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A NOTE ON THE APPLICATION OF TOPOLOGICAL TRANSVERSALITY TO NONLINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACES

D. O'REGAN

ABSTRACT. In this paper we suggest a new method, via Topological Transversality, for examining nonlinear differential equations in Hilbert Spaces. Furthermore, we show how this analysis can be used to obtain existence of solutions to certain integro-differential equations.

1. Introduction. The theory of nonlinear differential equations in abstract spaces became popular in the 1970's and is still being studied in great depth. For a detailed account of the subject see Deimling [**4**], Lakshmikantham and Leela [**12**] and Martin [**14**]. In this paper we present a new approach via the Topological Transversality Theorem, to studying problems of the form

$$(1.1) \qquad \begin{cases} y' = f(t,y), \ t \in [0,T] \\ y(0) = y_0. \end{cases}$$

Here y takes values in a real Hilbert space $(H, ||\cdot||)$, $y_0 \in H$ and $f : [0,T] \times H \to H$ is continuous.

For notational purposes let $C^1([0,T],H)$ denote the space of continuously differentiable functions g on $[0,T]$. Now $C^1([0,T],H)$ with norm

$$\begin{aligned} ||g||_1 &= \max\Big\{ \sup_{t\in[0,T]} ||y(t)||, \sup_{t\in[0,T]} ||y'(t)|| \Big\} \\ &= \max\Big\{ ||y||_0, ||y'||_0 \Big\} \end{aligned}$$

is a Banach space. Similarly we define $C([0,T],H)$. Finally, by a solution to (1.1) we mean a function $y \in C^1([0,T],H)$ together with y satisfying $y' = f(t,y), t \in [0,T]$, and $y(0) = y_0$.

Unlike the finite dimensional case, continuity assumptions on f alone will not guarantee even local existence; see Banas and Geobel [**2**]. In

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this paper, by placing compactness conditions on f, we obtain, with a restriction on T which depends on the nonlinearity of f, solutions to (1.1) in $C^1([0,T],H)$. Now the basic existence theorems available in the literature guarantee that a solution exists for $t < \varepsilon$ for some $\varepsilon > 0$ suitably small; however, from these theorems it is extremely difficult, and many times impossible, to produce a specific interval of existence of a solution. The results of this paper enable us to read off immediately from the differential equation an interval of existence of a solution. Furthermore, we show that this interval is maximal for a certain class of problems. In particular, we examine the dependence of the interval of existence on f and y_0.

2. Preliminary results. We begin with some standard theorems on the calculus of functions from an interval into a real Hilbert space; see Martin [**14**], Barbu [**3**] and Shilov [**16**] for details. Suppose for the remainder of this section that H is a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and J is a compact interval in R.

THEOREM 2.1. *Suppose f is a differentiable function from J into H and $f'(t) = 0$ for all $t \in J$. Then f is constant on J*

THEOREM 2.2. *Suppose f is a differentiable function from J into H. Then*

$$\frac{d}{dt}\langle f(t), f(t)\rangle = 2\langle f'(t), f(t)\rangle.$$

THEOREM 2.3. *Suppose $J = [a,b]$ and $f(u)$ is a continuous function from J into H. Also let $u = u(t)$ be a continuously differentiable function on $\alpha \le t \le \beta$, where $u(\alpha) = a$ and $u(\beta) = b$. Then*

$$\int_a^b f(u)du = \int_\alpha^\beta f(u(t))u'(t)dt.$$

To obtain our existence theorems in the following section we need a more general version of the Arzela Ascoli Theorem.

THEOREM 2.4. *Suppose M is a subset of $C(J,H)$. Then M is relatively compact in $C(J,H)$ (i.e., $\overline{M}$ is a compact subset of $C(J,H)$)*

if and only if M is bounded, equicontinuous and the set $\{f(t) : f \in M\}$ is relatively compact for each $t \in J$.

Topological methods based on essential maps (see [**5**] and [**6**]) are used to establish the existence results of this paper. For convenience, we summarize here the topological results needed. Let X and Y be metric spaces. A map (continuous function) $F : X \to Y$ is compact if $F(X)$ is contained in a compact subset of Y. F is completely continuous if the image of each bounded set in X is contained in a compact subset of Y. Let U be an open subset of a convex set K in a normed linear space E. Let $\overline{U}$ and ∂U be the closure and boundary of U in K. A compact map $F : \overline{U} \to K$ which is fixed point free on ∂U is *essential* if every compact map $G : \overline{U} \to K$ which agrees with F on ∂U has a fixed point in U. (In particular, F has a fixed point in U.) The Schauder fixed point theorem implies: *Let $u_0 \in U$ and define $F : \overline{U} \to K$ by $F(u) = u_0$. Then the constant map F is essential.*

Two compact maps $F, G : \overline{U} \to K$ which are fixed point free on ∂U are called homotopic if there is a compact homotopy $H : \overline{U} \times [0,1] \to K$ such that $H_\lambda(u) = H(u,\lambda)$ is fixed point free on ∂U for each λ in $[0,1], H_0 = F$, and $H_1 = G$. In this context, the *Topological Transversality Theorem* asserts: *If F and G are homotopic, then F is essential if and only if G is essential.*

3. Initial value problems in Hilbert Spaces. We begin by examining the homogeneous first order initial value problem

$$\begin{cases} y' = f(t,y), t \in [0,T] \\ y(0) = 0, \end{cases} \tag{3.1}$$

where y takes values in a real Hilbert space $(H, \langle \cdot\, , \cdot \rangle)$ and $f : [0,T] \times H \to H$ is continuous. Let $|| \cdot ||^2 = \langle \cdot\, , \cdot \rangle$.

Now the Topological Transversality Theorem and the Arzela Asocoli Theorem are used to extend Theorem 2.1 of [**7**] for initial value problems in Hilbert spaces.

THEOREM 3.1. *Let $f : [0,T] \times H \to H$ be continuous and $0 \le \lambda \le 1$.*

Suppose, in addition, f satisfies the following:

$$(3.2)\quad \begin{cases} \text{There is a continuous function } \psi : [0,\infty) \to (0,\infty) \\ \text{such that } \|f(t,y)\| \le \psi(\|y\|). \end{cases}$$

$$(3.3)\quad \begin{cases} f \text{ is completely continuous on } [0,T] \times H. \end{cases}$$

$$(3.4)\quad \begin{cases} \text{For } t,s \in [0,T] \text{ and } \Omega \text{ a bounded subset} \\ \text{of } C^1([0,T],H), \text{ there exist constants } \alpha > 0, \\ A \ge 0 \text{ (which can depend on } \Omega) \text{ such that} \\ \quad \|f(t,u(t)) - f(s,u(s))\| \le A|t-s|^\alpha \\ \text{for all } u \in \Omega. \end{cases}$$

Finally, suppose there is a constant K independent of λ such that $\|y\|_1 \le K$ for each solution $y(t)$ to

$$(3.1)_\lambda \qquad \begin{cases} y' = \lambda f(t,y), & t \in [0,T] \\ y(0) = 0. \end{cases}$$

Then the initial value problem (3.1) *has at least one solution in $C^1([0,T],H)$.*

PROOF. For notational purposes let $C_B^1([0,T],H) = \{u \in C^1([0,T], H) : u(0) = 0\}$. Also let $\overline{V} = \{u \in C_B^1([0,T],H) : \|u\|_1 \le K+1\}$ and define $F_\lambda : C_B^1([0,T],H) \to C([0,T],H)$, $0 \le \lambda \le 1$, by $F_\lambda[u](t) = \lambda f(t,v(t))$. Now, assumptions (3.2), (3.3) and (3.4) together with Theorem 2.4 imply that F_λ is completely continuous. To see this let Ω be a bounded subset of $C^1([0,T],H)$; then, for $u \in \Omega$, $\|F_\lambda u\| = \|\lambda f(t,u)\| \le \psi(\|u\|) \equiv M_0$, where $M_0 < \infty$ is a constant. Clearly, from (3.4), $F_\lambda(\Omega)$ is equicontinuous and we have also, for each $t \in [0,T]$, $F(\Omega(t)) = \{f(t,u(t)); u \in \Omega\}$ which is relatively compact in H since f is completely continuous.

Finally we define $L : C_B^1([0,T],H) \to C([0,T],H)$ by $Ly = y'$. It follows from Theorem 5.10 of [**15**] that L^{-1} is a bounded linear operator. Thus $H_\lambda = L^{-1}F_\lambda$ defines a homotopy $H_\lambda : \overline{V} \to C_B^1([0,T],H)$. It is clear that the fixed points of H_λ are precisely the solutions to $(3.1)_\lambda$. Moreover, the complete continuity of F_λ together with the continuity of L^{-1} imply that the homotopy H_λ is compact. Now, H_0 is essential, so Theorem 1.5 of [8] implies that H_1 is essential. Thus (3.1) has a solution. □

REMARK. If we replace the Hilbert space H with a Banach space $\tilde{B}$, then again Theorem 3.1 holds with $\tilde{B}$ replacing H.

In view of Theorem 3.1 we immediately obtain

THEOREM 3.2. *Suppose* $f : [0,T]\times H \to H$ *is continuous and satisfies* (3.2), (3.3) *and* (3.4). *Then the initial value problem* (3.1) *has a solution in* $C^1([0,T],H)$ *for each* $T < \int_0^\infty \frac{du}{\psi(u)}$.

PROOF. To prove existence of a solution in $C^1([0,T],H)$ we apply Theorem 3.1. To establish a priori bounds for $(3.1)_\lambda$, let $y(t)$ be a solution to $(3.1)_\lambda$. Then

$$||y'|| = ||\lambda f(t,y)|| \le \psi(||y||).$$

Now if $||y(t)|| \neq 0$, we have from Theorem 2.2 and the Cauchy Schwartz inequality that

$$||y||' = \frac{\langle y', y\rangle}{||y||} \le ||y'||,$$

and the inequality above yields

$$||y||' \le \psi(||y||)$$

at any point t where $||y(t)|| \neq 0$. Suppose $||y(t)|| \neq 0$ for some point $t \in [0,T]$. Then, since $y(0) = 0$, there is an interval $[a,t]$ in $[0,T]$ such that $||y(s)|| > 0$ on $a < s \le t$ and $||y(a)|| = 0$. Then the previous inequality implies

$$\int_a^t \frac{||y(s)||'}{\psi(||y(s)||)} ds \le t - a,$$

so

$$\int_0^{||y(t)||} \frac{du}{\psi(u)} \le T < \int_0^\infty \frac{du}{\psi(u)}.$$

This inequality implies there is a constant M_1 such that $||y||_0 \leq M_1$. Also, $(3.1)_\lambda$ and (3.2) imply $||y'(t)|| \leq \max_{0\leq u\leq M_1} \psi(u) \equiv M_2$ for some constant M_2. So $||y||_1 \leq K = \max\{M_1, M_2\}$ and the existence of a solution is established. □

Theorem 3.1 also holds for the inhomogeneous initial condition $y(0) = y_0 \in H$. In fact Theorem 5.1 of [**9**] and trivial adjustments in the above proof yield

THEOREM 3.3. *Suppose $f : [0,T]\times H \to H$ is continuous and satisfies* (3.2), (3.3) *and* (3.4). *Then the initial value problem*

$$\begin{cases} y' = f(t,y), \ t \in [0,T] \\ y(0) = y_0 \in H \end{cases}$$

has a solution in $C^1([0,T],H)$ for each

$$T < \int_{||y_0||}^{\infty} \frac{du}{\psi(u)}.$$

EXAMPLE 1. Consider the evolution equation

$$(*) \qquad \begin{cases} \frac{dy}{dt} + A(t)y = g(t), \quad t \in [0,T] \\ y(0) = 0, \end{cases}$$

where $A(t) \in L(H)$ (the space of bounded linear operators from H into H) for each $t \in [0,T]$ and the map $(t,y) \to A(t)y$ is continuous from $[0,T] \times H$ into H. Also assume $g : [0,T] \to H$ is continuous.

It follows from the Banach-Steinhaus Theorem that $A(t)$ is uniformly bounded (see [**11**, p. 10]), i.e., there exists a constant $||A|| \geq 0$ independent of t such that $||A(t)|| \leq ||A||$ for each $t \in [0,T]$. Here $||A(t)|| = \sup_{||x||\leq 1} ||A(t)x||$. Suppose in addition that A and g satisfy the following:

(i) $f(t,u) = g(t) - A(t)u$ is completely continuous on $[0,T] \times H$.

(ii) A and g are Lipschitz continuous on $[0,T]$, i.e., there exists constants $M, N < \infty$ such that, for $t,s \in [0,T]$, $||A(t)-A(s)|| \leq M|t-s|$ and $||g(t) - g(s)|| \leq N|t - s|$.

To apply Theorem 3.2 we need only show (3.4) is satisfied; to see this let Ω be a bounded subset of $C^1([0,T],H)$, i.e., there exists a constant $K \geq 0$ such that $||u||_1 \leq K$ for all $u \in \Omega$. Now, for $t, s, \in [0,T]$,

$$\begin{aligned}||f(t,u(t)) - f(s,u(s))|| &= ||g(t) - g(s) + A(s)u(s) - A(t)u(t)||\\ &\leq ||A(t)u(t) - A(t)u(s)|| + ||A(t)u(s) - A(s)u(s)|| + ||g(t) - g(s)||\\ &\leq ||A|| \left|\left| \int_s^t u'(z)dz \right|\right| + KM|t-s| + N|t-s|\\ &\leq ||A||K|t-s| + KM|t-s| + N|t-s|.\end{aligned}$$

Now since

$$\int_0^\infty \frac{dx}{A_0 x + B_0} = +\infty$$

for all constants $A_0, B_0 > 0$, then $(*)$ has a solution in $[0,T]$ for all $T > 0$.

REMARK. It is possible to replace (ii) with A and g being uniformly Hölder continuous on $[0,T]$.

EXAMPLE 2. The techniques above may be applied to integro-differential equations of the form

$$\text{(3.5)} \qquad \begin{cases} \frac{\partial}{\partial t} y(t,s) = \int_0^T g(t,s,r,y(t,r))dr, & t,s \in [0,T] \\ y(0,s) = \mu(s), \end{cases}$$

where $\mu : [0,T] \to R$ is continuous. Equations of this type arise quite naturally in transport and transfer models; see Anselone [**1**, p. 51] for details.

Let $H = L^2([0,T],R)$, with the usual inner product and define the mapping B from $[0,T] \times H$ into H by

$$[B(t,u)](s) = \int_0^T g(t,s,r,u(r))dr$$

for all $(t,s,u) \in [0,T] \times [0,T] \times E$ where $E \subset H$.

We begin by examining the initial value problem

$$\begin{cases} u' = B(t, u(t)), t \in [0, T] \\ u(0) = \mu \end{cases} \tag{3.6}$$

where $B : [0, T] \times H \to H$.

Various conditions on g insuring the continuity and complete continuity of B from $[0, T] \times H$ into H may be found in Krasnoselskii [**10**]. We also assume g satisfies certain growth conditions so that

$$||B(t, u)||_{L_2} \leq \psi(||u||_{L_2}),$$

where $\psi : [0, \infty) \to (0, \infty)$ is continuous. Now assume

$$T < \int_{||\mu||_{L_2}}^{\infty} \frac{du}{\psi(u)}.$$

Finally suppose conditions are put on g so that, for $t, t' \in [0, T]$ and Ω a bounded subset of $C^1([0, T], H)$, there exists a constant $A \geq 0$ such that

$$||B(t, u(t)) - B(t', u(t'))||_{L_2} \leq A|t - t'|$$

for all $u \in \Omega$. (For a discussion on how to put conditions of this form on g see Martin [**14**; v. 4, p. 172]. It should be remarked that the operator B occurs widely in the theory of nonlinear integral equations; special cases include the Uryson, Hammerstein and Voltera integral operators. For a detailed discussion on the subject see [**14**; Chapter V] and [**10**].)

Then Theorem 3.3 implies that (3.6) has a solution on $[0, T]$. Suppose $u(t)$ is a solution to (3.6) on $[0, T]$, then one sees that if $y(t, s) = [u(t)](s)$ for all $t \in [0, T]$ and $s \in (0, T]$, then $y(0, \alpha) = \mu(\alpha)$ and y is a solution to (3.5). To see this let $[u(t)](s) \equiv v(s)$, so

$$\begin{aligned} \frac{\partial}{\partial t} y(t, s) = B(t, v)(s) &= \int_0^T g(t, s, r, v(r))dr \\ &= \int_0^T g(t, s, r, y(t, r))dr. \end{aligned}$$

Now Theorem 3.3 yields the best possible result (maximal interval of existence) in the sense of the following theorem.

Suppose, in addition, any solution, y, to

$$\begin{cases} y' = f(t,y), & t \in [0,\overline{T}] \\ y(0) = 0 \end{cases} \tag{3.7}$$

satisfies

$$\left\| \int_0^t y'(z)dz \right\| = \int_0^t \|y'(z)\| dz. \tag{3.8}$$

Then the result in Theorem 3.2 *is the best possible in the sense that the initial value problem* (3.7) *can have a solution only if*

$$\overline{T} < \int_0^\infty \frac{du}{\psi(u)}.$$

PROOF. The existence of a solution to (3.7) is guaranteed by Theorem 3.2. Now, the differential equation and (3.2)* yields

$$\|y'\| = \|f(t,y)\| = \psi(\|y\|).$$

On the other hand, (3.8) yields

$$\|y(t)\| = \left\| \int_0^t y'(z)dz \right\| = \int_0^t \|y'(z)\| dz$$

and this together with the above equality implies

$$\frac{\|y'(t)\|}{\psi\left(\int_0^t \|y'(z)\| dz \right)} = 1.$$

Thus

$$\begin{aligned} \overline{T} &= \int_0^{\overline{T}} \frac{\|y'(t)\|}{\psi\left(\int_0^t \|y'(z)\| dz \right)} dt = \int_0^{\int_0^{\overline{T}} \|y'(z)\| dz} \frac{du}{\psi(u)} \\ &= \int_0^{\|y(\overline{T})\|} \frac{du}{\psi(u)} < \int_0^\infty \frac{du}{\psi(u)}. \end{aligned}$$

□

REMARK. A similar result can be obtained for the inhomogeneous initial value problem.

EXAMPLE 3. Suppose $H = \mathbf{R}^n$ with the usual norm,

$$\text{(3.9)} \qquad \begin{cases} y' = (\psi(|y|), 0, \dots, 0) = \overline{f}(t, y) \\ y(0) = 0, \end{cases}$$

where $\psi : [0, \infty) \to (0, \infty)$ is continuous. In addition, suppose $\overline{f}$ satisfies (3.3) and (3.4). Thus $y_2' = \cdots = y_n' = 0$, $y_1' = \psi(|y_1|) > 0$, yields

$$\left|\left| \int_0^t y'(z)dz \right|\right| = \left|\left| \left(\int_0^t y_1'(z)dz, 0, \dots, 0 \right) \right|\right|$$
$$= \int_0^t y_1'(z)dz = \int_0^t |y_1'(z)|dz = \int_0^t ||y'(z)||dz,$$

so (3.8) is satisfied. Hence, conditions of Theorem 3.4 are satisfied, which guarantees that (3.9) has a solution for

$$\overline{T} < \int_0^\infty \frac{du}{\psi(u)}$$

and this result is the best possible. In fact the above is also true if we remove assumptions (3.3) and (3.4); see Lee and O'Regan [**13**] for details.

REMARK. Condition (3.8) in Theorem 3.4 could have been stated as follows: Suppose, in addition, f satisfies

$$\text{(3.8)}^* \qquad \left|\left| \int_0^t f(s, y(s))ds \right|\right| = \int_0^t ||f(s, y(s))||ds$$

for all $t \in [0, T]$ and $y \in C^1([0, T], H)$.

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DEPARTMENT OF MATHEMATICS AND APPLIED STATISTICS, UNIVERSITY OF IDAHO, MOSCOW, ID 83843

SCHOOL OF MATHEMATICS, TRINITY COLLEGE, UNIVERSITY OF DUBLIN, DUBLIN 2, IRELAND