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RESOLVING SINGULAR NONLINEAR EQUATIONS

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ABSTRACT. This paper concerns the solutions of operator equations $G(z,\lambda) = 0$ having solutions (z_0, λ_0) for which $G'(z_0, \lambda_0)$ is not a surjection. More precisely, suppose $\lambda \in R^q, q \geq 0$ and $\dim N(G'(z_0, \lambda_0)) = m+q > 0$, where $N(\cdot)$ denotes the kernel. Several different kinds of singular problems can be treated in a unified way. Examples are parameter dependent problems with $q > 0$ and $m > 1$ and operator equations with $m > 0, q = 0$. In the latter case the corresponding discrete analogues also have some corresponding singularities which usually lead to the breakdown of numerical solution techniques. The former case includes multiple bifurcations for multi-parameter problems. The main results involve the construction of an inflated map $H(z, \lambda, \dots)$ (where $\dots$ denotes additional augmented variables). The map H has an invertible derivative at $(z_0, \lambda_0, \dots)$ and a component $F(z, \lambda, c)$ such that $F(z, \lambda, 0) = G(z, \lambda)$. This H may be used to define quadratically convergent Newton methods. Several examples of finite dimensional equations and operator equations are studied. In practical applications m is often not known a priori. Some ways of determining $m + q$ are described.

1. Introduction. In this paper we consider operators

$$G : \mathbf{E} := \mathbf{E}_0 \times \mathbf{R}^q \to \hat{\mathbf{E}}, \quad (\mathbf{E}_0, \hat{\mathbf{E}} \text{ are Banach spaces, } q \geq 0), \tag{1.1}$$

in the neighborhood of a zero point (z_0, λ_0) of G having a nontrivial null space of the Frechet derivative $G' = (G_z, G_\lambda)$. Here G_z, G_λ denote the partial derivatives with respect to z and λ, respectively. Letting $\mathbf{N}(L)$ denote the kernel of a linear operator L, we have

$$G(z_0, \lambda_0) = 0, \quad \dim \mathbf{N}(G'(z_0, \lambda_0)) = m + q > 0. \tag{1.2}$$

In case $q > 0$ we may have the usual bifurcation problem in continuation methods, which is discussed in the literature primarily for $m = 1$. For $q = 0$ and $m > 0$, the usual discretization methods for the computation of an isolated solution usually fail. Hence, modifications are necessary.

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Finally, the Newton methods for the computation of z_0 generally break down if (1.2) holds.

The aim of this paper is to suggest a uniform procedure to solve the three problems indicated above. We define an inflated operator H closely related to G such that, at the singular point (z_0, λ_0), the Frechet derivative H' of H is boundedly invertible. For H the usual discretization and Newton methods may be applied to obtain z_0, $\mathbf{N}(G'(z_0))$ for $q = 0$, and $(z_0, \lambda_0), \mathbf{N}(G'(z_0, \lambda_0))$ for $q > 0$. The discrete approximations converge with the order of the given method and the Newton iterates converge quadratically. For the bifurcation problem we obtain (z_0, λ_0) and $\mathbf{N}(G'(z_0, \lambda_0))$ and hence we are able to really obtain a "multiple bifurcation point" directly.

We assume that m in (1.2) is either known theoretically or empirically. One of the contexts in which such information is frequently available concerns problems involving bifurcation. In this case the determination of (z_0, λ_0) and $\mathbf{N}(G'(z_0, \lambda_0))$ might be used to simplify the task of solving the corresponding bifurcation equation arising in the method of Liapunov-Schmidt (see, e.g., [**42, 45**]). For the case $m = 1$ our inflated maps coincide with those of several authors (see [**29, 30, 35, 42-44, 52, 53**]).

The inflation H is introduced and discussed in §2 for Banach space mappings. §3 is devoted to the case $\dim \mathbf{E}_0 = \dim \mathbf{E} < \infty$, and §4 to the problem of discretization of operator equations, which is then applied in §5 to our special problems of inflated mappings.

An empirical determination of (m, q) for the case $\dim \mathbf{E}_0 = \dim \hat{\mathbf{E}} < \infty$ is studied in §3. The actual computations of the Gauss decomposition of the corresponding matrices yields "local" information about (m, q) which might have to be updated during the further computations. Via discretization, this approach may be used for operator equations as well.

The case $m = 1, q > 0$ in (1.2) is frequently discussed in the literature. The problem (1.1), (1.2) with $m > 1, \mathbf{E} = \hat{\mathbf{E}} = \mathbf{R}^\ell$ and a known trivial bifurcation point $z_0 = 0$ is treated under additional assumptions in [**8**]. In this paper an inflation H is formally introduced and described for some specific examples. To our knowledge no attempt to date has been made to give a general theory for operators including discretization and Newton methods. This approach may be used as well for the solution of nonlinear problems where the additional parameters $\lambda \in \mathbf{R}^q$ can introduce further difficulties, because the structure of the kernels are not known a priori.

2. Inflated maps.

2.1 *The problem.* We want to solve the problem

$$G(z, \lambda) = 0. \tag{2.1}$$

We make the following assumptions concerning the solution
(z_0, λ_0) of

(2.1)

i) $G : \mathbf{D}(G) \subset \mathbf{E} = \mathbf{E}_0 \times \mathbf{R}^q \to \mathbf{R}(G) \in \hat{\mathbf{E}}$;

ii) $\mathbf{E}, \hat{\mathbf{E}}$ are Banach spaces;

iii) $G \in C^2(\mathbf{D}(G)), \mathbf{D}(G)$ an open neighborhood of $x_0 := (z_0, \lambda_0)$;

(2.2)

iv) $\mathbf{N}(G'(z_0, \lambda_0)) = [\phi_1, \dots, \phi_{m+q}] \subseteq \mathbf{E}_0 \times \mathbf{R}^q = \mathbf{E}$;

v) $\mathbf{N}(G'(z_0, \lambda_0)^*) = [\varphi_1^*, \dots, \varphi_m^*] \subseteq \hat{\mathbf{E}}^*$;

vi) $G'(z_0, \lambda_0)$; and for every $w \in \mathbf{N}(G'(z_0, \lambda_0)), G''(z_0, \lambda_0)w$ are linear (bounded) operators with closed range.

Here $\mathbf{E}^*, \hat{\mathbf{E}}^*$, and L^* represent the dual spaces and operators, respectively, for the Banach spaces $\mathbf{E}, \hat{\mathbf{E}}$ and the linear operator $L : \mathbf{E} = \mathbf{E}_0 \times \mathbf{R}^q \to \hat{\mathbf{E}}, [w_1, \dots, w_i]$ indicates the span of $w_1, \dots, w_i$ and $\mathbf{N}(L)$ and $\mathbf{R}(L)$ indicate kernel and range of L, respectively.

Condition (2.2) iii) implies that $G'(z, \lambda)$ and $G''(z, \lambda)$ are bounded linear and bilinear operators if $\| (z, \lambda) - (z_0, \lambda_0) \|$ is sufficiently small. We could as well have started with a densely defined operator G. Then G' and G'' would be (or assumed to be) closed densely defined linear operators with closed range and appropriate $C^2(\mathbf{D}(G))$. The following theory essentially remains unchanged for this case. To avoid a more technical discussion we therefore choose (2.2).

As a consequence of (2.2) (iv), (v) the usual kinds of discretization and Newton methods applied to (2.1) will fail, since stability and convergence of Newton's method break down for nonexisting $(G'(z_0, \lambda_0))^{-1}$. Therefore, we introduce an inflated mapping. In this context we need a bounded linear operator

(2.3)

i) $L \in \ell(\mathbf{E} = \mathbf{E}_0 \times \mathbf{R}^q, \mathbf{R}^{m+q})$

such that

ii) $\mathbf{N}(L) \cap \mathbf{N}(G'(z_0, \lambda_0)) = \{0\}$,

an $(m+q) \times (m+q)$ matrix

(2.4)

i) $A = (a_1, \ldots, a_{m+q})$

such that

ii) $\operatorname{rank} A = m + q$,

and a variable $c \in \mathbf{R}^p$, which we will further specify below. We embed G into a family of operators such that, with

(2.5)

$$\begin{aligned} &x := (z, \lambda) \in \mathbf{D}(G) \subseteq \mathbf{E} = \mathbf{E}_0 \times \mathbf{R}^q, \quad c \in \mathbf{D}_F \subseteq \mathbf{R}^p, \quad 0 \in \mathbf{D}_F, \\ &F := \mathbf{D}(G) \times \mathbf{D}_F \subseteq \mathbf{E}_0 \times \mathbf{R}^q \times \mathbf{R}^p \to \hat{\mathbf{E}}, \ F(x, 0) = G(x) \end{aligned}$$

and

$$F \in C^2(\mathbf{D}(G) \times \mathbf{D}_F).$$

The question of how to choose F is treated in §2.4. Now we define the operator H, with F_x and F_c representing the partial derivatives of F, as

$$(2.6) \quad \left[\begin{array}{r} H : \mathbf{D}(G) \times E^{m+q} \times \mathbf{R}^p \subseteq \mathbf{F} := \mathbf{E}^{m+q+1} \times \mathbf{R}^p \to \hat{\mathbf{F}} \\ := \hat{\mathbf{E}} \times (\hat{\mathbf{E}} \times \mathbf{R}^{m+q})^{m+q}, \\ H(x, w_1, \ldots, w_{m+q}, c) := \begin{bmatrix} F(x, c) \\ F_x(x, c) w_i \\ L w_i = a_i \\ i = 1, \ldots, m+q \end{bmatrix} \end{array} \right].$$

The integer p is determined by the requirement that, for $\mathbf{E}_0 = \hat{\mathbf{E}} = \mathbf{R}^n$, the derivative of H represents a square matrix. Omitting the argument on the right-hand side of the equation, we find

$$
H'(x, w_1, \dots, w_{m+q}, c) =
$$

$$
(2.7)\qquad \begin{bmatrix}
F_x & 0 & 0 & \cdots & & 0 & F_c \\
F_{xx}w_1 & F_x & 0 & & & 0 & F_{xc}w_1 \\
0 & L & 0 & & & & 0 \\
F_{xx}w_i & 0 & 0 & \ddots & & 0 & F_{xc}w_i \\
0 & 0 & 0 & \ddots & F_x & 0 & 0 \\
F_{xx}w_{m+q} & 0 & & & L & F_x & F_{xc}w_{m+q} \\
0 & 0 & & \cdots & & L & 0
\end{bmatrix}.
$$

In this "matrix" the columns 1 to $m+q+2$ represent the partial derivatives with respect to $x, w_1, \dots, w_{m+q}$ and c, respectively. If $G : \mathbf{R}^{n+q} \to \mathbf{R}^n$, then (2.7) represents a matrix with

$$
(n+q)(m+q+1)+p \text{ columns and } n+(n+m+q)(m+q) \text{ rows.}
$$

A square matrix is then obtained if and only if

$$
(2.8)\qquad p = m(m+q) - q.
$$

For simplicity we require the perturbation $F(x, c)$ to satisfy the following conditions: (2.3)

$$
(2.9)\qquad \begin{array}{l} \text{for } F \text{ in (2.5) choose } p \text{ as in (2.8) and, for } x \in \mathbf{D}(G), \text{ let} \\ F(x,0) = G(x), \quad F_x(x,0) = G'(x), \quad F_{xx}(x,0) = G''(x). \end{array}
$$

Finally, we introduce the operator M for $x_0 = (z_0, \lambda_0)$ as

$$
(2.10)\qquad M := M(x_0) : \mathbf{E} \times \mathbf{R}^p \to \mathbf{R}^{m(m+q)}
$$

such that

$$
M(x, c) := \Big(\langle \varphi_j^*, F_{xx}(x_0, 0)\phi_i x + F_{xc}(x_0, 0)\phi_i c \rangle \Big)_{i=1}^{m+q}{}_{j=1}^{m},
$$

where $\langle \cdot\,, \cdot \rangle$ is defined on $\hat{\mathbf{E}}^* \times \hat{\mathbf{E}}$. We furthermore require

$$
(2.11)\qquad \mathbf{N}(M) \cap \mathbf{N}(F'(x_0, 0)) = \{0\}
$$

and

$$\mathbf{R}(F'(x_0,0)) = \hat{\mathbf{E}}. \tag{2.12}$$

2.2. *Comments and propositions.* Before we discuss H in more detail we want to comment upon the assumptions (2.2)-(2.4) and (2.11), (2.12) and give some propositions concerning them. We first describe $\mathbf{N}(G')$ in terms of G_x and G_λ. Since we will need a similar result for $\mathbf{N}(F')$ below, we prove

PROPOSITION 2.1. *Let $K := (K_z, K_\lambda) : \mathbf{E}_1 \times \mathbf{R}^s \to \hat{\mathbf{E}}$ be a bounded linear operator for Banach spaces $\mathbf{E}_1, \hat{\mathbf{E}}$, such that*

$$\begin{aligned} &K_z : \mathbf{E}_1 \to \hat{\mathbf{E}} \text{ has closed range } \mathbf{R}(K_z) \subseteq \hat{\mathbf{E}},\\ &\hat{\mathbf{E}} = \mathbf{R}(K_z) + U \text{for } U \text{the closed complement of } \mathbf{R}(K_z),\\ &P : \hat{\mathbf{E}} \to U \text{is a projector along } \mathbf{R}(K_z) \text{onto } U \ (\mathbf{N}(P) = \mathbf{R}(K_z),)\\ &\dim \mathbf{N}(K_z) = k, \dim PK_\lambda \mathbf{R}^s = j. \end{aligned} \tag{2.13}$$

Then

$$\begin{aligned} \mathbf{N}(K) &= \{(x,\lambda) : K_z z + K_\lambda \lambda = 0\}\\ &= \{(z,\lambda) : \lambda = \lambda_0 + \lambda_r, \lambda_0 \lambda_r \in \mathbf{N}(K_\lambda), K_\lambda \lambda_r \in \mathbf{R}(K_z),\\ &\quad z = z_0 + z_r, z_0 \in \mathbf{N}(K_z), K_z z_r = -K_\lambda \lambda_r\} \end{aligned} \tag{2.14}$$

and

$$\dim \mathbf{N}(K) = k + s - j.$$

PROOF. There is a system of $s = \ell + i + j$ linearly independent vectors $\lambda_{01}, \ldots, \lambda_{0\ell}, \lambda_{r1}, \ldots, \lambda_{ri}, \lambda_{u1}, \ldots, \lambda_{uj} \in \mathbf{R}^s$ such that

$$\begin{aligned} &\mathbf{N}(K_\lambda) = [\lambda_{01}, \ldots, \lambda_{0\ell}],\\ &\mathbf{R}_\lambda := \{\lambda \in \mathbf{R}^s : K_\lambda \lambda \in \mathbf{R}(K_z)\} = [\lambda_{r1}, \ldots, \lambda_{rq}],\\ &PK_\lambda \mathbf{R}^s = P[K_\lambda \lambda_{u1}, \ldots, K_\lambda \lambda_{uj}]\\ &K_\lambda \mathbf{R}_\lambda \subseteq \mathbf{R}(K_z), \quad PK_\lambda \mathbf{R}^s \subseteq U, \quad \dim PK_\lambda \mathbf{R}^s = j. \end{aligned} \tag{2.15}$$

Therefore, $\lambda \in \mathbf{R}^s$ is uniquely representable as

$$\lambda = \lambda_0 + \lambda_r + \lambda_u$$

with $\lambda_0 \in \mathbf{N}(K_\lambda), \lambda_r \in \mathbf{R}_\lambda, PK_\lambda\lambda_u \in PK_\lambda\mathbf{R}^s$. Now $(z,\lambda) \in \mathbf{N}(K')$ implies

$$\begin{aligned} 0 &= K_z z + K_\lambda \lambda = K_z z + K_\lambda\lambda_0 + K_\lambda\lambda_r + K_\lambda\lambda_u \\ &= K_z z + K_\lambda\lambda_r + K_\lambda\lambda_u. \end{aligned}$$

Applying P to this equation yields $PK_\lambda\lambda_u = 0$ or $\lambda_u = 0$ so that

$$\begin{aligned} \mathbf{N}(K) = \{(z,\lambda) : &\lambda = \lambda_0 + \lambda_r, z = z_0 + z_r \text{ with } z_0 \in \mathbf{N}(K_z), \\ &\lambda_0 \in \mathbf{N}(K_\lambda), \text{ and } K_z z_r = -K_\lambda\lambda_r\}. \end{aligned}$$

This implies (2.14).

REMARK 2.2

(i). In Proposition 2.1 the roles of K_Z and K_λ might as well have been exchanged. If $Q : \hat{\mathbf{E}} \to V$ is a projector along the (necessarily) closed $\mathbf{R}(K-\lambda)$ onto its closed complement and if

$$\dim \mathbf{N}(K_\lambda) = \ell, \quad \dim QK_z\mathbf{E}_1 = t, \quad \dim(\mathbf{R}(K_z) \cap \mathbf{R}(K_\lambda)) = i,$$

we find, analogously to (2.14) or symmetrically in K_z, K_λ,

$$\dim \mathbf{N}(K) = \ell + s - t = k + \ell + i.$$

(ii). We use $K = G' : \mathbf{E}_0 \times \mathbf{R}^q \to \hat{\mathbf{E}}$ in Proposition 2.1 and additionally postulate $G_z(z_0, \lambda_0)$ having closed range. Then a comparison between (2.2) (iv) and (2.14) shows that (2.2) (iv) is satisfied if and only if

$$\dim \mathbf{N}(G_z(x_0)) = m + j, \quad \dim PG_\lambda(x_0)\mathbf{R}^q = j \geq 0,$$

The latter case is particularly interesting for bifurcation problems, it means that bifurcations up to multiplicity $m + q$ are possible.

(iii). Using Banach's closed range theorem (2.2) (iv), (v) represents the fact that $G'(x_0)$ is a Fredholm operator of index 0 with condimension m. If furthermore, $G_z(x_0)$ is a Fredholm operator of index 0, then automatically $\dim \mathbf{N}(G'(x_0)^*) = m$ and (2.2) (v) is automatically satisfied. Thus, the conditions (2.2) (iv) , (v) represent a combination which is satisfied for may important classes of problems, e.g., finite-dimensional equations and differential, integral and integrodifferential equations depending upon a parameter.

COROLLARY 2.3. *Assume the conditions* (2.2), (2.9) *and* (2.12) *and omit the argument* $(x_0, 0)$ *here* (*and in the following proof*). *Then*

$$\begin{aligned} \mathbf{N}F') &= \mathbf{N}(F_x, F_c) \\ &= \{(x,c) : c = c_{00} + c_r, c_{00} \in \mathbf{N}(F_c), F_c c_r \in \mathbf{R}(F_x), \\ &\qquad x = x_{00} + x_r, x_{00} \in \mathbf{N}(f_x), F_x x_r = -F_c c_r\} \end{aligned} \tag{2.16}$$

and

$$\dim \mathbf{N}(F') = p + q = m(m+q).$$

PROOF. The conditions (2.2) and (2.9) show that the first two lines of (2.13) are satisfied. (2.2 v) and the closed range theorem shows that

$$\mathbf{R}(F_x) = \mathbf{R}(G') = [\varphi_1^*, \ldots, \varphi_m^*]^\perp,$$

where $\perp$ indicates that, with respect to the Banach space pairing $\langle \cdot , \cdot \rangle$ for $\hat{\mathbf{E}}^*, \hat{\mathbf{E}}$, we have $\langle \phi_j^*, G'x \rangle = 0$ for $j = 1, \ldots, m$ and arbitrary $x \in \mathbf{E}$. This result, combined with (2.12), yields $\dim PF_x\mathbf{R}^p = j = m$, and, with (2.2)(iv), (2.8) and (2.14)

$$\dim \mathbf{N}(F') = m + q + p - m = p + q = m(m+q).$$

The conditions $Lw_i - a_i = 0$ in (2.6) are normalizing conditions, realized by $m + q$ linear bounded functionals

$$\ell_j : \mathbf{E} = \mathbf{E}_0 \times \mathbf{R}^q \to \mathbf{R}. \tag{2.17}$$

As an example we might envision, for $\mathbf{E}_0 \subseteq C[a,b]$, a set of fixed real scalars ℓ_{ji}, fixed coordinates $t_i \in [a,b]$, $i = 1, \ldots, s$, fixed vectors $r_j \in \mathbf{R}^q$, and with the inner product $(\cdot , \cdot)$ in $\mathbf{R}^q$,

$$\ell_j x := \ell_j(z, \lambda) := \sum_{i=1}^{s} \ell_{ji} z(t_i) + (r_j, \lambda).$$

PROPOSITION 2.4. *For* L *as in* (2.3) (i) *and*

$$L_0 := (\ell_j \phi_j)_{i,j=1}^{m+q} \in \mathbf{R}^{(m+q)\times(m+q)}, \tag{2.18}$$

the condition (2.3) (ii) *is satisfied if and only if rank* $(L_0) = m + q$.

PROOF. The following statements are obviously equivalent:

(i) Rank $L_0 < m + q$.

(ii) There exists a $0 \neq \phi = \sum_{i=1}^{m+q} \alpha_i \phi_i \in \mathbf{N}(G'(z_0, \lambda_0))$ such that

$$\sum_{i=1}^{m+q} \alpha_i(\ell_j \phi_i) = 0, \quad j = 1, \ldots, m+q,$$

hence $\phi \in \mathbf{N}(L)$.

(iii) There exists a $\phi \neq 0$ such that $\phi \in \mathbf{N}(L) \cap \mathbf{N}(G'(z_0, \lambda_0))$.

PROPOSITION 2.5. *Let L in* (2.3)(i) *satisfy* (2.3)(ii) *and choose* $\overline{\phi}_i \in \mathbf{N}(G'(z_0, \lambda_0))$ *such that* $L\overline{\phi}_i = a_i \in \mathbf{R}^{m+q}$, $i = 1, \ldots, m+q$. *Then the* $\overline{\phi}_i$, $i = 1, \ldots, m+q$, *are linearly independent if and only if the* a_i *satisfy* (2.4).

PROOF. With the linearly independent $\phi_i, i = 1, \ldots, m+q$ in (2.2)(iv) we have the existence of $\alpha_{ni} \in \mathbf{R}$ such that

$$\overline{\phi}_i = \sum_{n=1}^{m+q} \alpha_{ni} \phi_n.$$

Now

$$L\overline{\phi}_i = a_i, \quad i = 1, \ldots, m+q,$$

if and only if

$$L_0 \begin{bmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{(m+q)i} \end{bmatrix} = a_i, \quad i = 1, \ldots, m+q, \quad L_0 \text{ in } (2.18).$$

Therefore, we have by Proposition 2.4, the $a_i, i = 1, \ldots, m+q$ are linearly independent if and only if

$$\begin{bmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{(m+q)i} \end{bmatrix}, \quad i = 1, \ldots, m+q$$

are linearly independent. This in turn is equivalent to the following statements:

$$\text{rank }[(\alpha_{ji})_{i,j=1}^{m+q}] = m+q,$$

$$\overline{\phi}_i, \quad i = 1,\dots,m+q \text{ are linearly independent.}$$

PROPOSITION 2.6. *Let $\rho_k = (\varepsilon_k, \tau_k), k = 1,\dots,m(m+q)$ be a basis for $\mathbf{N}(F')$ (see Corollary 2.3), $\phi_1,\dots,\phi_{m+q}$ and $\varphi_1^*,\dots,\varphi_m^*$ be bases for $\mathbf{N}(G'(x_0)) = \mathbf{N}(F_x(x_0,0))$ and $\mathbf{N}((G'(x_0))^*)$, respectively (see (2.2)). Then (2.11) is equivalent to the fact that the $m(m+q)\times m(m+q)$ matrix*

(2.19)

$$M_0 := (m_{\ell k})_{\ell,k=1}^{m(m+q)},$$

$$m_{\ell k} := \langle \varphi_j^*, F_{xx}(x_0,0)\phi_i\varepsilon_k + F_{xc}(x_0,0)\phi_i\tau_k\rangle, \quad \ell := (i-1)m+j,$$

has full rank $m(m+q)$. This property is independent of the special bases $\rho_k = (\varepsilon_k, \gamma_k), \phi_i, \varphi_j^*$ chosen.

PROOF. This follows immediately from (2.16).

REMARK 2.7.

(i). It follows from a theorem of Sard (see, e.g., [**34**]) that the condition (2.3) (ii) is satisfied with probability one if we choose the bounded $\ell_j, j = 1,\dots,m+q$ at random. This does not exclude, however, that, for a special $\mathbf{N}(F')$, which we do not know yet, we might have chosen ℓ_j such that (2.3)(ii) is violated. This will become obvious, however, during the computations (see §3). In such a case we would locally change some of the $\ell_j, j = 1,\dots,m+q$.

(ii). We have assumed $m+q$ to be known. This knowledge may be achieved in several ways: One might start computation for $Fz = 0$, e.g., by Newton's method, and then discover that $m > 0$. In this case one could gradually increase m, using Remark (i) until m is maximal. One might as well have chosen some embedding and used Sylvester's law of inertia (see, e.g., Birkhoff, MacLane [**9**] as was done in [**3**] and [**11**] to determine m. Finally, some theoretical or e.g., physical knowledge might provide the value of m.

(iii). If the conditions (2.2) and (2.9)-(2.12) are violated in the sense that an appropriate F does not exist (see discussions below and Remark 2.2), the following analysis breaks down.

2.3. *The main result.* To prove the (locally) unique solvability of $H = 0$ and the regularity of H' we use the following well-known corollary of the open mapping theorem (see, e.g., Yosida [**54**]).

COROLLARY 2.8. *If a bounded linear operator* $T : \mathbf{F} \to \hat{\mathbf{F}}$ $(\mathbf{F}, \hat{\mathbf{F}}$ *Banach spaces*), $\mathbf{R}(T) = \hat{\mathbf{F}}$ *is invertible, that is* $T^{-1}y = 0$ *implies* $y = 0$, *then* $T^{-1} : \hat{\mathbf{F}} \to \mathbf{F}$ *is a bounded linear operator.*

THEOREM 2.9. *For an operator* G *satisfying* (2.2) *let* L *and* A *be chosen so as to satisfy* (2.3), (2.4) *and let an inflation* H *be defined in* (2.6) *satisfying* (2.5) *and* (2.9) – (2.12). *Then there exists a locally unique solution* $(x_0, \overline{\phi}_1, \ldots, \overline{\phi}_{m+q}, 0)$ *of*

$$H(x, w_1, \ldots, w_{m+q}, c) = 0 \tag{2.20}$$

with $[\overline{\phi}_1, \ldots, \overline{\phi}_{m+q}] = \mathbf{N}(F_x(x_0, 0)) = \mathbf{N}(G'(x_0)), x_0 = (z_0, \lambda_0)$. *For this solution, the operator*

$$H'(x_0, \overline{\phi}_1, \ldots, \overline{p}_{m+q}, 0) : \mathbf{F} \to \hat{\mathbf{F}},$$

where

$$\mathbf{F} = \mathbf{E}^{m+q+1} \times \mathbf{R}^p = (\mathbf{E}_0 \times \mathbf{R}^q)^{m+q+1} \times \mathbf{R}^{m(m+q)-q},$$
$$\hat{\mathbf{F}} = \hat{\mathbf{E}} \times (\hat{\mathbf{E}} \times \mathbf{R}^{m+q})^{m+q}$$

defined in (2.7) *is a regular linear bounded operator. That is,*

$$H'(z_0, \lambda_0, \overline{\phi}_1, \ldots, \overline{\phi}_{m+q}, 0)]^{-1} : \hat{\mathbf{F}} \to \mathbf{F}$$

exists and is bounded.

PROOF. The combination of (2.1) and (2.5) yields $F(z_0, \lambda_0, 0) = 0$, (2.2) (iv), (2.3), (2.4), (2.9) and Proposition 2.5 show the existence of

$\overline{\phi}_i$ such that (2.20) is satisfied. The local uniqueness is a consequence of the regularity of H'.

We want to show, omitting the argument $(x_0, \overline{\phi}_1, \dots, 0)$ in F_x, F_{xx}, F_{xc}, that

$$H'(x_0, \overline{\phi}_1, \dots, \overline{\phi}_{m+q}, 0) \cdot \begin{bmatrix} u^H \\ v_1 \\ v_{m+q} \\ d \end{bmatrix} =$$

(2.21)

$$\begin{bmatrix} F_x u + F_c d \\ F_{xx}\overline{\phi}_i + F_x v_i + F_{xc}\overline{\phi}_i d \\ L v_i \\ i = 1, \dots, m+q \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{v}_i \\ r_i \\ i = 1, \dots, m+q \end{bmatrix}$$

has a unique solution $(u, v_1, \dots, v_{m+q}, d) \in \mathbf{F}$ for every $(\hat{u}, \hat{v}_1, r_1, \dots, \hat{v}_{m+q}, r_{m+q}) \in \hat{\mathbf{F}}$. By applying the φ_j^* to the equations (2.21) we are able to eliminate the v_i and obtain the system for u, d:

(2.22)

$$F_x u + F_c d = \hat{u}$$
$$\langle \varphi_j^*, F_{xx}\overline{\phi}_i + F_{xc}\overline{\phi}_i d \rangle = \langle \varphi_j^* d \rangle, \quad i = 1, \dots, m+q, \; j = 1, \dots, m.$$

Because of (2.12) there is a fixed pair (u^{in}, d^{in}) such that, with $(x, c) \in \mathbf{N}(F')$, we have the general solution (see Corollary 2.3)

$$F_x(u^{in} + x) + F_c(d^{in} + c) = \hat{u}.$$

Now we can use the equations in the second line of (2.22) to compute (x, c) from

$$\langle \varphi_j^*, F_{xx}\overline{\phi}_i x + F_{xc}\overline{\phi}_i c \rangle = \langle \varphi_j^*, \hat{v}_i - F_{xx}\overline{\phi}_i u^{in} - F_{xc}\overline{\phi}_i d^{in} \rangle, \quad i = 1, \dots, m+q, \; j = 1, \dots, m. \tag{2.23}$$

Corollary 2.3, Proposition 2.6, and (2.11) show that (2.23) is uniquely solvable for arbitrary $\hat{v}_i, i = 1, \dots, m+q$.

Since u and d are known, we are left with the following equations (see (2.21)):

$$\begin{aligned} &\text{(i) } F_x v_i = \hat{v}_i - F_{xx}\overline{\phi}_i u - F_{xc}\overline{\phi}_i d, \\ &\text{(ii) } L v_i = r_i, \; i = 1, \dots, m+q. \end{aligned} \tag{2.24}$$

As a consequence of (2.22) we have, for the right-hand side in (2.24), the relations

$$\langle \varphi_j^*, \hat{v}_i - F_{xx}\overline{\phi}_i - F_{xc}\overline{\phi}_i d\rangle = 0, \quad i = 1, \ldots, m+q, \; j = 1, \ldots, m.$$

The conditions (2.2)(v) and (2.9) show that this implies that the right-hand sides in (2.24)(i) are indeed in $\mathbf{R}(F_x)$, so there exists a v_i^{in} such that $v_i^{in} + v_{io}$ with arbitrary $v_{io} \in \mathrm{N}(F_x)$ is the general solution for (2.24)(i). Then a combination of (2.3), (2.4), and Proposition 2.4 shows the unique existence of $v_1, \ldots, v_{m+q}$ satisfying (2.24). A combination of Corollary 2.8 with this result completes the proof.

REMARK 2.10. In the proofs of Theorem 2.9 and the preceding propositions, we have seen that the conditions which we have imposed are well balanced so they are in some sense "necessary and sufficient".

It seems to be necessary to extend the argument in $G, x \in \mathbf{D}(G) \subseteq \mathbf{E}$, into an argument $(x, c) \in \mathbf{E} \times \mathbf{R}^p$. To show this, we introduce in (2.6) and (2.7) a modified $\tilde{H}$ (by avoiding c) so that

(2.25)

$$\tilde{H}(x, w_1, \ldots, w_{m+q}) := (G(x), G'(x)w_i, Lw_i - a_i, i = 1, \ldots, m+q)^T.$$

In the formulas (2.7) and (2.10), the F_x, F_{xx}, F_{xc} have to be replaced by $G'(x), G''(x), 0$, respectively, the last column of H' in (2.7) disappears and M in (2.10) only depends upon x. Then we may use the following theorem (see, e.g., Taylor-Lay [**51**]) to obtain Theorem 2.12.

THEOREM 2.11. *Let $T : \mathbf{D}(T) \subseteq \mathbf{E} \to \mathbf{R}(T) \subseteq \hat{\mathbf{E}}$ be a closed, densely defined linear operator, with Banach spaces, $\mathbf{E}, \hat{\mathbf{E}}$, and let $T^{-1} : \mathbf{R}(T) \to \mathbf{D}(T)$ exist. Then T^{-1} is bounded if and only if $\mathbf{R}(T)$ is closed.*

THEOREM 2.12. *Let G, L, M satisfy*

(2.2)(i)-(v) *is satisfied for G with a dense $\mathbf{D}(G) \subseteq \mathbf{E}$;*

(2.2)(vi) *$G'(x_0)$ and, for every $w \in \mathrm{N}(G'(z_0)), G''(x_0)w$ are closed densely defined linear operators with closed ranges;*

(2.3), (2.11) *are satisfied for* $\mathbf{N}(G'(z_0,)), L, M$; *and let* $\tilde{H}$ *be defined as in* (2.25). *Then the modified* $\tilde{H}'$ *is a closed densely defined operator with closed range,* $\mathbf{R}(\tilde{H}')$ *in* $x_0, \overline{\phi}_1, \ldots, \overline{\phi}_{m+q}$, *such that*

$$(\tilde{H}'(x_0, \overline{\phi}_1, \ldots, \phi_{m+q}))^{-1} : \mathbf{R}(\tilde{H}'(x_0, \overline{\phi}_1, \ldots, \overline{\phi}_{m+q})) \to \mathbf{E}^{m+1}$$

exists and is bounded.

Comparing Theorems 2.9 and 2.11 we see that by introducing $c \in \mathbf{R}^p$ we avoid the hard problem of finding $\mathbf{R}(\tilde{H}')$ for the "simple" $\tilde{H}$ in (2.25). Instead we obtain that, for the "complicated" H in (2.6), the derivative H' in (2.7) has $\mathbf{R}(H') = \hat{\mathbf{F}}$, is the whole Banach space, and $(H')^{-1} : \hat{\mathbf{F}} \to \mathbf{F}$ is continuous. Certainly we have to pay the price of introducing $c \in \mathbf{R}^p$ as the cost of this crucial advantage.

2.4 *Construction of Embedding.* Until now we have assumed the existence of an embedding F in (2.5) such that the above conditions are satisfied. Usually an operator G as in (2.1), (2.2) will be given and we have to construct an F. Certainly it would be possible to introduce perturbations such that the bifurcation point of multiplicity $m+q$ is totally or nearly unfolded. However, we do not want to change the multiplicity nor the exact solution (z_0, λ_0). Furthermore, it makes sense to define F to be as "simple as possible". For this reason we introduce

$$F(x,c) := G(x) + B(x,c) + Qc, \tag{2.26}$$

with continuous bilinear and linear operators B and Q into $\hat{\mathbf{E}}$, respectively, and $x \in \mathbf{E}, c \in \mathbf{R}^p$. If $x_0 = 0$, we need $Q \neq 0$, otherwise $Q = 0$ may be chosen. For the operator F in (2.26) we have

$$F(x,0) = G(x),$$

$$F'(x,c)\begin{bmatrix} u \\ d \end{bmatrix} = G'(x)u + B(u,c) + B(x,d) + Qd,$$

$$F_x(x,c)u = G'(x)u + B(u,c), \quad F_x(x,0)u = G'(x)u, \tag{2.27}$$

$$F_c(x,c)d = B(x,d) + Qd, \quad F_c(x,0)d = Qd + B(x,d)$$

$$F_{xx}(x,c)uv = G''(x)uv,$$

$$F_{xc}(x,c)ud = B(u,d).$$

For this type of F the conditions (2.5) and (2.9) are satisfied and (2.10)-(2.12) have the form

(2.28)

$$M(x_0,0)\begin{bmatrix} u \\ d \end{bmatrix} = \langle \varphi_j^*, G''(x_0)\phi_i u + B(\phi_i, d)\rangle$$

$$\text{for } 1 \le i \le m+q, \; 1 \le j \le m,$$

$$\mathbf{N}(F'(x_0,0)) = \mathbf{N}(G'(x_0), B(x_0,\cdot) + Q)$$

$$= \{(u,d) : G'(x_0)u + B(x_0,d) + Qd = 0\} (\text{ see (2.16)}),$$

$$\mathbf{R}(F'(x_0,0)) = \{G'(x_0)u + B(x_0,d) + Qd\} = \hat{\mathbf{E}}.$$

Since, by (2.2)(v),

$$\mathbf{R}(G'(x_0)) = [\varphi_1^*, \ldots, \varphi_m^*]^\perp,$$

the last condition implies that, with an appropriate projector P,

$$PR(B(x_0,\cdot) + Q) \supseteq [\varphi_1^*, \ldots, \varphi_m^*]^\perp. \tag{2.29}$$

We shall not discuss (2.29) in detail. However, it is obvious that in the special situation where

$$\mathbf{N}(G'(x_0)) \cap \{u : G''(x_0)\phi_i u = 0, i = 1, \ldots, m+q\} \neq \{0\} \tag{2.30}$$

our choice (2.26) cannot satisfy (2.11). In this (very exceptional) case we have to modify (2.26) into the form

$$F(x,c) := G(x) + C(x,x,c-\bar{c}) + B(x,c) + Qc$$

with a trilinear operator $C : \mathbf{E}^2 \times \mathbf{R}^p \to \hat{\mathbf{E}}$ and $\bar{c} \neq 0$. In §3, §4 and §5 we will present, for the special case of finite systems of finite equations and of operator equations and their discretizations, respectively, some hints on how to choose B, Q and (if necessary) C.

3. Finite-dimensional Equations.

3.1 *Determination of* m, Q *and* B. In this section we want to specify the preceding general results for the finite-dimensional case. We will be able to discuss more explicitly the conditions (2.3) and (2.11), (2.12). Instead of (2.2) we have

$$G : \mathbf{R}^{n+q} \to \mathbf{R}^n, \quad G(x_0) = 0, \quad x_0 = (z_0, \lambda_0). \tag{3.1}$$

In this case $\mathbf{E}_0 = \hat{\mathbf{E}} = \mathbf{R}^n = \mathbf{E}_0^* = \hat{\mathbf{F}}^*$. We want to write down the system (2.21) in slightly altered form. The bordering numbers in (3.2) (and in the sequel) indicate the numbers of rows and columns of the corresponding matrices. In contrast to (2.21) we do not use $H'(x_0, \overline{\phi}_1, \ldots, \overline{\phi}_{m+q}, 0)$, since these arguments are not yet known (except $= 0$); however, we use $H(x, w_1, \ldots, w_{m+q}, c)$, omitting these arguments in F_x, F_c, F_{xx} and F_{xc}.

$$\begin{array}{llclclcll} & n+q & & n+q & & p & & & \\ n & F_x v_i & + & F_{xx} w_i u & + & F_{xc} w_i d & = & \hat{v}_i & m+q \text{ times for} \\ m+q & L v_i & & & & & = & r_i & i = 1, \ldots, m+q \\ n & & & F_x u & + & F_c d & = & \hat{u}. & \end{array} \tag{3.2}$$

By (2.8) this is a square matrix, which has full rank $(n+m+q)(m+q)+n$ if (2.3) and (2.11) are satisfied. In this context (2.12) is a consequence of (2.3) and (2.11).

Throughout the following we restrict our discussion to the case that the different linear equations are solved by some type of Gaussian algorithm. This assumption is appropriate for (3.1). If operator equations are discretized (see §4 and §5), n in (3.1) might become so large that other techniques have to be applied. For multigrid methods (see, e.g., Hackbusch-Trottenberg [**28**]) the following considerations may be used on the coarsest grid. For other discretizations the mesh independence principle (see Allgower-Böhmer [**1**] and Allgower-Böhmer-Potra-Rheinboldt [**2**]) again yield the information on the coarser grids.

Given (3.1) we do not know a priori whether singularities of the form (2.2) with $m > 0$ are to be expected. If Newton's method is used for the solution of (3.1), a defect m in the rank of G' will become apparent in the neighborhood of (z_0, λ_0). Although the theorem of Sard guarantees that, for a random choice of L in (2.3)(i), the condition (2.3)(ii) will be satisfied, we proceed slightly differently. By one of the usual decomposition strategies we can transform $G'(z, \lambda)$ into the form

$$P_G G'(z,\lambda) = \begin{array}{|c|c|l} \multicolumn{1}{c}{n-m} & \multicolumn{1}{c}{m+q} & \\ \hline \diagdown & G_0(x) & n-m \\ \hline 0 & \approx 0 & m \\ \hline \end{array} \tag{3.3}$$

where the last m rows are nearly 0, and the matrix $G_0 = G_0(x)$ has nonvanishing diagonal elements. Then we choose a matrix L such that

$$\text{rank} \begin{array}{|c|c|l} \multicolumn{1}{c}{n-m} & \multicolumn{1}{c}{m+q} & \\ \hline 0 \;\diagdown & G_0 & n-m \\ & & = n+q \\ \hline \multicolumn{2}{|c|}{L} & m+q \\ \hline \end{array} \tag{3.4}$$

Thus, we obtain a square matrix with full rank.

Unless (2.30) is satisfied, Sard's theorem again implies that a random choice of B and Q satisfies (2.11) and (2.12). However, using (3.3) we are again able to do better. From the last m nearly vanishing lines in (3.3), approximations w_i for the

$$\phi_i \in \mathbf{N}(G'(x_0, 0)), \quad i = 1, \ldots, m+q,$$

and v_j^* for the

$$\varphi_j^* \in \mathbf{N}(G'(x_0, 0)^*), \quad j = 1, \ldots, m,$$

may be obtained (see the end of §3.3 for more details). With these approximations we choose B, Q (and C if necessary) such that (2.11) and (2.12) are satisfied for $w_i, x_j^*, x = (z, \lambda)$ instead of ϕ_i, φ_j^*, x_0. In any case, it is advisable to choose the B, Q (and C) such that R_* in (3.5) has "maximal rank". Thus, by combining (2.28) with (2.11), (2.12) we obtain the approximate conditions

$$R\begin{pmatrix} u \\ d \end{pmatrix} := \begin{bmatrix} \langle v_j^*, G''(x) w_i u + B(w_i, d) \rangle \\ G'(x)u + B(x, d) + Qd \end{bmatrix}, \tag{3.5}$$

$1 \le i \le m+q, 1 \le j \le m$, rank $R = n+q+p = m(m+q)+n$, and thereby (via (2.29)) we have

$$\mathbf{R}(B(x,\cdot)+Q\cdot) \supseteq [v_1^*,\dots,v_m^*]. \tag{3.6}$$

If we should realize during the computation that (3.4)-(3.6) or (2.30) is violated, we have to correct locally. We are going to present two different numerical approaches to solve (3.2). A naive solution via the Gaussian algorithm using column pivoting relative to the ℓ_1-norm of the rows would essentially require $(n(m+q+1))^3$ operations. We can, however, do much better. The two ways differ by the fact that, in §3.2, the dual kernel, introduced into the calculation via (2.22), is avoided, whereas it is used in §3.3. The approach in §3.3 only works under a very special assumption (see (3.18)) and furthermore turns out to be more costly than the method in §3.2. We discuss the problem elsewhere as to whether it is worthwhile to use the known value $c = 0$ in the solution of $H(x_0, \overline{\phi}_1, \dots, \overline{\phi}_{m+q}, 0)$ already during some iteration processes such as Newton's method.

3.2 *Solution without dual kernel.* With F defined, we once more want to study (3.2). At first we treat the $(m+q)$ systems for the v_i. Since we have assumed $F_x(x,0) = G'(x)$ we only discuss the case that $\| c \|$ and $\| x - x_0 \|$ are small enough to guarantee (see (3.3))

$$P_G F_x(x,c) = \begin{array}{|c|c|} \multicolumn{1}{c}{n-m} & \multicolumn{1}{c}{m+q} \\ \hline \diagdown & F_0(x) \\ \hline 0 & \approx 0 \\ \hline \end{array} \begin{array}{l} \\ n-m \\ m \end{array}$$

Now let P_F be the matrix corresponding to (3.3) such that

$$\begin{array}{r} \\ n \\ m+q \end{array} \begin{array}{|ccc|} \multicolumn{1}{c}{n+q} & \multicolumn{1}{c}{n+q} & \multicolumn{1}{c}{p} \\ \hline F_x & F_{xx}w_i & F_{xc}w_i \\ l & 0 & 0 \\ \hline \end{array} \begin{pmatrix} v_i \\ u \\ d \end{pmatrix}$$

is transformed by P_F into

$$\begin{array}{r|ccc|cc|}
 & n+q & & & n+q & p \\
\hline
n-m & \diagdown & & F_0 & & \\
 & 0\cdots & \hat{F} & & M_i^0 & \\
m+q & & 0 & \diagdown & & \\
\hline
m & & 0 & & M_i^u & M_i^d \\
\hline
\end{array}
\qquad \begin{pmatrix} v_i \\ u \\ d \end{pmatrix}$$

with a matrix $\hat{F}$ of full rank $n+q$, obtained from F_0 and L in a manner analogous to (3.4). Then (3.2) is equivalent to the system

$$\hat{F}v_i = P_F \begin{bmatrix} \hat{v}_i \\ r_i \end{bmatrix} - M_i^0 \begin{bmatrix} u \\ d \end{bmatrix}, \quad i = 1, \ldots, m+q,$$

$$P_G F_x u + P_G F_c d = P_G \hat{u}, \quad P_G F_x \approx F_0,$$

$$M_i^u u + M_i^d d = R P_F \begin{bmatrix} v_i \\ r_i \end{bmatrix}, \quad i = 1, \cdots, m+q,$$

where R indicates the restriction to the last m components of $P_F \begin{bmatrix} \hat{v}_i \\ r_i \end{bmatrix}$ corresponding to the matrix (M_i^u, M_i^d) in (3.7); the statement $P_G F_x \approx F_0$ indicates that the last m lines in $P_G F_x$ are nearly zero. So we obtain for (u, d) a system of the following structure:

(3.8)

$$\begin{array}{r|cc|}
 & n+q & p \\
\hline
n & F_x & F_c \\
m & M_1^u & M_1^d \\
m & M_{m+q}^u & M_{m+q}^d \\
\hline
\end{array}
\qquad \begin{pmatrix} u \\ d \end{pmatrix}$$

is transformed by P_G into

$$
\begin{array}{r|c|c|}
 & n+q & p \\ \cline{2-3}
n-m & \begin{matrix} \diagdown & F_0 \\ & 0 \\ 0 & \diagdown \end{matrix} & P_G F_c \\ \cline{2-2}
m & \approx 0 & \\ \cline{2-3}
m & M_1^u & M_1^d \\ \cline{2-3}
 & --- & \\ \cline{2-3}
m & M_{m+q}^u & M_{m+q}^d \\ \cline{2-3}
\end{array}
\begin{pmatrix} u \\ d \end{pmatrix}.
$$

This system (3.8) has, by (2.8), $n + m(m + q) = n + q + p$ equations for the $n + q + p$ unknowns. Condition (2.11) is equivalent to the fact that the corresponding matrix has full rank $n + q + p$.

Let us now study the number of elementary arithmetic operations required to solve (3.2). Under the assumption that

$$n >> (m + p + q) \tag{3.9}$$

and only terms multiplied by n^3, n^2 and high powers of m are taken into account. We furthermore assume n to be small enough that we may apply the following procedure:

(3.10)

$$\left\{\begin{array}{l}
\text{Solve (3.2) in the form (3.7), (3.8) with a} \\
\text{Gaussian algorithm and use, for the } F_x\text{-part in(3.7),} \\
\text{(3.8), column pivoting with respect to the } \ell_1\text{-norm} \\
\text{of the first } n+q \text{ elements. For the last } m+q \text{ and } m(m+q) \\
\text{rows in (3.7) and (3.8), respectively, use column pivoting} \\
\text{relative to the } \ell_1\text{-norm of the full rows.}
\end{array}\right.$$

We study the number of necessary of operation subject to the assumptions (3.9) and (3.10). The terms mentioned in (3.9) are dominating terms, which in the sequel are indicated by $\doteq$

(3.11)

	Partial problem	Number of operations
(i)	LR decomposition for F_x with ℓ_1-norm relative pivoting	$\doteq n^3 + \frac{3}{2}n^2(q+1)$
(ii)	operations additional to (i) for full LR decomposition of $\binom{F_x}{L}$ in (3.7)	$\doteq n^2(m+q)$
(iii)	operations, induced in $F_{xx}w_i, F_{xc}w_i$ by LR decomposition of F_x, yielding M_i^0, M_i^u, M_i^d in (3.7), $i = 1,\ldots,m+q$	$\doteq n^3(m+q)$ $+ n^2(m+q)(p+q-2)$
(iv)	operations additional to (i), (iii), for full LR decomposition of (3.8)	$\doteq n^2m(m+q)$ $+ (m(m+q))^3$

Adding the numbers in (i)-(iv) we obtain for the total amount (= number of elementary operations):
(3.12)
Total amount for the LR decomposition of (3.2) $\doteq$

$$n^3(m+q+1) + n^2((m+q)(p+q+m-1) + \frac{3}{2}(q+1)) + (m(m+q))^3$$

In (3.11) and (3.12) we have assumed that the $F_{xx}w_i$ and $F_{xc}w_i$ in (3.7) have already been computed. Since

$$(3.13) \qquad \begin{aligned} &F_{xx} \in \mathcal{L}((\mathbf{R}^{n+q})^2, \mathbf{R}^n), \quad F_{xc} \in \mathcal{L}(\mathbf{R}^{n+q} \times \mathbf{R}^p, \mathbf{R}^n), \\ &F_x, F_{xx}w_i \in \mathcal{L}(\mathbf{R}^{n+q}, \mathbf{R}^n), \quad F_{xc}w_i \in \mathcal{L}(\mathbf{R}^p, \mathbf{R}^n), \end{aligned}$$

we find
(3.14)
the total amount for computing $F_x, F_{xx}, F_{xc} \doteq 2(n^3 + n^2(2q+p+1))$

and
(3.15)
the total amount for computing $F_{xx}w_i, F_{xc}w_i, i = 1,\ldots,m+q,$
$\doteq 2n^3(m+q) + 2n^2(m+q)(p+2q).$

Having solved (3.8) for (u,d) we have to use these known values to compute the $F_{xx}w_iu + F_{xc}w_id, i = 1,\ldots,m+q$, which are needed in (3.7) to determine the v_i.
(3.16)
The total amount for computing $F_{xx}w_iu + F_{xc}w_id,\ i = 1,\ldots,m+q,$
$\doteq 2n^2(m+q).$

To transform the right-hand sides and solve the equations (3.7) and (3.8) we obtain:

(3.17) The total amount for computing the solutions of (3.7), (3.8) $\doteq 2n^2(m+q+1)$.

The summation of (3.12) and (3.14)-(3.17) yields

PROPOSITION 3.1. *Under the assumption* (3.9) *the procedure* (3.10) *requires*

$$\doteq n^3 3(m+q+1) + n^2((m+q)(m+5q+3p+3) + \frac{7}{2}(q+1) + p + 1) + (m(m+q))^3$$

operations to solve (3.2).

The solution of (3.2) via (3.7) and (3.8) may be interpreted as follows: By splitting (3.7) into the first $n+q$ equations for v_i and combining the last m equations in (3.7), $M_i^u u + M_i^d d = \dots$ with $F_x u + D_c d$ into (3.8), we have separated the total system into $m+q+1$ systems with $n+q$ unknowns each if only (3.8) is solved first and then the known value (u,d) is used in (3.7). In this approach the related structure of the different systems is taken into account. Another way to separate (3.2) into small systems is via the use of (2.22), i.e., via the use of the dual kernel.

3.3 *Solution using the dual kernel.* The following procedure only works for the special case that condition (3.18) is satisfied. We will see, however, that even for this case the amount in §3.3 is higher than in §3.2.

(3.18) For $\| x - x_0 \| + \| c \|$ small enough, let $\dim \mathbf{N}(F_x(x,c)) = m+q, \dim \mathbf{N}(F_x(x,c)^*) = m$ and (2.3)(ii), (2.11), (2.12) be satisfied for $F'(x,c)$.

Again omitting the arguments $(x, w_1, \dots, w_{m+q}, c)$ as in (3.2) we first compute approximations $y_j \in \mathbf{R}^n$ for the φ_j with an appropriate operator $\hat{L} : \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\mathbf{N}(\hat{L}) \cap \mathbf{N}(F_x^*) = \{0\}, \tag{3.19}$$

and a matrix

$$B = (b_1, \ldots, b_m), \quad b_j \in \mathbf{R}^m, \text{ rank } B = m, \tag{3.20}$$

from

$$\begin{aligned} F_x^* y_j &= 0 \\ \hat{L} y_j &= b_j, \quad j = 1, \ldots, m. \end{aligned} \tag{3.21}$$

We will see below that the LR decomposition of F_x may be used to solve (3.21) and to see how to choose $\hat{L}$. As a consequence of (3.18)-(3.20) the systems (3.21) are uniquely solvable. Taking the inner product $(\cdot,\cdot)$ of the y_j in (3.21) with the first equations in (3.2) we find, as in (2.21), (2.22) that

$$\begin{aligned} F_x u + F_c d &= \hat{u} \\ (y_j, F_{xx} w_i u + F_{xc} w_i d) &= (y_j, \hat{v}_i), \quad i = 1, \ldots, m+q, \ j = 1, \ldots, m. \end{aligned} \tag{3.22}$$

From the proof of Theorem 2.9 we know that this system for (u, d) is uniquely solvable by (3.18). With the known (u, d) we finally compute v_i from (3.2)(i), (ii) as the solution of

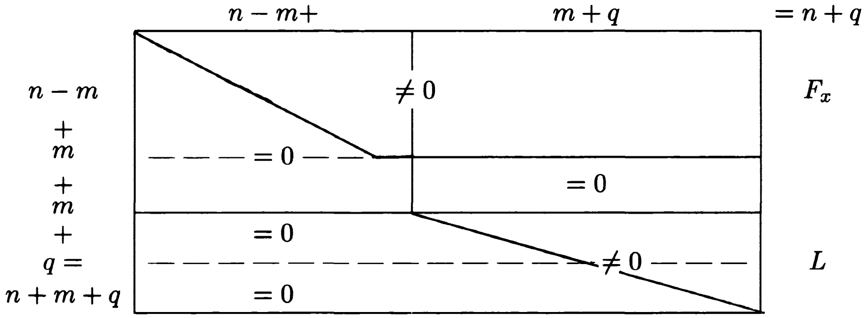
$$\begin{aligned} F_x v_i &= \hat{v}_i - F_{xx} w_i u - F_{xc} w_i d, \\ L v_i &= r_i, \quad i = 1, \ldots, m+q. \end{aligned} \tag{3.23}$$

By (3.18) this overdetermined system is uniquely solvable as we have seen in the proof of Theorem 2.9.

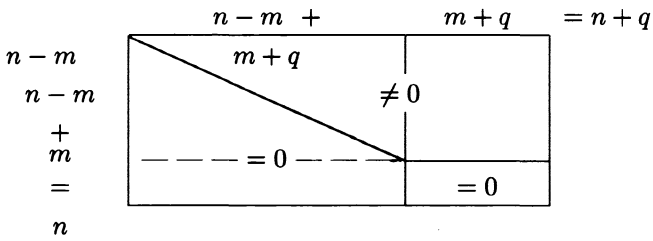
For the solution of (3.22) and (3.23) we may use nearly the same LR decomposition. We use the strategy in (3.10), where obviously (3.7) and (3.8) must be replaced by (3.23) and (3.22), respectively. Thus, the LR decomposition for $\begin{bmatrix} F_x \\ L \end{bmatrix}$ is obtained as

$$G_{n+q-1} P_{n+q-1} \cdots F_{n-m+1} P_{n-m+1} P G_{n-m} P_{n-m} \cdots G_1 P_1 \begin{bmatrix} F_x \\ L \end{bmatrix}$$

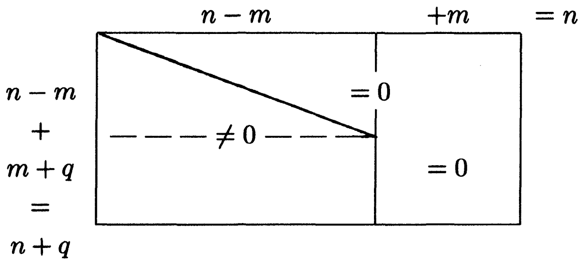
which has the structure



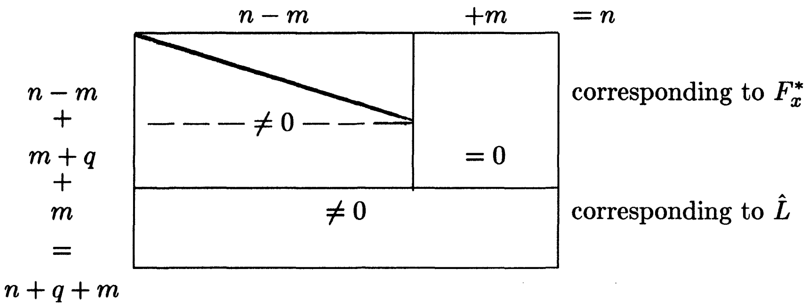
In (3.24) the P_j, G_j represent the well-known permutation and combination matrices in the Gaussian algorithm (see, e.g., [**48**, p. 135]). By (3.10) we allow only "restricted" permutations P_j; they permute, for $j = 1, \dots, n-m$, only the rows $1, \dots, n$ of F_x. Then P exchanges the last (now trivial) rows of the transformed F_x with the transformed L. Necessarily, the $P_j, j = n-m+1, \dots, n+q-1$, only exchange the lines of L. Now, we denote for simplicity the "restrictions" of the P_j, G_j to the lines of F_x instead of the full $\begin{bmatrix} F_x \\ L \end{bmatrix}$ with the same symbols P_j, G_j again, $j = 1, \dots, n-m$. $G_{n-m}P_{n-m} \cdots G_1P_1F_x$ has the structure:



With $P_j^* = P_j$ we find that $F_x^* P_1 G_1^* \dots P_{n-m} G_{n-m}^*$ has the structure:



With the $n \times n$ matrices P_j, G_j^* we finally obtain that $\left[\begin{smallmatrix} F_x^* \\ \hat{L} \end{smallmatrix}\right] P_1 G_1^* \ldots P_{n-m} G_{n-m}^*$ has the structure:



Let us choose, in particular, $\hat{L}$ such that

$$\hat{L} P_1 G_1^* \cdots P_{n-m} G_{n-m}^* = \overset{\quad n-m \qquad m}{\boxed{\;0\;} \boxed{\;I_{m,m}\;}}\; m,$$

where $I_{m,m}$ denotes the $m \times m$ identity matrix, and let us denote the matrix in (3.25) by A^*. Then with appropriate $\hat{b}$, and with $(G_j^*)^{-1}$ immediately obtained from G_j^* by inverting the nondiagonal terms,

$$b = \begin{bmatrix} F_x^* \\ \hat{L} \end{bmatrix} y \Leftrightarrow \hat{b} = {A^*}^{-1} G_{n-m}^* P_{n-m} {G_1^*}^{-1} y =: A^* x.$$

Now the equation $A^*x = \hat{b}$ is solved very easily, since the first $n+q$ components in b and in $\hat{b}$ are zero and the lines $n-m+1,\dots,n+q$ in the upper part of A^* depend linearly upon the first $n-m$ lines of A^*. Hence,

(3.26)
calculating $y = P_1 G_1^* \dots P_{n-m} G_{n-m}^{*n}$ requires $\doteq n^2$ operations.

A full use of the decomposition in (3.24) would be possible; however, it would require much bookkeeping and a complicated argumentation, which we wish to avoid.

Now we obtain the number of operations to solve (3.21)-(23.23) very similarly to §3.2. Using the approach for the LR decomposition of F_x^* indicated above, and with known matrices in (3.27) (see (3.22), (3.11) (i), (ii), (iv), (3.12)), we get:

(3.27) The total amount of operations required for the LR decomposition of (3.21)-(3.23)
$$\doteq n^3 + n^2((m+q)(1+m) + \frac{3}{2}q) + (m(m+q))^3.$$

Furthermore, we have to compute the

$$(3.28) \quad (y_j, F_{xx}w_i\cdot), \quad (y_j, F_{xc}w_i\cdot), \quad i=1,\dots,m+q, \ j=1,\dots m,$$

(see (3.13)) which requires in addition to (3.14) and (3.15) that

(3.29) total amount of operations for computions (3.28)
$$\doteq 2mn^3 + 2m(p+q)n^2.$$

Finally, we have to transform the right-hand sides and solve the equations (3.21)-(3.23), requiring a total amount of operations

(3.30)
for computing the solution of (3.21)-(3.23) $\doteq 2n^2(2m+2q+1)$.

The summation of the numbers in (3.14)-(3.16), (3.26) m times, and (3.27)-(3.30) yields

PROPOSITION 3.2. *Under the assumption* (3.9), (3.18) *the procedure corresponding to* (3.10) *via* (3.21)-(3.23) *requires*

$$\doteq n^3(4(m+\frac{q}{2})+3)+n^2((m+q)(m+2p+4q+3)+2m(p+q+\frac{5}{2}) \\ +2p+\frac{19}{2}q+3)+(m(m+q))^3$$

operations.

3.4 *Examples.* We conclude this section with several examples.

EXAMPLE 3.3. *Let* $G:\mathbf{R}^2 \to \mathbf{R}^2$ *and*

$$G\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \frac{1}{2}\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}, \quad G\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then we have

$$G'\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \quad G'\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{N}\left(G'\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \mathbf{R}^2,$$

and we find

$$G'\begin{bmatrix} 0 \\ 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = G'\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{N}\left(G'\begin{bmatrix} 0 \\ 0 \end{bmatrix}^*\right) = \mathbf{R}^2,$$

since

$$\left[G'\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right] = 0 = \left[\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, G'\begin{bmatrix} 0 \\ 0 \end{bmatrix}^*\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right].$$

So we have to define

$$F:\mathbf{R}^2 \times \mathbf{R}^4 \to \mathbf{R}^2$$

using 4×2 matrices B_1, B_2, Q to obtain with the inner product $(\cdot,\cdot)$ in $\mathbf{R}^2$

$$F(x,c) := G(x) + B(x,c) + Qc \text{ with } B(x,c) := \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} B^1c \\ B_2c \end{pmatrix}\right].$$

Then we need the following partial derivatives at $x_0 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, c_0 := (0,0,0,0)^T$,

(3.31)
$$\begin{cases} F_x(x_0, c_0)u = G' \begin{bmatrix} 0 \\ 0 \end{bmatrix} u + B(u, c_0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ F_c(x_0, c_0)d = B(0, d) + Qd = Qd, \\ F_{xx}(x_0, c_0)(u, v) = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = G''(x_0)(u, v) = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \end{bmatrix}, \\ F_{xc}(x_0, c_0)(u, d) = B(u, d) = \left[\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \right]. \end{cases}$$

Now we choose (see (2.18)) $L = L_0$ such that rank $L_0 = 2$, e.g., $L_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and (2.3) is satisfied. To guarantee (2.12) we need $\mathbf{R}(F'(x_0, c_0)) = \mathbf{R}(F_x, (x_0, c_0), F_c(x_0, c_0)) = \mathbf{R}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, Q \right) = \mathbf{R}^2$, so Q can be any matrix of rank 2. Finally, we require, with $\phi_i = e_i, \varphi_j^* = e_j^*, e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the above results, the conditions (2.11) or (2.27). This amounts to

	1	0	β_{11}^1	β_{12}^1	β_{13}^1	β_{14}^1	$\langle \varphi_1^*, B_1 phi_1 \cdot \rangle$
$M : \langle \varphi_j^*, G''\phi_i \cdot \rangle$	0	0	β_{21}^1	β_{22}^1	β_{23}^1	β_{24}^1	
(3.32)	0	0	β_{11}^2	β_{12}^2	β_{13}^2	β_{14}^2	$\langle \varphi_2^*, B_2 \phi_i \cdot \rangle$
	0	1	β_{21}^2	β_{22}^2	β_{23}^2	β_{24}^2	
	0	0	q_{11}	q_{12}	q_{13}	q_{14}	Q
	0	0	q_{21}	q_{22}	q_{23}	q_{24}	
(variable)	u			d			

having the full rank 6. So the following choice would satisfy all of the conditions

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.33}$$

By the above statements, any random choice would have sufficed with probability one as well.

EXAMPLE 3.4. Let $G\mathbf{R}^2 \to \mathbf{R}^2$ be

$$G\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is an example satisfying (2.30) and we have to use

$$F(x,c) = G(x) + C(x,x,c-\overline{c}) + B(x,c) + Qc.$$

If we choose for simplicity a trilinear $C(x,y,c-\overline{c})$¡ which is symmetric in x, y, the only difference compared to (3.31)is F_{xx} which has to be replaced by

$$F_{xx}(x_0,c_0)(u,v) = 2C(u,v,-\overline{c}).$$

For example, if

$$C(x,y,c) := \begin{bmatrix} x_1y_1c_1 + x_2y_2c_2 \\ x_1y_1c_3 + x_2y_2c_4 \end{bmatrix} \quad \text{and} \quad -\overline{c} = (1,1,1,1),$$

the partial matrix $(\varphi_j^*, G_i''\cdot)$ in (3.32) has to be replaced by

$$\begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 0 & 2 \end{bmatrix},$$

and B_1, B_2, Q have to be chosen such that the modified matrix in (3.32) has full rank 6. That is the case, e.g., with the B_1, B_2, Q in (3.33), and would also have been correct with probability one for any random choice of $Q, B_1, B_2^{\bullet}, C$.

The preceding examples illustrate the inflation technique for the case $q = 0$ and $E = \mathbf{R}^n = \hat{E}$. In this case the inflation technique is related to the tensor Newton method recently described by Schnabel and Frank [**41**]. For additional references on this case see [**41, 20**] and the bibliographies therein.

The inflation technique for $q = 1$ and $m = 1$ involving simple bifurcation has recently been treated by several authors (see, e.g., [**29, 30, 35, 42-44, 52, 53**, and **8a**]).

To illustrate an example with $q = 1$ and $m > 1$ we will briefly outline a case recently studied in [**3**] at the conclusion of this paper.

Discretization methods applied to H. As was mentioned in the introduction, we want to use the results in §2, especially Theorem 2.9 in two different ways. Either, for $q > 0$, in continuation methods where it is important to notice the existence of branching points. Whenever these branching points have to be computed, our results apply. Often it is not necessary to really compute them. Then unfolding techniques can be used. Or we may, for $q = 0$, use our results to compute z_0 and $\mathbf{N}(G'(z_0))$ for (2.1) which is not tractable with the common discretization methods, since $G'(z_0)$ is not invertible.

Let us briefly introduce the formalism for defining discretization methods. Instead of the original problem (2.1) we want to solve

$$HX = 0 \text{ with } X := (x, w_1, \dots, w_{m+q}, c)^T \in \mathbf{F}, \tag{4.1}$$

where H is defined in (2.6). We know from Theorem 2.9 that

$$H'(Z_0) : \mathbf{F} = \mathbf{E}^{m+q+1} \times \mathbf{R}^p \to \hat{\mathbf{F}} = \hat{\mathbf{E}} \times (\hat{\mathbf{E}} \times \mathbf{R}^{m+q})^{m+q}, \tag{4.2}$$

where $Z_0 := (z_0, \overline{\phi}_1, \dots, \overline{\phi}_{m+q}, 0)^T$ (the exact solution of (4.1)) is continuously invertible.

This fact yields the possibility of using the general setting of discretization methods as presented, e.g., in Stetter [**46**], Böhmer [**10**]. Let, for (4.1), (4.2),

$$\begin{aligned} &\Delta^h : \mathbf{F} \to \mathbf{F}^h := (\mathbf{E}^h)^{m+q+1} \times \mathbf{R}^p \\ &\hat{\Delta}^h : \hat{\mathbf{F}} \to \hat{\mathbf{F}}^h := \hat{\mathbf{E}}^h \times (\hat{\mathbf{E}}^h \times \mathbf{R}^{m+q})^{m+q} \end{aligned} \tag{4.3}$$

be bounded linear operators onto finite dimensional spaces, e.g., $\mathbf{E}^h$ or $\hat{\mathbf{E}}^h$ Banach spaces of grid functions, or of finite elements used in the discretization, where h indicates a discretization parameter in an appropriate $\mathbf{R}^k$. Then (4.1) is transformed into

$$\begin{aligned} &H^h X^h = 0 \in \hat{\mathbf{F}}^h \\ &X^h = (x^h, w_1^h, \dots, w_{m+q}^h, c^h) \in \mathbf{F}^h, \end{aligned} \tag{4.4}$$

having the exact solution $Z_0^h = (x_0^h, \phi_1^{-h}, \dots, \phi_{m+q}^{-1}, c_0^h)$.

Whenever H is given in a form where the usual discretization methods apply, the general concepts of consistency, stability and convergence are available and read in this case as (choose appropriate norms)

$$\begin{aligned} &\| H^h \Delta^h Z_0 - \hat{\Delta}^h H Z_0 \| = \| H^h \Delta^h Z_0 \| = o(1)(O(h^\ell)) \\ &\text{consistency (of order } \ell\text{), for bounded } \| z_0 \|_*, \end{aligned} \tag{4.5}$$

with a suitable Sobolev-norm $\| \cdot \|_*$,

$$\| X_1^h - X_2^h \| \leq S \, \| H^h X_1^h - H^h X_z^h \| \quad \text{stability,} \tag{4.6}$$

where S is independent of h and the right-hand side and $\| X_i^h - \Delta^h Z \|$ are sufficiently small, and

$$\| \Delta^h Z_0 - Z_0^h \| = o(1)(0(h^\ell)) \text{ convergence (of order } \ell\text{).} \tag{4.7}$$

If, with $\Delta^h, \hat{\Delta}^h$ in (4.3), the relations (4.5), (4.6) are satisfied and H^h is continuous in $\| X^h - \Delta^h Z_0 \| \leq \rho, \rho$ sufficiently small and independent of h, then (4.7) is true.

It is well known (see [**46**]) that under rather general conditions the bounded invertibility of $(H^h)'$ in (4.2) implies the stability (4.6), a fact which is usually much harder to prove than the consistency (4.5).

In some recent papers (see, e.g., Stummel [**49, 50**]) for integral equations of the second kind, Beyn [**7**] and Grigorief [**23, 24**] for initial and boundary value problems in ordinary differential equations or integro-differential equations, and Hackbusch [**25, 26**] for certain elliptic boundary value problems) results of the following type have been shown:

$$\lim_{h \to 0} \| (H^{h'}(Z^h))^{-1} \| = \| H'(Z))^{-1} \|; \tag{4.8}$$

the operator norms in (4.8) are based on limit relations for the norms related via Δ^h and $\hat{\Delta}^h$ as

$$\lim_{h\to 0} \| \Delta^h X \| = \| X \| \quad \text{for any } x \in \mathbf{F}, \tag{4.9}$$
$$\lim_{h\to 0} \| \hat{\Delta}^h Y \| = \| Y \| \quad \text{for any } Y \in \hat{\mathbf{F}}.$$

Under well-known smoothness or piecewise smoothness conditions for H and appropriate discretizations H^h, asymptotic expansions for the "local discretization error"

$$H^h \Delta^h X - \hat{\Delta}^h H X = \hat{\Delta}^h \Big(\sum_{j=1}^{q} h^j V^j + O(h^{q+1}) \Big)$$

for small enough $\| X - Z \|_*$ are observed. For a consistent and stable discretization, satisfying additional technical conditions (see [**10, 46**]), the discrete approximation admits an asymptotic expansion of the form

$$Z^h = \Delta^h \Big(Z + \sum_{j=p}^{q} h^j W^j + O(h^{q+1}) \Big). \tag{4.10}$$

where the W^j (and the V^j above) are independent of h. For "symmetric" discretizations we usually have h^2 expansions, so the powers of h have to be replaced by powers of h^2. This establishes the possibility for using Richardson extrapolation or any kind of defect and deferred corrections (see, e.g., Böhmer [**10, 11**], Böhmer-Hemker-Stetter [**12**], Böhmer-Römer [**13**], Böhmer-Stetter [**14**], Frank-Hertling-Ueberhuber [**18**], Frank-Ueberhuber [**19**], Hackbusch [**27**], Lindberg [**32, 33**], Pereyra [**36-38**], and Stetter [**47**]).

5. Computation of the discrete approximation. We start the computation of

$$H^h Z^h = 0 \tag{5.1}$$

(see (4.4)) by first considering only the original equation

$$G^h x^h = 0, \tag{5.2}$$

which we assume has an exact solution x_0^h. Starting with a point $x^{h,0}$ near x_0^h, we might use a Newton-type method, e.g.,

(5.3) $$G^{h'}(x^{h,\nu})(x^{h,\nu+1} - x^{h,\nu}) = -G^h(x^{h,\nu}), \quad \nu = 0, 1, \ldots.$$

Since $\mathbf{N}(G'(x_0)) \neq \{0\}$, for a sufficiently good approximation $x^{h,\nu}$ to x_0 the $(G^h)'(x^{h,\nu})$ should be nearly singular. Because of the perturbation caused by the discretization, neither $(G^h)'(x_0^h)$ nor $(G^h('(\Delta^h x_0)$ are necessarily singular. Another problem results from the fact that, for a discretization, we usually only have, for a suitably restricted Δ^h,

(5.4) $$(F_x(x,c))^h = (F^h(\Delta^h(x,c)))_x h + O(h^\ell),$$

when $\| (x,c) - (z,0) \|$ is sufficiently small. The $O(h^\ell)$ term indicates that (omitting the arguments (x,c) and $\Delta^h(x,c)$)

$$\| (F_x)^h - (F^h)_x u^h \| \leq C \cdot h^\ell \cdot \| u^h \|_*^h$$

for some C independent of h and where $\| \cdot \|_*^h$ is a discrete Sobolev norm corresponding to $\| \; \|_*$ in (4.5). Therefore, for the following discussion we make

ASSUMPTION 5.1. *Let the discretization for H satisfy the following relations*

(5.5) $$H^h X^h = \begin{bmatrix} F^h(x^h, c^h) \\ (F^h)_x h(x^h, c^h) w_i^h \\ L^h w_i^h - a_i^h \end{bmatrix} + O(h^\ell),$$

$i = 1, \ldots, m+q$, for $\| x^h - \Delta^h z \|_*^h + \| c^h \|$ small enough, where $F^h, (F^h)_x h, L^h$ are the corresponding discretizations of the single components of H. Furthermore, omitting the arguments on the right-hand side,

(5.6) $$(H^h)'(X^h, w_1^h, \ldots, w_{m+q}^h, c^h) = \begin{bmatrix} F^h_{x^h} & 0 & \cdots & 0 & F^h_{c^h} \\ F^h_{x^h x^h} w_1^h & F^h_{x^h} & \cdots & 0 & F^h_{x^h c^h} q_1^h \\ 0 & L^h & \cdots & 0 & 0 \\ & & & & \\ F^h_{x^h x^h} w^h_{m+q} & 0 & \cdots & F^h_{x^h} & F^h_{x^h c^h} w^h_{m+q} \\ 0 & 0 & \cdots & L^h & 0 \end{bmatrix}$$

for $\| x^h - \Delta^h z \|_*^h + \| c^h \|$ *sufficiently small and* $\| w_i^h \|_*^h$ *bounded.*

REMARK 5.2.

(i). To the knowledge of the authors the properties (5.5) and (5.6) are satisfied for all important discretization methods. They are closely related to the so-called admissible $(H^h)'$ and F_x, F_{xx} and F_{xc} approximations introduced in Böhmer [**11**], especially if the discretization of H works componentwise.

(ii). If stability results hold for H^h and $(H^h)'$, then a simple application of perturbation arguments shows that they are valid for the right-hand side approximations in (5.5) and (5.6) as well. The existence of asymptotic expansions is not touched upon since we need (5.5) and (5.6) only for the numerical solution of the approximation X_0^h in combination with a method in §3.2. The approach in §3.3 does not work here because of the problems mentioned following (5.3).

Based on (5.5), (5.6) it is now straightforward to compute a numerical solution for a problem with singular $G'(z_0, \lambda_0)$. We finish with some examples.

EXAMPLE 5.3. We discuss the nonlinear boundary value problem

$$\begin{aligned} &z'' + \sin z + z^2 = 0, \\ &\sin(z(0) + z(\pi)) = 0, \\ &z'(0) + z'(\pi) = 0. \end{aligned} \tag{5.7}$$

We apply the general theory for the case of

(5.8) closed, densely defined operators G with closed range.

If we want to compute locally unique solutions for (5.7), we have to require

$$\begin{aligned} x(0) + x(\pi) &= 0, \\ x'(0) + x'(\pi) &= 0, \end{aligned} \tag{5.9}$$

or some integer multiples of π instead of 0 for the function values. Now, in $H^2[0,\pi]$, let (see (2.2))

(5.10)

$$\mathbf{D}(G) := \{x \in C^2[0,\pi], x \text{ satisfies (5.9) } \} \subseteq L^2[0,\pi] =: \mathbf{E} =: \mathbf{E}^*$$

$$=: \hat{\mathbf{E}} =: \hat{\mathbf{E}}^* \text{ with } \langle u, v\rangle = \int_0^\pi u(\tau)v(\tau)d\tau = (u,v)_2,$$

$$G(x) := x'' + \sin x + x^2, G(0) = 0.$$

For $x, u \in \mathbf{D}(G)$ we then have

$$\begin{aligned} &G'(x)u = u'' + (2x + \cos x)u, \quad G''(x)uv = (2 - \sin x)uv, \\ &G'(0)u = u'' + u, \quad G''(0)uv = 2uv. \end{aligned} \tag{5.11}$$

Then $G'(x)$ and $G''(x)$ satisfy (5.8) and we are able to apply our general theory. We have immediately

$$\mathbf{N}(G'(0)) = [\sin\cdot, \cos\cdot], \text{ so } m = 2, q = 0. \tag{5.12}$$

To compute $G'(0)^*$ we observe that, for $u \in \mathbf{D}(G), v \in H^2[0,\pi]$ (see (5.10), (5.11)),

$$\begin{aligned} (G'(0)u, v)_2 - (u, G'(0)v)_2 &= \int_0^\pi [v(u''+u) - u(v''+v)]d\tau \\ &= u'(\pi)v(\pi) - u'(0)v(0) + u(0)v'(0) - u(\pi)v'(\pi) \\ &= u'(\pi)[v(\pi) + v(0)] + u(0)[v'(0) + v'(\pi)]. \end{aligned}$$

So we have $\mathbf{D}(G'(0)^*) = \mathbf{D}(G) = D(G'(0))$, and hence

$$G'(0) = G'(0)^*, \ \mathbf{N}(G'(0)) = \mathbf{N}(G'(0)^*) = [\sin\cdot, \cos\cdot]. \tag{5.13}$$

To define L in (2.3) we have to choose two continuous linear functionals defined on $\mathbf{D}(G)$, e.g.,

$$Lx := \begin{bmatrix} x(\pi/2) \\ x'(\pi/2) \end{bmatrix}. \tag{5.14}$$

From (5.12) we see that $\mathbf{N}(L) \cap \mathbf{N}(G'(0)) = \{0\}$. Corresponding to §2.4 we define (see (2.8))

$$F(x,c) := G(x) + B(x,c) + Qc, \quad x \in \mathbf{D}(G), c \in \mathbf{R}^4. \tag{5.15}$$

Then

$$\begin{aligned} F'(0,c)\begin{bmatrix} u \\ d \end{bmatrix} &= G'(0)u + Qd, \\ \mathbf{R}(F'(0,c)) &= \mathbf{R}(G'(0)) + \mathbf{R}(Q). \end{aligned} \tag{5.16}$$

Because of

$$\mathbf{R}(G'(0)) = [\sin\cdot, \cos\cdot]^{\perp 2}, \tag{5.17}$$

(see (5.13) and e.g., [**51, 54**]) we need a Q (see (2.12)) such that

$$\begin{aligned} &\dim \mathbf{R}(Q) \geq 2, \text{ and linearly independent } x^{(1)}, x^{(2)} \in \mathbf{R}(Q), \\ &\text{with } \mid (x^{(i)}, \sin\cdot)_2 \mid + \mid (x^{(i)}, \cos\cdot \,|_2| > 0, \quad i = 1, 2. \end{aligned} \tag{5.18}$$

For our simple case with $G'(0)$ as in (5.11), we may verify (5.17) directly and we would not need a general theorem. The general solution for

$$y'' + y = f$$

is given as

$$y(t) = \left(\alpha_1 + \int_0^t f(\tau)\cos\tau d\tau\right)\sin t + \left(\alpha_2 + \int_0^t f(\tau)\sin\tau d\tau\right)\cos t.$$

Since we only admit $y \in \mathbf{D}(G)$ we have to require

$$y(0) + y(\pi) = \int_0^\pi f(\tau)\sin\tau \; d\tau = 0$$

$$y'(0) + y'(\pi) = -\int_0^\pi f(\tau)\cos\tau \; d\tau = 0,$$

that is, (5.17).

To satisfy (5.18) we may, e.g., choose Q as

(5.19)

$$\begin{aligned} &Q(c_1, \ldots, c_4) := (c_1 + c_2)1 + c_3 t + c_4(2 - \pi t + t^2), \text{ with} \\ &\text{functions } x^{(1)}(t) = 1, x^{(2)}(t) = t^2, x^{(3)}(t) = 2 - \pi t + t^2 \end{aligned}$$

which are linearly independent and

$$(x^{(1)}, \sin\cdot)_2 \neq 0, (x^{(2)}, \cos\cdot)_2 \neq 0,$$
$$(x^{(3)}, \sin\cdot)_2 = (x^{(3)}, \cos\cdot)_2 = 0, \text{ so } x^{(3)} \in \mathbf{R}(G'(0)).$$

For our choice of L in (5.14) and Q in (5.18) and (5.19) the conditions (2.3) (ii) and (2.12) are satisfied. To discuss (2.11) first study $N(F'(0,0)) = \mathbf{N}(G'(0), Q)$. We have, by Proposition 2.1 (see (5.16)),

$$\begin{aligned} \mathbf{N}(F'(0,0)) &= \mathbf{N}(G'(0), Q) \\ &= \{(u, d) \mid u'' + u + Qd = 0, u \in \mathbf{D}(G), d \in \mathbf{R}^4\} \\ &= \{(u, d) \mid d = d_0 + d r, d_0 \in \mathbf{N}(Q), Qd_r \in \mathbf{R}(G'(0)), \\ &\quad u = u_0 + u_r, u_0 \in \mathbf{N}(G'(0)), G'(0)u_r = -Qd_r\}. \end{aligned}$$

Now

$$\begin{aligned}\mathbf{N}(Q) &= \{d = (d_1, d_2, d_3, d_4)^T \mid d_1 + d_2 = d_3 = d_4 = 0\}\\ &= [(1, -1, 0, 0)],\\ Q\begin{bmatrix}0\\0\\0\\1\end{bmatrix} &= (2 - \pi t + t^2) = G'(0)u_p,\end{aligned}$$

$$u_p'' + u_p = (2 - \pi t + t^2) \text{ implies } u_p = t^2 - \pi t \in \mathbf{D}(G),$$

and finally

$$\tag{5.20} \begin{aligned}\mathbf{N}(F'(0,0)) = \{(u,d) : d &= (d_1, -d_1, 0, d_4),\\ u(t) &= \alpha \sin t + \beta \cos t - d_4 u_p\}.\end{aligned}$$

To satisfy (2.11) we have to define a bilinear operator

$$B : \mathbf{D}(G) \times \mathbf{R}^4 \to \hat{\mathbf{E}} = L^2[0, \pi].$$

We choose, e.g.,

$$\tag{5.21} B(u, d) := d_2 u + d_4 u.$$

With (5.20) this B satisfies (2.11) if and only if the rank of the matrix

$$\int_o^\pi \varphi_j(t)\Big(2\phi_i(t)[\alpha \sin t = \beta \cos t - d_4 u_p] + d_2\phi_i(t) + d_4\phi_i'(t)\Big)dt,$$

$i, j = 1, 2$, is 4. We obtain the matrix

$$\begin{array}{rcccc} j = i = 1: & 8/3 & 0 & -\pi/2 & \pi^3/6 + \pi/2,\\ j = 1, i = 2: & 0 & 4/3 & 0 & -\pi/2,\\ j = 2, i = 1: & 0 & 4/3 & 0 & \pi/2,\\ j = 2 = i: & 4/3 & 0 & -\pi/2 & \pi^3/6 - \pi/2\end{array}$$

which indeed has the rank 4.

It seems as if the choice of L, Q and B in (5.14), (5.19) and (5.21) required to define F in (5.15) relies very much upon the knowledge of $\mathbf{N}(G'(0))$ and $\mathbf{R}(G'(0)) = \mathbf{N}(G'(0)^*)^\perp$. Indeed, we have chosen

L, Q and B such that the condition (2.3)(ii), (2.11), (2.12) are satisfied with the known $\mathbf{N}(G'(0)) = \mathbf{N}(G'(0)^*)$. However, the choice of L in (5.14) would have been straightforward whenever any moderate approximation for $\mathbf{N}(G'(0))$ would have been available. The same is true essentially for Q in (5.19). Since $x^{(1)}$ and $x^{(2)}$ are arbitrary anyway and for any $x^{(3)} \in \mathbf{R}(G'(0))$, the component of $x^{(3)}$ in $\mathbf{R}(G'(0))$ or in an approximation would have done as well. Finally, the B in (5.21) has not been related with $\mathbf{N}(G'(0))$ at all.

Now we apply a symmetric divided difference of formula to (5.7). We introduce a grid

$$\mathbf{C}^h := \{t_\nu := \nu\pi/n, \nu = -1, \dots, 2n+1\}$$

and

$$z_\nu^h := z^h(t_\nu), w_{i,\nu}^h := w_i^h(t_\nu).$$

The discretization of (5.7) with (5.9) would then be

$$(5.7^h)\quad (z_{\nu+1}^h - 2z_\nu^h + z_{\nu-1}^h)/h^2 + \sin z_\nu^h + (z_\nu^h)^2 = 0, \quad \nu = 0, 1, \dots, 2n,$$

$$(5.9^h)\qquad z_0^h + z_{2h}^h = 0, \quad \frac{z_1^h - z_{-1}^h}{2h} + \frac{z_{2n+1}^h - z_{2n-1}^h}{2h} = 0.$$

For $F(x, c)$ (see (5.15), (5.19) and (5.21)) we would have

$$\begin{aligned} F^h(x^h, x^h) =& \frac{x_{\nu+1}^h - 2x_\nu^h + x_{\nu-1}^h}{h^2} + \sin x_\nu^h + (x_\nu^h)^2 + c_2 x_\nu^h \\ &+ c_4 \frac{x_{\nu+1}^h - x_{\nu-1}^h}{2h} + (c_1 + c_2) + c_3 t_\nu + c_4(2 - \pi t_\nu + t_\nu^2), \\ &\nu = 0, 1, \dots, 2n, x_0^h + x_{2h}^h = 0, \quad \frac{x_1^h - x_{-1}^h}{2h} + \frac{x_{2n+1}^h - x_{2n-1}^h}{2h} = 0. \end{aligned}$$

The $(F_x)^h(x^h, c^h)w_i^h$ would be

$$\begin{aligned} (F_x)^h(x^h, c^h)w_i^h =& \frac{w_{i,\nu+1}^h - 2w_{i,\nu}^h + w_{i,\nu-1}^h}{h^2} + (\cos x_\nu^h) w_{i,\nu}^h + 2x_\nu^h w_{i,\nu}^h \\ &+ c_2 w_{i,\nu}^h + c_r \frac{w_{i,\nu+1}^h - w_{i,\nu-1}^h}{2h}, \quad \nu = 0, 1, \dots, 2n, \end{aligned}$$

and

$$L^h x^h = \begin{bmatrix} x_n^h \\ (x_{n+1}^h - x_{n-1}^h)/(2h) \end{bmatrix}.$$

With these results we see that (5.5) and (5.6) are satisfied even without the $O(h^\ell)$-terms.

EXAMPLE 5.4. Consider the equation

$$G(z, \lambda) = 0$$

where $G : C_0^2([0,1]^2) \times \mathbf{R}_+ \to C([0,1]^2)$ is of the form

$$G(z, \lambda) = \Delta z + \lambda f(z).$$

We assume $C_0^2([0,1]^2)$ is the space of twice continuously differentiable functions defined on the unit square and vanishing on the boundary. In addition, let us assume that

$$f'(0) = 1, \quad f(0) = 0.$$

Then $G'(z, \lambda)(w) = \Delta w + \lambda f'(z)w$ and, in particular,

$$G'(0, \lambda)(w) = \Delta w + \lambda w.$$

Thus, on the λ-axis, the eigenvalues of $G'(0, \lambda)$ are given by

$$\lambda_{s,t} = (s^2 + t^2)\pi^2$$

where s, t are positive integers. Now suppose that $s^2 + t^2$ is factored (uniquely) into the form

$$s^2 + t^2 = 2^\alpha \prod_{i=1}^{k} p_i^{r_i} \prod_{i=1}^{\ell} q_i^{s_i},$$

where p_i and q_i are of the form $p_i = 4i + 1, q_i = 4i + 3$. Then it can be shown [**31**] that the multiplicity $m_{s,t}$ of $\lambda_{s,t}$ is given by

$$m_{s,t} = \prod_{i=1}^{k} (1 + r_i).$$

Thus, if $s^2 + t^2 = 8$, then $m = 1$; if $s^2 + t^2 = 10$, then $m = 2$; if $s^2 + t^2 = 50$, then $m = 3$; if $s^2 + t^2 = 65$, then $m = 4$, etc. If, in addition, f is an odd map, then according to **[6]** bifurcations of order m actually occur at $(0, \lambda_{s,t})$.

It is not our aim at this point to carry through the entire discussion for the extension of $G(x) = G(z, \lambda)$ to $f(x, c) = F(z, \lambda, c)$. We confine this discussion to a few additional remarks. First of all, since $q = 1$, we have by (2.8) that

$$P_{s,t} = m_{s,t}(m_{s,t} + 1) - 1.$$

By (2.9), (2.26) and the subsequent discussion we have $x = (z, \lambda), x_0 = (0, \lambda_{s,t})$. Then

$$G'(x_0)u = \Delta u + \lambda_{s,t} u$$

$$G''(x_0)(u, v) = \lambda_{s,t} f''(0)uv$$

and

$$F(x, c) = G(x) + B(x, c) + Qc$$

where

$$B : E \times \mathbf{R}^p \to \hat{E} \text{ is bilinear}$$

$$Q : \mathbf{R}^p \to \hat{E} \text{ is linear.}$$

In general it will suffice to choose a random $\alpha \in \mathbf{R}^p - \{0\}$ and random linearly independent $z_i \in C_0^2([0,1]^2), i = 1, \ldots, p$. Then we may take

$$F(x, c) = F(z, \lambda, c) = G(z, \lambda) + \sum_{i=1}^{p} c_i z_i + \sum_{i=1}^{p} \alpha_i c_i.$$

Since for practical numerical considerations it is not in general possible to work with this extension, we shall not continue the discussion of this example.

Although we will not carry forth the further discussion of this example here, a few additional remarks concerning the numerical aspects of the case $q = 1$ ought to be made. First of all, for $q = 1$, a continuation method may be used to traverse the solution sets, since they are 1-manifolds near regular points. In traversing these 1-manifolds one may monitor algebraic invariants of the Jacobians (e.g., the signature) in order to empirically determine m at a singular point x_0. Secondly, in many examples arising from physical applications a great deal of a priori information concerning properties of the solutions may be available, e.g., symmetry, oscillations, etc. In order to avoid the numerically irrelevant solutions (which are likely to be present) and to possibly reduce slightly the dimension of the extension, these known properties ought to be incorporated into the extension $F(x, c)$.

For $G : \mathbf{R}^n \times \mathbf{R}^q \to \mathbf{R}^n$ with $q > 1$ there are at the present time few general numerical techniques available for reliably tracing a q-dimensional manifold M_G defined by

$$G(z, \lambda) = 0$$

and these are still in a research state. One technique involves the use of continuation methods applied to restrictions so that varieties of 1-manifolds are traced out. For a discussion of this approach, see the recent monograph of Rheinboldt [**40**]. Another approach involves the approximation of M_G by a piecewise-linear manifold which can then be iteratively refined in any local region. Discussions of this approach may be found in [**4**] and [**5**]. In both of these approaches it is possible to begin to isolate singular points and singular manifolds. However, it seems that these problems still require further exploration.

REFERENCES

1. E.L. Allgower and K. Böhmer, *A mesh independence principle for operator equations and their discretizations,* Arbeitspapiere an GMD, No. 129, 1985.

2. ———,———, F. Potra, and W. Rheinboldt, *A mesh independence principle for operator equations and their discretizations,* SIAM J. Numer. Anal. **23** (1986), 160-169.

3. ——— and C.-S. Chien, *Continuation and local perturbation for multiple bifurcations,* SIAM J. Sci. Stat. Computing **7**(1986), 1265-1281.

4. ——— and S. Gnutzmann, *An algorithm for piecewise linear approximation of implicitly defined 2-dimensional surfaces,* SIAM J. Numer. Anal. **24** (1987), 452-469.

5. ——— and P.H. Schmidt, *An algorithm for piecewise linear approximation of an implicitly defined manifold,* SIAM J. Numer. Anal. **22** (1985), 322-346.

6. M.S. Berger, *On one parameter families of real solutions of nonlinear operator equations,* Bull. Amer. Math. Soc. **75** (1969), 456-459.

7. W.-J. Beyn, *Discrete Green's functions and strong stability properties of the finite difference method,* Applicable Analysis **14** (1982), 73-98.

8. ———, *Defining equations for singular solutions and numerical applications,* in Proc of the Conf. Numerical Methods for Bifurcation Problems, T. Küpper, H. Mittlemann and H. Weber, ISNM **70**, Birkhäuser Verlag, Basel, 1984, 42-56.

8a. ———, *Zur numerischen Berechnugmehrfacher V Verzweigunspunkte,* ZAMM **64** (1984).

9. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra,* MacMillan, New York, 1954.

10. K. Böhmer *Fehlerasymptotik von Diskretisierungverfahren und ihre numerische Anwendung,* Universität Karlsryhe, Inst. f. Prakt. Math., Interner Berischt **77**/2, 1977.

11. ———, *Discrete Newton methods and iterated defect corrections,* Numerische Mathematik **37** (1981), 167-192.

12. ———, P. Hemker and H. Stetter, *The defect correction approach,* Computing Supplementum **5** (1984), 1-32.

13. ——— and Th. Römer, *A mesh strategy for the Kreiss-Kreiss method for stiff boundary value problems,* preprint.

14. ——— and H. Stetter, *Defect correction methods, theory and applications,* Computing Supplementum **5** (1984).

15. D.W. Decker, H.B. Keller and C.T. Kelley, *Convergence rates for Newton's method at singular points,* SIAM J. Numer. Anal. **20** (1983), 296-314.

16. ———, and C.T. Kelley, *Newton's method at singular points* I, SIAM J. Numeri. Anal. **17** (1980), 66-70.

17. ——— and ———, *Newton's method at singular points* II, SIAM J. Numeri. Anal. **17** (1980), 465-471.

18. R. Frank, J. Hertling and Chr. W. Ueberhuber, *An extension of the applicability of iterated deferred corrections,* Math. Comp. **31** (1977), 907-915.

19. ——— and Chr. W. Ueberhuber, *Iterated defect correction for the efficient solution of stiff systems of ordinary differential equations,* BIT **17** (1977), 46-159.

20. A. Griewank, *On solving nonlinear equations with simple singularities or nearly singular solutions,* SIAM Review **27** (1985), 537-562.

21. ——— and M.R. Osborne *Newton's method for singular problems when the dimension of the null space is* > 1, SIAM J. Numer. Anal. **18** (1981), 145-149.

22. ——— and ———, *Analysis of Newton's method at irregular singularities,* SIAM J. Numer. Anal. **20** (1983), 747-773.

23. R.D. Grigorieff, *On the convergence of stability constants,* manuscript, 1983.

24. ———, *Differenzenapproximationen von Integro-Differentialgleichungen einer Veranderlichen,* Bericht Fachbereich Mathematik, Technische Universität Berlin, 1984.

25. W. Hackbusch, *On the regularity of difference schemes,* Arkiv for Matematik **19** (1928), 71-95.

26. ———, *Regularity of difference schemes:* Part II. *Regularity estimates for linear and nonlinear problems,* Report 80-13, Mathematisches Institut, Universität zu Köln, 1980.

27. ———, *On multi-grid iterations with defect correction,* aus: *Multi-Grid Methods* - Proc. of the Conference held at Köln-Porz in 1981, Springer Lecture Notes In Math. **960** (1982), 461-473.

28. ——— and U. Trottenberg eds., *Multigrid Methods,* Proc. of the Conference held at Köln-Porz, 1981, Springer Lecture Notes in mathematics **960** (1982), Springer, Berlin.

29. A.D. Jepson and A. Spence, *The numerical solution of nonlinear equations having several parameters* I: *Scalar Equations,* SIAM J. Numer. Anal. **22** (1985), 736-759.

30. H.B. Keller, *Numerical solution of bifurcation and nonlinear eigenvalues problems,* in Applications of Bifurcation Theory, P.H. Rabinowitz, ed., Academic Press, New York, 1977, 359-384.

31. J.R. Kuttler and V.G. Sigillito, *Eigenvalues of the Laplacianin two dimensions,* SIAM Review **26** (1984), 163-194.

32. B. Lindberg, *Error estimation and iterative improvement for the numerical solution of operator equations,* preprint UIUCDS-R-76-8200 (1976).

33. ———, *Error estimation and iterative improvement for discretization algorithms,* BIT **20** (1980), 486-500.

34. J.W. Milnor, *Topology from the Differential Viewpoint,* University Press of Virginia, Charlottesville, VA, 1969.

35. G. Moore and A. Spence, *The calculation of turning points of nonlinear equations,* SIAM J. Numer. Anal. **17** (1980), 569-576.

36. V. Pereyra, *On improving an approximate solution of a functional equation by deferred corrections,* Numer. Math. **8** (1966), 373-391.

37. ———, *Iterated deferred corrections for nonlinear boundary value problems,* Numer. math. **11** (1968), 111-125.

38. ———, *High-order finite difference solution of differential equations,* preprint, STAN-CS-73-348, April 1973.

39. G.W. Reddien, *On Newton's method for singular problems,* SIAM J. Numer. Anal. **15** (1978), 993-996.

40. W.S. Rheinboldt, *Numerical Analysis of Prametrized Nonlinear Equations,* Wiley-Interscience, New York, 1986.

41. R.B. Schnabel and P.D. Frank, *Tensor methods for nonlinear equations,* SIAM J. Numer. Anal. **21** (1984), 815-843.

42. R. Seydel, *Numerical computation of branch points in nonlinear equations,* Numer. Math. **33** (1979), 339-352.

43. ———, *A continuation algcrithm withstep control,* in *Numerical Methods in Bifurcation Problems,* eds. T. Küpper, H. Mittlemann, and H. Weber, Birkhäuser Verlag, Basel, 1984, 480-494.

44. A. Spence and B. Werner, *Nonsimple turning points and cusps,* IMAJ of Numer. Anal. **2** (1982), 413-427.

45. I. Stakgold, *Branching of solutions of nonlinear equations,* SIAM Review **13** (1971), 289-332.

46. H. Stetter, *Analysis of Discretization Methods for Ordinary Differential Equations,* Springer, Berlin-Heidelberg-New York, 1973.

47. ———, *The defect correction principle and discretization methods,* Numer. Math. **29** (1978), 425-443.

48. J. Stoer, *Einführung in die Numerische Mathematik,* Springer, Berlin-Heidelber-New York, 1976.

49. Fr. Stummel, *Diskrete Konvergenz Linearer Operatoren.* I, Math Ann. **190**

(1970), 45-92.

50. ———, *Diskrete Konvergenz Linearer Operatoren.* II, Math. Zeit. **120** (1971), 231-264.

51. A. Taylor and D. Lay, *Introduction to Functional Analysis,* J. Wiley and Sons, New York, 1980.

52. B. Werner and A. Spence, *The computation of symmetry-breaking bifurcation points,* SIAM J. Numer. Anal. **21** (1984), 388-399.

53. ——— and Voss, *Berechnung von Nullstellen mit fast singularer Jacobi-Matrix,* private communication.

54. K. Yosida, *Functional Analysis,* Springer, Berlin-Heidelberg-New York, 1971.

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