ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 3, Summer 1988

# THE AFFINE CLASSIFICATION OF CUBIC CURVES

DAVID A. WEINBERG

ABSTRACT. The orbits of the action of the affine group of $\mathbf{R}^2$ on the real plane cubic curves given by $Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$ are computed.

**Introduction.** Since the pioneering work of Isaac Newton, there have been many classifications of cubic curves, based on a variety of criteria. The turning point in the approach to the classification problem came with the Klein Erlangen Program, which places geometry on a group theoretic foundation. Until that time the criteria used can best be described as non-group theoretic and were primarily due to Newton, Plucker, and Cayley. (See [**1, 3, 7, 9**].) For classifications that fall within the scope of the Klein Erlangen Program see [**2, 4, 6, 11, 12**]. The classification in [**2**] is cased on the Euclidean motion group of the plane, while that in [**4, 6, 11,**] or [**12**] is based on the complex projective group. Recent classifications by equisingularity due to C.T.C. Wall combine certain aspects of both major approaches. (See [**13** and **14**].)

In this paper, a Kleinian approach based on the affine group of $\mathbf{R}^2$ is presented. The purpose of this paper is to compute the orbits of the action of the affine group of $\mathbf{R}^2$ on the real plane cubic curves and to exhibit a complete set of equivalence class representatives. Formulated in an algebraic way, the central question of this paper is: (Q) How much can the equation $Ax^3+Bx^2y+Cxy^2+Dy^3+Ex^2+Fxy+Gy^2+Hx+Iy+J=0$ be simplified by performing the following two types of operations:

AMS (MOS) *Subject Classifications* (1980): Primary 14H05, 51N10, 51N25, 51N35, 14N99; Secondary 10C10, 15A72, 14B05, 14E15, 32C40, 58C27.
Received by the editors on July 19, 1985, and in revised form on August 4, 1986.

1. Making substitutions of the form

$$x = \lambda_1 x' + \lambda_2 y' + \lambda_3$$
$$y = \lambda_4 x' + \lambda_5 y' + \lambda_6,$$

where $\lambda_1\lambda_5 - \lambda_2\lambda_4 \neq 0$,

2. Multiplying the equation by $\mu \neq 0$.

In addition to displaying a complete list of orbits, a major goal of the Klein Erlangen Program is the computation of invariants. In the present context the relevant question is: How can the equivalence class of a given cubic curve be determined from the coefficients? To a large extent, this question is left unanswered in this paper. However, the invariants associated with the terms of degree three are given.

The main result of this paper is contained in the following

THEOREM. *A complete set of equivalence class representatives for the action of the affine group of* $\mathbf{R}^2$ *on the set of real plane cubic curves is given in the following table.*

The invariants are as follows:

$$\begin{aligned} \Delta &= 27A^2D^2 - 18ABCD + 4AC^3 \\ P_1 &= 3AC - B^2 \\ P_2 &= 3DB - C^2 \\ Q_1 &= 27A^2D - 9ABC + 2B^3 \\ Q_2 &= 27D^2A - 9DCB + 2C^3 \end{aligned}$$

Different values of the parameters in the table correspond to distinct equivalence classes.

The author wishes to thank Bill Gustafson for several helpful conversations and the referee for pointing out reference [**2**] and for making suggestions to help improve the presentation in this paper.

Calculations. Given the equation

$$(1)\quad Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0,$$

| Equivalence class representatives | Invariants |
|---|---|
| $x^3 + xy^2 + x^2 + Hx + Iy + J = 0$ | |
| $-\infty < H < \infty,\ 0 \leq I < \infty,\ -\infty < J < \infty$ | |
| $x^3 + xy^2 + y + Hx + J = 0$ | |
| $-\infty < H < \infty,\ -\infty < J < \infty$ | $\Delta > 0$ |
| $x^3 + xy^2 + x + J = 0,\ 0 \leq J < \infty$ | |
| $x^3 + xy^2 + 1 = 0$ | |
| $x^3 + xy^2 = 0$ | |
| $x^3 - xy^2 - y^2 + Hx + Iy + J = 0,$ | |
| $-\infty < H < \infty,\ -\infty < I \leq 0,\ -\infty < J < \infty$ | $\Delta < 0$ |
| $x^3 - xy^2 - y + Hx + J = 0,$ | |
| $-1 < H < \infty,\ 0 \leq J < \infty$ | |
| $x^3 - xy^2 + 1 = 0$ | |
| $x^3 - xy^2 = 0$ | |
| $x^2y + y^2 - x + y + J = 0,\ -\infty < J < \infty$ | |
| $x^2y + y^2 - x + y + J = 0,\ -\infty < J < \infty$ | $\Delta = 0$ |
| $x^2y + y^2 + y + J = 0,\ -\infty < J < \infty$ | |
| $x^2y + y^2 - 1 = 0$ | $P_1^2 + Q_1^2 + P_2^2 + Q_2^2 \neq 0$ |
| $x^2y + y^2 = 0$ | |
| $x^2y - x + y + J = 0,\ 0 \leq J < \infty$ | |
| $x^2y - x = 0$ | |
| $x^2y - x + 1 = 0$ | |
| $x^2y + y = 0$ | |
| $x^2y + y + 1 = 0$ | |
| $x^2y = 0$ | |
| $x^2y - 1 = 0$ | |
| $x^3 - y^2 + x + J = 0,\ -\infty < J < \infty$ | |
| $x^3 - y^2 - x + J = 0,\ -\infty < J < \infty$ | |
| $x^3 - y^2 + 1 = 0$ | $\Delta = 0$ |
| $x^3 - y^2 = 0$ | $P_1 = 0,\ Q_1 = 0,$ |
| $x^3 - y^2 - 1 = 0$ | $P_2 = 0,\ Q_2 = 0$ |
| $x^3 - y = 0$ | |
| $x^3 + x + J = 0,\ 0 \leq J < \infty$ | |
| $x^3 - xy = 0$ | |
| $x^3 - xy + 1 = 0$ | |
| $x^3 + 1 = 0$ | |
| $x^3 = 0$ | |

Make the substitution

$$x = \lambda_1 x' + \lambda_2 y' + \lambda_3$$
$$\tag{2} y = \lambda_4 x' + \lambda_5 y' + \lambda_6,$$

where $\lambda_1\lambda_5 - \lambda_2\lambda_4 \neq 0$. Combining terms and dropping the primes for notational convenience, the transformed equation is

$$\begin{aligned}
&x^3(A\lambda_1^3 + B\lambda_1^2\lambda_4 + C\lambda_4^2\lambda_1 + D\lambda_4^3)\\
&+ x^2y(3A\lambda_1^2\lambda_2 + B\lambda_1^2\lambda_5 + 2B\lambda_1\lambda_2\lambda_4\\
&\quad + C\lambda_4^2\lambda_2C\lambda_1\lambda_4\lambda_5 + 3D\lambda_4^2\lambda_5)\\
&+ xy^2(3A\lambda_1\lambda_2^2 + B\lambda_2^2\lambda_4 + 2B\lambda_1\lambda_2\lambda_5\\
&\quad + C\lambda_1\lambda_5^2 + 2C\lambda_2\lambda_4\lambda_5 + 3D\lambda_4\lambda_5^2)\\
&+ y^3(A\lambda_2^3 + B\lambda_2^2\lambda_5 + C\lambda_5^2\lambda_2 + D\lambda_5^3)\\
&+ x^2(3A\lambda_1^2\lambda_3 + B\lambda_1^2\lambda_6 + 2B\lambda_1\lambda_3\lambda_4\\
&\quad + C\lambda_4^2\lambda_3 + 2C\lambda_1\lambda_4\lambda_6 + 3D\lambda_4^2\lambda_6\\
&\quad + E\lambda_1^2 + F\lambda_1\lambda_4 + G\lambda_4^2\\
&+ xy(6A\lambda_1\lambda_2\lambda_3 + 2B\lambda_1\lambda_2\lambda_6 + 2B\lambda_2\lambda_3\lambda_4\\
&\quad + 2B\lambda_1\lambda_3\lambda_5 + 2C\lambda_3\lambda_4\lambda_5 + 2C\lambda_1\lambda_5\lambda_6\\
&\quad + 2C\lambda_2\lambda_4\lambda_6 + 6D\lambda_4\lambda_5\lambda_6 + 2E\lambda_1\lambda_2\\
&\quad + F\lambda_2\lambda_4 + F\lambda_1\lambda_5 + 2G\lambda_4\lambda_5)\\
&+ y^2(3A\lambda_2^2\lambda_3 + B\lambda_2^2\lambda_6 + 2B\lambda_2\lambda_3\lambda_5\\
&\quad + C\lambda_5^2\lambda_3 + 2C\lambda_2\lambda_5\lambda_6 + 3D\lambda_5^2\lambda_6\\
&\quad + E\lambda_2^2 + F\lambda_2\lambda_5 + G\lambda_5^2)\\
&+ x(3A\lambda_1\lambda_3^2 + 2B\lambda_1\lambda_3\lambda_6 + B\lambda_3^2\lambda_4\\
&\quad + 2C\lambda_3\lambda_4\lambda_6 + C\lambda_6^2\lambda_1 + 3D\lambda_4\lambda_6^2\\
&\quad + 2E\lambda_1\lambda_3 + F\lambda_1\lambda_6 + F\lambda_3\lambda_4\\
&\quad + 2G\lambda_4\lambda_6 + H\lambda_1 + I\lambda_4)\\
&+ y(3A\lambda_2\lambda_3^2 + 2B\lambda_2\lambda_3\lambda_6 + B\lambda_3^2\lambda_5\\
&\quad + 2C\lambda_3\lambda_5\lambda_6 + C\lambda_6^2\lambda_2 + 3D\lambda_5\lambda_6^2\\
&\quad + 2E\lambda_2\lambda_3 + F\lambda_2\lambda_6 + F\lambda_3\lambda_5\\
&\quad + 2G\lambda_5\lambda_6 + H\lambda_2 + I\lambda_5)\\
&+ (A\lambda_3^3 + B\lambda_3^2\lambda_6 + C\lambda_6^2\lambda_3 + D\lambda_6^3\\
&\quad + E\lambda_3^2 + F\lambda_3\lambda_6 + G\lambda_6^2 + H\lambda_3 + I\lambda_6 + J)\\
&\quad = 0.
\end{aligned} \tag{3}$$

The crucial observation here is that $E, F, G, H, I, J, \lambda_3, \lambda_6$ have no effect whatsoever on the terms of degree three. In fact, the terms of degree three transform as if there were no other terms around, exactly as they would transform under a homogeneous linear change of variables. Therefore, by referring to canonical forms for real homogeneous cubic forms in two variables under the action of $GL(2, R)$, we may assume that the terms of degree three are in one of the following four forms: $x^3$, $x^2y$, $x^3 - xy^2$, $x^3 + xy^2$. (See [**5, 9**] or [**15**].) This is what makes the calculation work. Furthermore, the analogous observation is true for equations of any degree, i.e., the first step is to determine canonical forms for the action of $GL(n, R)$ on the homogeneous forms of degree $n$.

The next observation is that since we are working with equations, we may multiply each side by a nonzero parameter $\mu$. This will help simplify the equation.

Finally, the plan of the calculation is that, given one of the four possibilities for terms of degree three, work from left to right in the equation, and try to make as many coefficients zero as is possible. The next best thing to a zero is a $+1$ or $-1$, but we shall see that sometimes a coefficient will take on a continuum of values, each value yielding a different orbit.

The calculations in each of the four cases are similar in spirit. Therefore it is sufficient to present here only one of the four cases. Consider the case in which the terms of degree three can be reduced to $x^3$.

Consider an affine transformation (2) which fixes the terms of degree three, in this case $x^3$. Denote the coefficients of the transformed equation by primes. Thus,

$$A' = 1, \quad B' = 0, \quad C' = 0, \quad D' = 0.$$

By referring to (3) we see that

$$\begin{aligned} 1 &= \mu\lambda_1^3, \quad 0 = 3\lambda_1^2\lambda_2, \\ 0 &= 3\lambda_1\lambda_2^2, \\ \therefore \lambda_2 &= 0. \end{aligned}$$

Thus, we must consider an affine transformation (2) with $\lambda_2 = 0$ and try to simplify

$$x^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0.$$

Work from left to right. Let us now show that we can make $E' = 0, F' = 0, G' = -1$. By referring to (3) we see that

$$
\begin{aligned}
E' &= 0 = 3\lambda_1^2\lambda_3 + E\lambda_1^2 + F\lambda_1\lambda_4 + \lambda_4^2 G \\
F' &= 0 = F\lambda_1\lambda_5 + 2G\lambda_4\lambda_5, \\
G' &= -1 = \mu\lambda_5^2 G.
\end{aligned}
$$

Suppose $G \neq 0$. At this point we cannot make $G' = 0$; since $\mu \neq 0$, we would have to take $\lambda_5 = 0$. But we already know that $\lambda_2 = 0$, so if $\lambda_5 = 0$, then $\lambda_1\lambda_5 - \lambda_2\lambda_4 = 0$. This violates the definition of an affine transformation (2). We must return to the case $G = 0$ separately:

$$F' = 0 = \lambda_5(F\lambda_1 + 2G\lambda_4).$$

Since $\lambda_5 \neq 0$, simply choose $\lambda_4 = -F\lambda_1/2G$, which is okay unless $G = 0$.

$$E' = 0 = \lambda_1^2\left(3\lambda_3 + E - \frac{F^2}{2G} + \frac{F^2}{4G}\right).$$

Since $\lambda_1 \neq 0$, simply choose

$$\lambda_3 = \frac{1}{3}(-E + \frac{F^2}{4G}).$$

We have now simplified the equation to

$$x^3 - y^2 + Hx + Iy + J = 0.$$

Now consider an affine transformation (2) which preserves the $x^3 - y^2$ part.

$$
\begin{aligned}
1 &= \mu\lambda_1^3 \\
\lambda_2 &= 0. \\
E' &= 0 = 3\lambda_1^2\lambda_3 + \lambda_4^2 \\
F' &= 0 = -2\lambda_4\lambda_5 \\
G' &= -1 = -\mu\lambda_5^2.
\end{aligned}
$$

Since $0 = 2\lambda_4\lambda_5$ and $\lambda_5 \neq 0$, we have $\lambda_4 = 0$; since $0 = 3\lambda_1^2\lambda_3$ and $\lambda_1 \neq 0$, we have $\lambda_3 = 0$. Furthermore, $\lambda_5^2 = \frac{1}{\mu}$. We will now show that we can make $I' = 0$, but $H' = 0$, or 1, or $-1$. By referring to (3),

$$I' = 0 = \lambda_5 I - 2\lambda_5\lambda_6 = \lambda_5(I - 2\lambda_6).$$

Since $\lambda_5 \neq 0$, choose $\lambda_6 = I/2$. Now $H' = 1 = \mu\lambda_1 H$. Since $\mu \neq 0, \lambda_1 \neq 0$, we must return to the case $H = 0$ separately. So $\lambda_1 = 1/\mu H$, but since $\mu\lambda_1^3 = 1$, we have $1/\mu = 1/(\mu^3 H^3)$. Hence $\mu^2 = 1/H^3$. Since $\lambda_5^2 = 1/\mu = \lambda_1^3$, $\mu > 0, \lambda_1 > 0$ (unless we are dealing with complex coefficients). Therefore we must have $H > 0$. If $H < 0$, then we can make $H' = -1$. So now we have $\lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_6 = 0, \mu = 1, \lambda_1 = 1, \lambda_5 = \pm 1$. Finally we come to $J'$. By referring to (3), we see that $J' = \mu J$. Thus, $-\infty < J' < \infty$. We now return to the case $H = 0$. Then $\mu, \lambda_1, \lambda_5$ are not yet determined. So we can choose $\mu = J'/J$, and since $\mu > 0$ (unless we have complex coefficients) we can choose $J' = \pm 1$ or 0. At this point we can summarize by saying that if $G \neq 0$, then we have the canonical forms

$$\begin{aligned}
&x^3 - y^2 + x + J = 0, \quad -\infty < J < \infty \\
&x^3 - y^2 - x + J = 0, \quad -\infty < J < \infty \\
&x^3 - y^2 + 1 = 0 \\
&x^3 - y^2 = 0 \\
&x^3 - y^2 - 1 = 0.
\end{aligned}$$

We now return to the case $G = 0$. Then

$$\begin{aligned}
E' &= \mu(3\lambda_1^2\lambda_3 + E\lambda_1^2 + F\lambda_1\lambda_4) \\
F' &= \mu F\lambda_1\lambda_5.
\end{aligned}$$

Thus, $F' \neq 0$ unless $F = 0$. We will return to the case $F = 0$ later. For the present, we can make $F' = -1$. For

$$E' = \lambda_1(3\lambda_1\lambda_3 + E\lambda_1 - \lambda_4),$$

by choosing $\lambda_4 = 3\lambda_1\lambda_3 + E\lambda_1$, we can make $E' = 0$. We have now simplified the equation to

$$x^3 - xy + Hx + Iy + J = 0$$

and have $\mu\lambda_1^3 = 1$, $\lambda_2 = 0$, $\mu\lambda_1\lambda_5 = 1$, $\lambda_4 = 3\lambda_1\lambda_3$. Now observe that, by (3),

$$\begin{aligned}
I' &= \mu(-\lambda_3\lambda_5 + I\lambda_5) \\
&= \mu\lambda_5(I - \lambda_3).
\end{aligned}$$

By taking $\lambda_3 = I$ we can make $I' = 0$. Then we also have $\lambda_3 = 0, \lambda_4 = 0$. Now

$$\begin{aligned}
H' &= \mu(-\lambda_1\lambda_6 + H\lambda_1) \\
&= \mu\lambda_1(-\lambda_6 + H).
\end{aligned}$$

By taking $\lambda_6 = H$ we can make $H' = 0$. Then we also have $\lambda_6 = 0$. Now $J' = \mu J$. So $J' = 0$ or 1. We have now reduced to the canonical forms

$$x^3 - xy = 0,$$

$$x^3 - xy + 1 = 0.$$

We now return to the case $F = 0$. Then

$$E' = \mu\lambda_1^2(3\lambda_3 + E).$$

By taking $\lambda_3 = -E/3$, we can make $E' = 0$. Now we have

$$x^3 + Hx + Iy + J = 0$$

with $\lambda_2 = 0, \lambda_3 = 0, \mu\lambda_1^3 = 1$. Now

$$\begin{aligned} I' &= \mu I\lambda_5 \\ -1 &= \mu I\lambda_5 \text{ unless } I = 0. \end{aligned}$$

We will return to the case $I = 0$. Having made $I = -1$, we now have $\mu\lambda_5 = 1$. Now $H' = \mu(\lambda_1 H - \lambda_4)$. By taking $\lambda_4 = \lambda_1 H$, we have $H' = 0$. If $H = 0$, then $\lambda_4 = 0$. For $J' = (-\lambda_6 + J)\mu$, by taking $\lambda_6 = J$, we have $J' = 0$. Thus, the only canonical form here is $x^3 - y = 0$. We return to the case $I = 0$. Then

$$\begin{aligned} H' &= \mu H\lambda_1 \\ 1 &= \mu H\lambda_1 \text{ unless } H = 0. \end{aligned}$$

We will return to the case $H = 0$. If $H = 1$, then $1 = \mu\lambda_1$, but since $1 = \mu\lambda_1^3$, $\lambda_1^2 = 1$. Hence $|\mu| = 1$. Since $J' = \mu J$, the only simplification that can be done is $0 \le J' < \infty$. Thus, we have the canonical forms

$$x^3 + x + J = 0, \quad 0 \le J < \infty.$$

Finally, we return to the case $H = 0$. $J' = \mu J$, so we can obtain $J' = 0$ or 1. Thus, the canonical forms are

$$x^3 = 0$$

$$x^3 + 1 = 0.$$

To summarize this calculation here is the list of real canonical forms:

$$
\begin{aligned}
x^3 - y^2 + x + J &= 0, \ -\infty < J < \infty \\
x^3 - y^2 - x + J &= 0 \ -\infty < J < \infty \\
x^3 - y^2 + 1 &= 0 \\
x^3 - y^2 &= 0 \\
x^3 - y^2 - 1 &= 0 \\
x^3 - y &= 0 \\
x^3 + x + J &= 0, \ 0 \leq J < \infty \\
x^3 - xy &= 0 \\
x^3 - xy + 1 &= 0 \\
x^3 + 1 &= 0 \\
x^3 &= 0.
\end{aligned}
\tag{4}
$$

## REFERENCES


**1.** W.W. Rouse Ball, *Newton's classification of cubic curves,* Proc. of the London Math. Soc. **50** (1891), 104-143.

**2.** R.S. Burington, and H.K. Holt, *Canonical forms of plane cubic curves under Euclidean transformations,* Annals of Math. (2) (1928), 52-60.

**3.** A Cayley, *Collected mathematical papers,* Vols. 1-12, Cambridge Univ. Press, Cambridge, 1889-1897.

**4.** W. Fulton, *Algebraic curves,* Benjamin, Reading, Mass, 1969.

**5.** G. Gurevich, *Foundations of the theory of algebraic invariants,* Noordhoff, Groningen, 1964.

**6.** R. Hartshorne, *Algebraic geometry,* Springer-Verlag, New York, 1977.

**7.** I. Newton, *The collected mathematical papers of I. Newton,* Vols. II, IV, V. (Edited by D.T. Whiteside, Cambridge Univ. Press, 1968), 1665-1695.

**8.** ———, *Enumeratio linearum tertii ordinis,* appendix to *Treatise on Optics,* Engl. Transl. by C.R.M. Talbot, London, 1861, (1704).

**9.** T. Poston and I.N. Stewart, *Taylor expansions and catastophes,* Pitman, London, 1976.

**10.** G. Salmon, *A treatise on the higher plane curves,* 2nd ed., Hodges, Foster, and Figgis, Dublin, 1879.

**11.** A Seidenberg, *Elements of the theory of algebraic curves,* Addison-Wesley, 1968.

**12.** R. Walker, *Algebraic curves,* Springer-Verlag, New York, 1978.

**13.** C.T.C. Wall, *Affine cubic functions* I, Math. Proc. of Camb. Phi. Soc. **85** (1979), 387-401.

**14.** ———, *Affine cubic functions* III, Math. Proc. of Camb. Phil. Soc. **87** (1980), 1-14.

**15.** D. Weinberg, *Canonical forms for symmetric tensors,* Linear algebra and its applications, **57**, (1984), 271-282.

**16.** ———, *The topological classification of cubic curves,* to appear in Rocky Mount. J. of Math. **18** (1988).

DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TX 79409