section $\Theta = \Theta_0$ to itself. The rotation number

$$r(\delta, \beta) = \lim_{k \to \infty} \frac{\Psi(\Theta_0 + 2k\pi, \Theta_0, \Psi_0, \delta, \beta) - \Psi_0}{2k\pi}$$

of this map is defined and continuous on $\mathcal{D}_\varepsilon \equiv \{(\delta, \beta) \mid 0 < \delta < 1/4(1-\varepsilon), \beta > 4\varepsilon\delta\}$ and is independent of Θ_0 and Ψ_0. The relationship between the eigenvalues of $W_1(T_\varepsilon(\delta, \beta)/2)$ and the rotation number is given by the following lemma, which is proven in [**10**].

LEMMA 1. *$W_1(T_\varepsilon(\delta, \beta)/2)$ has a real eigenvalue if and only if there are Θ_0 and Ψ_0 for which*

$$\Psi(\Theta_0 + 2\pi, \Theta_0, \Psi_0, \delta, \beta) = \Psi_0 + 2\eta\pi$$

for some integer n.

Said otherwise, $W_1(T_\varepsilon(\delta, \beta)/2)$ has a real eigenvalue if and only if the circle map C has a fixed point. If $r(\delta, \beta)$ is not an integer the eigenvalues of $W_1(T_\varepsilon(\delta, \beta))$ are either complex conjugates, or, if they are real, they must be equal. The following proposition follows from Lemma 1 and this observation.

PROPOSITION 3. *If $(\delta, \beta) \in \mathcal{D}_\varepsilon \cap \{(\delta, \beta) \mid \delta < 1/8(1-\varepsilon)\}$ and $r(\delta, \beta)$ is not an integer, then ω_π is asymptotically orbitally stable with asymptotic phase.*

At sufficiently small δ or sufficiently large β the rotation number is always less than two. The proofs of the following two lemmas, which make this statement precise, are analogous to the proofs of similar results given in [**10**], although the details in Lemma 3 are different.

LEMMA 2. *Let*

$$\mathcal{D}_{\varepsilon,1} = \left\{(\delta, \beta) \in \mathcal{D}_\varepsilon \mid \frac{4\varepsilon\delta}{1 - 4(1-\varepsilon)\delta} \leq 1 - \mid k \mid \right\}.$$

Then, for each $(\delta, \beta) \in \mathcal{D}_{\varepsilon,1}$, $r(\delta, \beta) = 1$.

LEMMA 3. *There exists* $C > 0$ *independent of* $\varepsilon \in (0, 1/2]$ *such that if*

$$\mathcal{D}_{\varepsilon,2} = \left\{(\delta, \beta) \in \mathcal{D}_\varepsilon \mid \beta \geq \frac{3\varepsilon}{\sqrt{5}(1-\varepsilon)}, \delta \geq \frac{1}{4} + \frac{\varepsilon(1-\varepsilon) - C\sqrt{\beta^2(1-\varepsilon)^2 + \varepsilon^2}}{4(1-\varepsilon)^2}\right\},$$

then, for all $(\delta, \beta) \in \mathcal{D}_{\varepsilon,2}, r(\delta, \beta) \leq 7/4$.

Next we consider the behavior of the rotation number near the half line $\mathcal{L} = \{(\delta, \beta) | \delta = 1/4(1-\varepsilon), \beta > \varepsilon/(1-\varepsilon)\}$, on which the Hopf bifurcations occur. For $\beta > \varepsilon/(1-\varepsilon)$, let $\mu = 1 - 4(1-\varepsilon)\delta$, and write (29) in the form

$$\frac{d\Psi}{d\Theta} = \overline{\omega} + \mu f(\Psi, \Theta, \mu), \tag{30}$$

where f is 2π-periodic in Ψ and Θ and the average frequency of the unperturbed flow is

$$\overline{\omega} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\beta(1-\varepsilon)d\Theta}{\beta(1-\varepsilon) - \varepsilon \sin \Theta} = \frac{\beta(1-\varepsilon)}{\sqrt{\beta^2(1-\varepsilon)^2 - \varepsilon^2}}.$$

When $\mu = 0$ the $\Psi - \Theta$ flow is periodic if and only if $\overline{\omega}$ is rational, i.e., if and only if $r(1/4(1-\varepsilon), \beta)$ is rational, and the question is whether such periodic solutions can be continued for $\mu > 0$. Equations of the form (30) have been studied in **[17]**, where it is shown that the set

$$H_p = \{(\delta, \beta) \in \mathcal{D}_\varepsilon \mid r(\delta, \beta) = p \text{ for } p \text{ rational }\}$$

is a cusp-like region with apex at

$$(\delta, \beta) = \left(\frac{1}{4(1-\varepsilon)}, \frac{\varepsilon p}{(1-\varepsilon)\sqrt{p^2 - 1}}\right).$$

In light of Proposition 3, it is necessary to determine where the rotation number is an integer in order to determine the stability of ω_π. In fact, if

$$\left(\mathcal{D}_\varepsilon \backslash \cup_{n=2}^{\infty} H_n\right) \cap \left\{(\delta, \beta) \mid \delta < \frac{1}{8(1-\varepsilon)}\right\}$$

is not empty, then ω_π is stable for some $(\delta, \beta) \in \mathcal{D}_\varepsilon$. The sets $H_n, n = 2, 3, 4, \ldots$ are called resonance horns in [**10**], and it remains to determine the behavior of these sets in $\mathcal{D}_\varepsilon$.

Firstly, a horn cannot terminate in $\mathcal{D}_\varepsilon$, for this would violate the continuity properties of the rotation number as a function of (δ, β). Moreover by the uniqueness of the rotation number of (29) different horns cannot intersect each other. By Lemmas 2 and 3 H_n must remain in the region

$$\mathcal{H}_\varepsilon = (\mathcal{D}_\varepsilon \backslash \mathcal{D}_{\varepsilon,1}) \cap (\mathcal{D}_\varepsilon \backslash \mathcal{D}_{\varepsilon,2}).$$

From these facts it follows that each horn must terminate on the line $\beta = 4\varepsilon\delta$ for some $\delta \in (\delta_\varepsilon^*, 1/4(1-\varepsilon))$, where δ_ε^* is the value of δ for which $\partial\mathcal{D}_{\varepsilon,1}$ intersects the line $\beta = 4\varepsilon\delta$. Furthermore, it can be shown that $r(\delta, \beta_0) \to \infty$ as $\delta \to \beta_0/4\varepsilon$ for any $\beta_0 \in (\varepsilon, \varepsilon/(1-\varepsilon))$. Thus all the horns must terminate on the open interval $\mathcal{I} = \{(\delta, \beta) \mid \beta = 4\varepsilon\delta, \delta_\varepsilon^* < \delta < 1/4)\}$, and it can be shown that they accumulate only at $(\delta, \beta) = (1/4, \varepsilon)$ (cf. Figure 3). On the other hand, $\mathcal{I}_0 \equiv \mathcal{I} \cap \{(\delta, \beta) \mid \beta > 0, 0 < \delta < 1/8(1-\varepsilon)\}$ coincides with $\mathcal{I}$ when $\varepsilon = 1/2$, but is a strict subset of $\mathcal{I}$ for any $\varepsilon < 1/2$. Thus only finitely many of the resonance horns intersect the region of the (δ, β)-plane in which $\lambda_1\lambda_2 < 1$ for any $\varepsilon < 1/2$. Furthermore, one can show that there is an $\varepsilon_0 \in (0, 1/2]$ such that for all $\varepsilon < \varepsilon_0, \mathcal{I}_0$ is empty, which implies that all the resonance horns are confined to the region in which ω_π is unstable. Finally, if we let δ_1 be the δ coordinate at which the left boundary of H_2 intersects $\mathcal{I}$, then we can show that there exists an ε_1 such that $\delta_1 < 1/8(1-\varepsilon)$ when $\varepsilon \in [\varepsilon_1, 1/2]$. We may summarize these conclusions as follows.

PROPOSITION 4. *For each $\varepsilon \in [\varepsilon_1, 1/2]$ there is an open set in $\mathcal{D}_\varepsilon$ on which ω_π is asymptotically orbitally stable with asymptotic phase. For each $\varepsilon \in [0, 1/2]\omega_\pi$ is unstable for $\delta > 1/8(1-\varepsilon)$.*

Note that we have not ruled out the possibility that ω_π is stable in some subset of $\mathcal{D}_\varepsilon$ when the resonance horns do not intersect the region in which $\lambda_1\lambda_2 < 1$. From the foregoing it is easy to see that the closure of $\mathcal{H}_\varepsilon$ converges to the point $(\delta, \beta) = (1/4, 0)$ as $\varepsilon \to 0$, i.e., all the resonance horns collapse to a point at $\varepsilon = 0$. Furthermore,

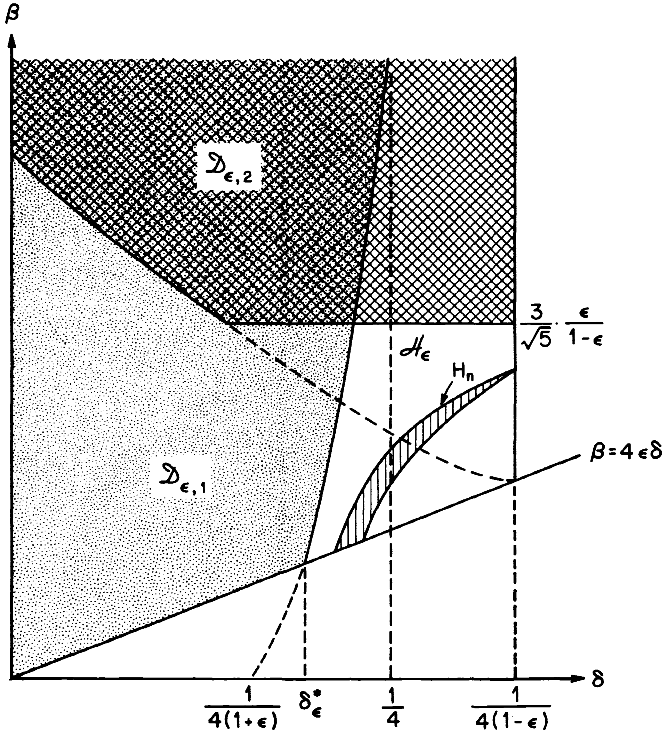

Figure 3. Schematic of the region in the (δ, β)-plane in which the resonance horns exist. The rotation number is 1 in $\mathcal{D}_{\varepsilon,1}$ and less than 7/4 in $\mathcal{D}_{\varepsilon,2}$. $\lambda_1\lambda_2 < 1$ for $\delta < 1/8(1-\varepsilon)$. As $\varepsilon \to 0$ the Hopf bifurcation line ($\delta = 1/4(1-\varepsilon)$) approaches $\delta = 1/4$, the infinite period line ($\beta = 4\varepsilon\delta$) approaches $\beta = 0$, and $\mathcal{H}_\varepsilon$ shrinks to the point $(\delta, \beta) = (1/4, 0)$.

the period-doubling cascades suggested by numerical work in [**10**] must also disappear at this point.

One finds that at $(\delta, \beta, \varepsilon) = (1/4, 0, 0)$ the Jacobian at the rest point $(0, 0, 0, 0)$ is similar to $I_2 \oplus O$, where O is the zero matrix, and thus there is a codimension-four singularity at this point. Our results show that the resonance structure found in [**10**] for $\varepsilon = 1/2$ arises from the three-parameter partial unfolding of this singularity analyzed herein. Clearly it would be desired to have a complete unfolding of this singularity.

5. Analytical results on the first bifurcation from ω_π. The uncoupled system also has periodic solutions of the form $((0,0)\times\eta(t))$ and $(\eta(t)\times(0,0))$, where $\eta(t)$ is given by (20). Numerical computations done in [10] show that the continuations of this pair of solutions for $\delta > 0$ connects to ω_π for some $\delta \in (0, 1/2)$. In this section we obtain this result analytically when $\varepsilon = 0$, i.e., when the coupling matrix is a multiple of the identity.

When $\varepsilon = 0$, the representation of ω_π in z coordinates is given by

$$z_\delta(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \sqrt{1-4\delta}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \\ -\cos\beta t \\ \sin\beta t \end{pmatrix}.$$

This solution exists for $\delta \in (-\infty, 1/4)$ and bifurcates from the origin at $\delta = 1/4$. In order to analyze the bifurcations from ω_π we perform the reduction done in §2. For each $(F(\cdot,\delta), z_\delta(\cdot))$, the matrix $\Psi(\cdot)$ given by

$$\Psi(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}\sin\beta s & \cos\beta s & 0 \\ \frac{1}{\sqrt{2}}\cos\beta s & -\sin\beta s & 0 \\ \frac{1}{\sqrt{2}}\sin\beta s & 0 & \cos\beta s \\ \frac{1}{\sqrt{2}}\cos\beta s & 0 & -\sin\beta s \end{pmatrix} \tag{31}$$

is admissible, and (7) takes the form

$$P(s,\delta) = \Psi(s)^T[DF(\psi_\delta(s),\delta)\Psi(s) - \Psi'(s)] \equiv \begin{pmatrix} 4\delta & 0 & 0 \\ 0 & 2(5\delta-1) & 2\delta \\ 0 & 2\delta & 2(5\delta-1) \end{pmatrix}.$$

Since the eigenvalues of $P(s,\delta)$ are $4\delta, 2(6\delta-1)$, and $2(4\delta-1)$, the Floquet multipliers of the variational equation

$$\frac{dy}{ds} = P(s,\delta)y$$

are $e^{8\delta\pi/\beta}, e^{4(6\delta-1)\pi/\beta}$, and $e^{4(4\delta-1)\pi/\beta}$. Thus periodic solutions may bifurcate from ω_π at $\delta = 1/6$. To facilitate the bifurcation analysis, let

$\mu = 1/6 - \delta$ and let $F(\cdot, \mu)$ be the vector field in (19). Then one finds that the equations for the normal components are

$$\begin{aligned}
\frac{dy_1}{ds} &= \frac{\sqrt{2}\beta y_1}{6[\sqrt{3}\beta + y_1(y_2+y_3)]}[4 - 3y_1^2 - 2\sqrt{3}(y_2 - y_3) + 6y_2y_3] \\
\frac{dy_2}{ds} &= \frac{\sqrt{2}\beta}{6\sqrt{3}[\sqrt{3}\beta + y_1(y_2+y_3)]}[4\sqrt{3}y_2 + 2\sqrt{3}y_3 + 2 \\
&\quad - (\sqrt{3}y_2 + 1)(3y_1^2 + 2(\sqrt{3}y_2+1)^2) \\
&\quad - 3\sqrt{3}y_1^2(y_2+y_3) + 12\mu(\sqrt{3}y_2 - \sqrt{3}y_3 + 2)] \\
\frac{dy_3}{ds} &= \frac{\sqrt{2}\beta}{6\sqrt{3}[\sqrt{3}\beta + y_1(y_2+y_3)]}[2\sqrt{3}y_2 + 4\sqrt{3}y_3 - 2 \\
&\quad - (\sqrt{3}y_3 - 1)(3y_1^2 + 2(\sqrt{3}y_3-1)^2) \\
&\quad - 3\sqrt{3}y_1^2(y_2+y_1) - 12\mu(\sqrt{3}y_2 - \sqrt{3}y_3 + 2)].
\end{aligned}$$

One steady-state solution of these equations is

$$\begin{aligned}
y_1 &= 0 \\
y_2 &= \frac{1}{\sqrt{3}}(\sqrt{12\mu+1} - 1) \\
y_3 &= -\frac{1}{\sqrt{3}}(\sqrt{12\mu+1} - 1),
\end{aligned}$$

and this solution corresponds to ω_π. It is easy to verify that another pair of solutions is given by

$$\begin{aligned}
y_1 &= 0 \\
y_2 &= \frac{1}{6}[\sqrt{2-3\mu} - \sqrt{2} \pm 3\sqrt{\mu}] \\
y_3 &= \frac{1}{6}[-\sqrt{2-3\mu} + \sqrt{2} \pm 3\sqrt{\mu}]
\end{aligned} \tag{32}$$

whenever $\mu \in [0, 2/3]$ or, equivalently, when $\delta \in [-1/2, 1/6]$. These solutions are the normal components of periodic solutions that bifurcate from ω_π at $\mu = 0$ ($\delta = 1/6$) and connect with ω_0 at $\mu = 2/3$ (i.e., at $\delta = -1/2$). For $(F(\cdot,\mu), z_{1/6})$ as before and $\Psi(s)$ as in (31), equation (4) becomes

$$\frac{dt}{ds} = \frac{\sqrt{2}\beta}{\sqrt{2}\beta + y_1(y_2+y_3)}.$$

Consequently,

$$\frac{dt}{ds} \equiv 1$$

along the solution given by (32). It follows that the two periodic solutions that bifurcate from ω_π at $\delta = 1/6$ have the representation

$$z_\delta^1(t) = \begin{pmatrix} r_1 \cos\beta t \\ -r_1 \sin\beta t \\ -r_1 \cos\beta t \\ r_1 \sin\beta t \end{pmatrix}, \quad z_\delta^2(t) = \begin{pmatrix} r_1 \cos\beta t \\ -r_1 \sin\beta t \\ -r_1 \cos\beta t \\ r_1 \sin\beta t \end{pmatrix},$$

where

$$r_1 = \frac{\sqrt{1+2\delta} + \sqrt{1-6\delta}}{2} \text{ and } r_2 = \frac{\sqrt{1+2\delta} - \sqrt{1-6\delta}}{2}.$$

These solutions are symmetry pairs under interchange of the oscillators and inversion through the origin. It is clear that

$$z_0^1(t) = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \\ 0 \\ 0 \end{pmatrix}, \quad z_0^2(t+\pi/\beta) = \begin{pmatrix} 0 \\ 0 \\ \cos\beta t \\ -\sin\beta t \end{pmatrix},$$

and that these coincide with the periodic solutions $(\eta(t)) \times (0,0)$ and $(0,0) \times (\eta(t))$, respectively.

To analyze the stability of the solutions $z_\delta^i, i = 1,2$, we define

$$\Psi_\delta(s) = \begin{pmatrix} \frac{r_2}{\sqrt{r_1^2+r_2^2}} \sin\beta s & \cos\beta s & 0 \\ \frac{r_2}{\sqrt{r_1^2+r_2^2}} \cos\beta s & -\sin\beta s & 0 \\ \frac{r_1}{\sqrt{r_1^2+r_2^2}} \sin\beta s & 0 & \cos\beta s \\ \frac{r_1}{\sqrt{r_1^2+r_2^2}} \cos\beta s & 0 & -\sin\beta s \end{pmatrix}.$$

This matrix is admissible for $(F(\cdot, 1/6), z_\delta^1(\cdot))$, and (7) becomes

$$\begin{aligned} P_\delta(s) &= \Psi_\delta(s)^T[DF(\psi_{1,\delta}(s),\delta)\Psi_\delta(s) - \Psi_\delta'(s)] \\ &\equiv \begin{pmatrix} 1-2\delta & 0 & 0 \\ 0 & 1-2\delta-3r_1^2 & 2\delta \\ 0 & 2\delta & 1-2\delta-3r_2^2 \end{pmatrix}. \end{aligned}$$

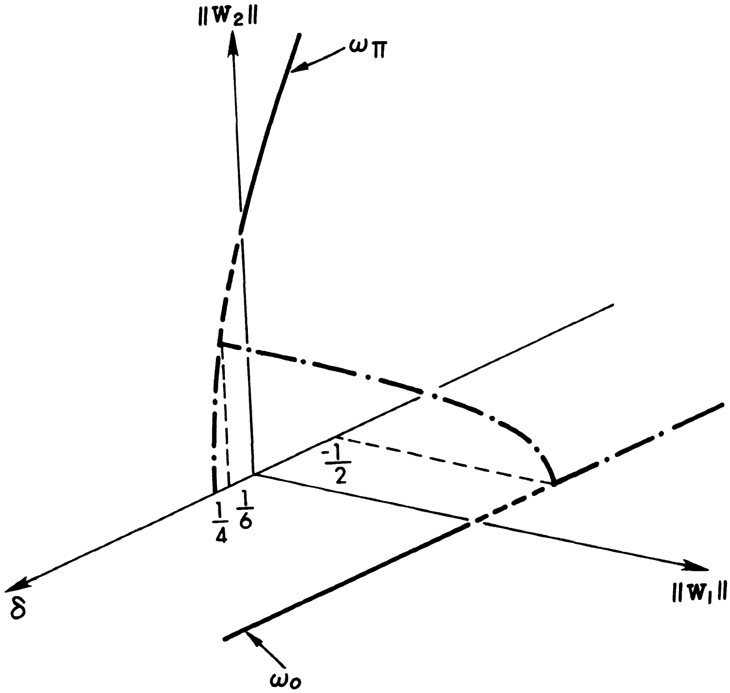

Figure 4. The global branches of periodic solutions that exist at $\varepsilon = 0$. ————: Floquet signature $= (0,-,-,-)$; — — —: Floquet signature $= (0,-,-,+)$; — · — ·: Floquet signature $= (0,-,+,+)$

Since the solutions z_δ^i exist for $\delta \in (-1/2, 1/6)$ it follows that

$$\operatorname{tr}\begin{pmatrix} 1-2\delta-3r_1^2 & 2\delta \\ 2\delta & 1-2\delta-3r_2^2 \end{pmatrix} = -(1-2\delta) < 0$$

and

$$\det\begin{pmatrix} 1-2\delta-3r_1^2 & 2\delta \\ 2\delta & 1-2\delta-3r_2^2 \end{pmatrix} = 2(2\delta+1)(6\delta-1) < 0.$$

Thus the Floquet signature of z_δ^1 is

(33) $$(0,-,+,+) \text{ for } -1/2 < \delta < 1/6.$$

A similar argument shows that the Floquet signature of z_δ^2 is also given by (33), which leads to the following proposition.

PROPOSITION 5. *When $\varepsilon = 0$ there is a supercritical bifurcation of unstable periodic solutions from ω_0 at $\delta = -1/2$. These periodic solutions coincide with the periodic solutions $(\eta(t)) \times (0,0)$ and $(0,0) \times (\eta(t))$ at $\delta = 0$, and disappear via a second Hopf bifurcation from ω_π at $\delta = 1/6$. The Floquet signature of solutions on this secondary branch satisfy* (33).

The global branch of solutions that bifurcates from ω_0 at $\delta = -1/2$ is shown in Figure 4. It is noteworthy that this branch varies between solutions on ω_0 and solutions on ω_π as δ varies in $[-1/2, 1/6]$. Thus there is a smooth transition between in-phase and out-of-phase oscillations. Numerical results in **[10]** for $\varepsilon = 1/2$ show that the periodic solutions $(\eta(t)) \times (0,0)$ terminates either by connecting to a periodic solution that bifurcates from ω_π as for $\varepsilon = 0$, or via an infinite-period bifurcation, depending on the magnitude of β. At present it is not understood how the solution structure found at $\varepsilon = 0$ relates to the structure found numerically at $\varepsilon = 1/2$.

REFERENCES

1. A.I. Selverston and M. Moulins, *Oscillatory neural networks,* Ann. Rev. Physiol. **47** (1985), 29.

2. *Nonlinear Oscillations in Biology and Chemistry,* (H.G. Othmer, ed.), Lect. Notes in Biomathematics, **66** Springer-Verlag, New York, 1986.

3. J. Jalife, *Mutual entrainment and electrical coupling as mechanisms for synchronous firing of rabbit sino-atrial pace-maker cells,* J. Physiol. **356** (1984), 221.

4. *Basic Mechanisms of the Epilepsies,* (H.Jasper, A. Ward, and A. Pope, eds.), Little, Brown and Co., Boston, 1969.

5. A. Winfree, *The Geometry of Time,* Springer-Verlag, New York, 1980.

6. M. Ashkenazi and H.G. Othmer, *Spatial patterns in coupled biochemical oscillators,* J. Math. Biol. **5** (1978), 305.

7. G.B. Ermentrout and N. Kopell, *Frequency plateaus in a chain of weakly coupled oscillators,* SIAM J. Appld. Math. **15** (1984), 215.

8. H.G. Othmer, D.G. Aronson and E.J. Doedel, *Resonance and Bistability in Coupled Oscillators,* Physics Letts., **113A** (1986), 349.

9. D.G. Aronson, E.J. Doedel and H.G. Othmer, *Bistable Behavior in Coupled Oscillators,* in Nonlinear Oscillations in Biology and Chemistry, (H.G. Othmer, ed.), Lect. Notes in Biomathematics, **66** Springer-Verlag, New York, 1986.

10. D.G. Aronson, E.J. Doedel and H.G. Othmer, *An analytical and numerical study of the bifurcations in a system of linearly-coupled oscillators,* Physica D **25** (1987), 20.

11. H.G. Othmer and J.A. Aldridge, *The effects of cell density and metabolite flux on cellular dynamics,* J. Math. Biol **5** (1978), 169.

12. H.G. Othmer, *Synchronization, Phase-Locking and Other Phenomena in Coupled Cells,* in Temporal Order (L. Rensing and N.I. Jaeger, eds.) Springer-Verlag, Heidelberg, 1985.

13. M. Urabe, *Nonlinear Autonomous Oscillations,* Academic Press, New York, 1967.

14. E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

15. S.P. Diliberto and G. Hufford, *Perturbation theorems for nonlinear ordinary differential equations,* in Contributions to the Theory of Nonlinear Oscillations, **3**, Princeton University Press, Princeton, 1956.

16. O.E. Lanford, *Bifurcation of periodic solutions into invariant torii: The work of Ruelle and Takens,* Lect. Notes in Math. **322** (1973), 159.

17. L. Bushard, *Periodic solutions and locking-in on the periodic surface,* Int. J. Nonlinear Mechanics **8** (1973), 129.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH, 84112