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BIVARIATE CARDINAL INTERPOLATION ON THE 3-DIRECTION MESH: l^p-DATA

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The analogue of the unvariate cardinal spline theory of Schoenberg has been successfully carried out for bivariate box splines on a three direction mesh [**1,2,3,4**]. However, there is one result that had eluded us: The convergence theory for bivariate cardinal spline operators from $l^p(\mathbf{Z}^2)$ to $L^p(\mathbf{R}^2)$. In [**5**] it was shown that the sequence of univariate cardinal spline interpolants, indexed by degree, has uniformly bounded norm when considered as a sequence of operators from $l^p(\mathbf{Z})$ to $L^p(\mathbf{R}), 1 < p < \infty$, and that these operators converge strongly in $L^p(\mathbf{R})$ to the classical Whittaker cardinal series. The analogous result for the bivariate case has been established only in the relatively trivial case $p = 2$ [**1**]. The aim of this paper is to complete this result, at least in the case of equal direction multiplicities.

The (centered) box spline M_n corresponding to the three directions $e_1 = (1,0), e_2 = (0,1), e_3 = e_1 + e_2 = (1,1)$ with equal multiplicities n may be defined by its Fourier transform,

$$\hat{M}_n(x) = \prod_{\nu=1}^{3} (\operatorname{sinc}(xe_\nu/2))^n$$

where $\operatorname{sinc}(t) := \sin t/t$. Thus, M_n is the n-fold convolution of the piecewise linear "hat-function" which indicates clearly the connection between box splines and univariate cardinal splines.

It was shown in [**3**] that the trigonometric polynomial

$$P_n(x) := \sum_{j\in Z^2} M_n(j)e^{-ijx} = \sum_{j\in Z^2} \hat{M}_n(x + 2\pi j)$$

is strictly positive and attains its minimum at $(2\pi/3, 2\pi/3)\bmod 2\pi\mathbf{Z}^2$. This implies that cardinal interpolation with the translates of the

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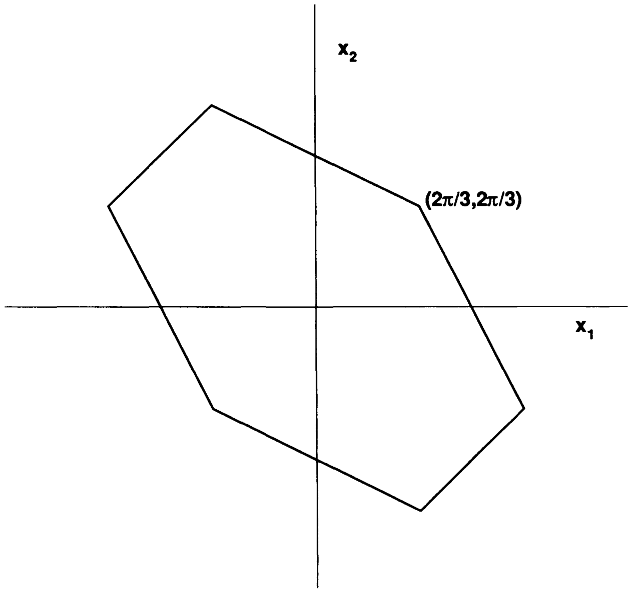

FIGURE 1.

box splines M_n is always well posed. That is, for given bounded data $y = \{y_j : j \in \mathbf{Z}^2\}$, there exists a unique bounded spline $I_n y \in S_n := \operatorname{span}\{M_n(\cdot - j), j \in \mathbf{Z}^2\}$ which interpolates y at the lattice points

$$I_n y(j) = y_j, \quad j \in \mathbf{Z}^2.$$

The cardinal spline interpolation operator I_n has the Lagrange representation

$$I_n y(w) = \sum_{j \in Z^2} y_j L_n(w - j), \quad w \in \mathbf{R}^2,$$

where L_n is the fundamental spline defined via its Fourier transform as

$$L_n(w) := \frac{1}{(2\pi)^2} \int_{R^2} \frac{\hat{M}_n(x)}{P_n(x)} e^{iwx} dx.$$

Since P_n is a non-vanishing trigonometric polynomial, $|L_n(w)|$ has exponential decay as $|w| \to +\infty$. Hence, if $y \in l^p(\mathbf{Z}^2)$, then $I_n y \in L^p(\mathbf{R}^2)$.

Denote by Ω the convex hull of $\pm(2\pi/3, 2\pi/3), \pm(4\pi/3, -2\pi/3), \pm(2\pi/3, -4\pi/3)$ (cf. Figure 1).

This set is a fundamental domain, i.e., its translates $2\pi j + \Omega$, $j \in \mathbf{Z}^2$, form an essentially disjoint partition of $\mathbf{R}^2$. in [**1**,**2**] we showed that the cardinal interpolants of a function f converge, as the degree tends to infinity, if the Fourier transform of f is a distribution with support contained in the interior of Ω. Our main Theorem strengthens this result.

THEOREM 1. *The bivariate cardinal spline interpolation operators I_n have uniformly bounded norms as operators from $l^p(\mathbf{Z}^2)$ to $L^p(\mathbf{R}^2), 1 < p < +\infty$. Moreover, for each $y \in l^p(\mathbf{Z}^2)$,*

$$||I_n y - Wy||_p \to 0 \quad \text{as } n \to +\infty$$

where $W : y \to \sum_{j \in Z^2} y_j \hat{\chi}_\Omega(\cdot - j)$ with χ_Ω the characteristic function of the set Ω. The Fourier transform of χ_Ω can be calculated explicitly,

$$\hat{\chi}_\Omega(w) = \frac{-6}{(2\pi)^2}\left[\frac{\cos 2\pi(w_1 + w_2)/3}{(w_1 - 2w_2)(w_2 - 2w_1)} + \frac{\cos 2\pi(w_2 - 2w_1)/3}{(w_1 + w_2)(w_1 - 2w_2)} + \frac{\cos 2\pi(w_1 - 2w_2)/3}{(w_1 + w_2)(w_2 - 2w_1)}\right].$$

The proof of this theorem is based on estimates for certain derivatives of $\hat{L}_n$. To formulate these estimates we need some auxiliary notation (cf. [**1**]). For $x = (u, v)$ and $j = (k, l)$ we set

$$a_j(x) := \frac{\hat{M}_n(x + 2\pi j)}{\hat{M}_n(x)} = \left(\frac{u}{u + k}\right)^n \left(\frac{v}{v + l}\right)^n \left(\frac{u + v}{u + v + k + l}\right)^n.$$

By straightforward, but tedious, computation one verifies that

$$\Omega = \{2\pi x : 0 \le a_j(x) \le 1 \text{ for } j \in J\},$$

where $J = \{\pm j_\nu : \nu = 1, 2, 3\}$ with $j_1 = (1, 0)$, $j_2 = (0, 1)$, $j_3 = (1, -1)$. The line segments $\Gamma_j, j \in J$, making up the boundary of Ω are subsets of $\{2\pi x : a_j(x) = 1\}$.

Because of the equal multiplicities, the box spline is invariant under linear changes of variables which do not alter the mesh generated by the three directions e_ν. The group $\mathbf{A}$ of such transformations is generated by the matrices

$$A^+_{(12)} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad A_{(13)} := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$
$$A_{(23)} := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A^- := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, any permutation σ of the three directions $\pm e_\nu$ corresponds to a linear transformation $A_\sigma \in \mathbf{A}$. It follows from the definitions that

$$\hat{M}_n(A^* x) = \hat{M}_n(x), \qquad \hat{L}_n(A^* x) = \hat{L}_n(x).$$

Moreover, if $R := \{(u,v) : u, v \geq 0\}$ denotes the positive orthant, then

$$\Omega = \cup_\sigma \Omega \cap R_\sigma$$

with $R_\sigma := A_\sigma R$.

With this notation, we now state the estimates needed for the proof of Theorem 1.

THEOREM 2. *Let $R_{\sigma,\varepsilon} := \{x : \operatorname{dist}(R_\sigma, x) \leq \varepsilon\}$ and let $D_{\sigma,1}, D_{\sigma,2}$ denote differentiation parallel to the two boundary segments of Ω which intersect R_σ. Then*

$$\sup_n \int_{R_{\sigma,\varepsilon}} |D_{\sigma,1} D_{\sigma,2} \hat{L}_n(x)| dx < \infty.$$

Note that $D_{\sigma,\nu} = (A_\sigma \gamma_\nu) \cdot \nabla$ with $\gamma_1 = (2,-1), \gamma_2 = (1,-2)$.

THEOREM 3. *There exist positive constants c_1 and c depending only on α such that*

$$|D^\alpha[\hat{L}_n(x) - \chi_\Omega(x)]| \leq \frac{c_1 n^{|\alpha|}}{[1 + c \operatorname{dist}(x, \partial\Omega)^n}.$$

The proof of Theorem 3 is analogous to that of Theorem 3 in [**2**]. It uses the following estimates of the derivatives of a_j.

LEMMA 1. *There exist positive constants c_1 and c depending only on α such that, for $2\pi x \in \Omega$,*

$$|D^\alpha a_j(x)| \le c_1 n^{|\alpha|}|j|^{|3\alpha|} \begin{cases} [1 + c\,\mathrm{dist}\,(x, \Gamma_j)]^{-n}, j \in J \\ [1 + c|j|]^{-n}, j \in Z^2 \backslash \{J \cup \{0\}\} \end{cases}.$$

This Lemma is easily proved by induction using Leibnitz's rule. The improvement over the corresponding result in [**2**] is possible because the set Ω does not depend on n.

LEMMA 2. *Let $x' = x + j, j \in \mathbf{Z}^2\backslash 0$, and let $2\pi x \in \Omega$. There exist positive constants c_1 and c depending only on α such that*

$$|D^\alpha a_j(x)| \le c_1 n^{|\alpha|}[1 + c\,\mathrm{dist}\,(2\pi x', \partial\Omega)]^{-n}.$$

Lemma 2 is a consequence of Lemma 1 and is used in turn to prove Theorem 3. The arguments follow those for the corresponding results in [**2**].

For the proof of Theorem 2 we need to examine the dependence of the estimates in Lemmas 1 and 2 on n more carefully.

LEMMA 3. *Denote by d_i the distance of x from Γ_{j_i}. Let $R_\varepsilon := \{x : \mathrm{dist}\,(R, x) \le \varepsilon\}$ and let D_i denote differentiation parallel to Γ_{j_i}. Then there exist positive constants c_1 and c such that, for $2\pi x \in \Omega$,*

$$|a_{j_i}| \le \frac{c_1}{(1 + cd_i)^n}, \quad i = 1, 2,$$

$$|D_i a_{j_i}| \le \frac{c_1 n d_i}{(1 + cd_i)^n}, \quad i = 1, 2,$$

$$|D_k a_{j_i}| \le \frac{c_1 n}{(1 + cd_i)^n}, \quad i = 1, 2, \quad k \ne i$$

$$|D_k D_i a_{j_i}| \le \frac{c_1 n^2 d_i + c_1 n}{(1 + cd_i)^n}, \quad i = 1, 2, \quad k \ne i.$$

PROOF OF LEMMA 3. The first assertion follows from Lemma 1 with $\alpha = 0$. We have

$$a_{j_1} = \left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n,$$

$$D_1 = \frac{1}{\sqrt{5}}(1,-2)\cdot\nabla, \quad D_2 = \frac{1}{\sqrt{5}}(2,-1)\cdot\nabla, \quad D_3 = \frac{1}{\sqrt{2}}(1,-1)\cdot\nabla.$$

Since

$$(a,b)\cdot\nabla a_{j_1} = n\Big[\frac{a}{u(1-u)} + \frac{a+b}{(u+v)(1-u-v)}\Big]a_{j_i}$$

it follows that

$$\begin{aligned} D_1 a_{j_1} &= \frac{ncv(1-2u-v)}{u(1-u)(u+v)(1-u-v)}a_{j_1}, \\ D_2 a_{j_1} &= \frac{nc[3u(1-u)+2v(1-2u-v)]}{u(1-u)(u+v)(1-u-v)}a_{j_1}, \\ D_3 a_{j_1} &= \frac{nc}{u(1-u)}a_{j_1}. \end{aligned}$$

Since $d_1 = \frac{2\pi}{\sqrt{5}}|1-2u-v|$, the two middle assertions hold for a_{j_1}. A similar analysis of $(a,b)\cdot\nabla(D_1 a_{j_1})$ gives the final assertion for a_{j_1}. The corresponding assertions for a_{j_2} follow by symmetry. □

PROOF OF THEOREM 2. By Theorem 3 we may assume that x is within δ of the boundary of Ω. By symmetry, we may also assume that $R_{\sigma,\varepsilon} = R_\varepsilon$ and that $x \in R_\varepsilon \cap \{(u,v) : v \le u\}$.

(Proof inside Ω) We use the notation of Lemma 3. Since $\hat{L}_n = 1/\sum a_j$, it follows that

$$\begin{aligned} D_1 D_2 \hat{L}_n &= \frac{2(D_1\sum a_j)(D_2\sum a_j) - (\sum a_j)(D_1 D_2 \sum a_j)}{(\sum a_j)^3} \\ &= O(1)\Big[2(D_1 a_{j_1} + D_1 a_{j_2})(D_2 a_{j_1} + D_2 a_{j_2}) \\ &\quad - (1 + a_{j_1} + a_{j_2})(D_1 D_2 a_{j_1} + D_1 D_2 a_{j_2}) + O\Big(\frac{c_3 n^2}{(1+c_2)^n}\Big)\Big] \end{aligned}$$

for some positive c_2, c_3 as $n \to +\infty$ in view of Lemma 1. Thus, as $n \to +\infty$, Lemma 3 implies

$$\begin{aligned} |D_1 D_2 \hat{L}_n| \le c_4 \Big[&\Big(\frac{nd_1}{(1+cd_1)^n} + \frac{n}{(1+cd_2)^n}\Big)\Big(\frac{n}{(1+cd_1)^n} + \frac{nd_2}{(1+cd_2)^n}\Big) \\ &+ \Big(\frac{n^2 d_1 + n}{(1+cd_1)^n} + \frac{n^2 d_2 + n}{(1+cd_2)^n}\Big) + O\Big(\frac{c_3 n^2}{(1+c_2)^n}\Big)\Big] \end{aligned}$$

for some positive c_4. Since

$$\frac{n^2}{(1+c_2)^n} + \int_0^\delta \frac{n^2 z}{(1+cz)^n} dz + \int_0^\delta \frac{n}{(1+cz)^n} dz = O(1)$$

as $n \to +\infty$, the contribution to $\int_{R_\varepsilon} |D_1 D_2 \hat{L}_n|$ from within Ω is finite.

(Proof outside Ω). Let $x = (u,v)$ in $R_\varepsilon \cap \{(u,v) : v \le u\}$ and $\operatorname{dist}(2\pi x, \partial\Omega) \le \delta$, but $2\pi x$ outside of Ω. Map x to $x' = (u-1, v)$. Then $2\pi x'$ is inside Ω so that, for an appropriate permutation σ, $2\pi x'' = 2\pi A_\sigma x'$ is in $R_{2\varepsilon}$ with $\operatorname{dist}(2\pi x'', \partial\Omega) \le 2\delta$ (The changes $\varepsilon \to 2\varepsilon$ and $\delta \to 2\delta$ allow for some distortion if $-\varepsilon < v < 0$ or $1/3 < v < u$.). Using the symmetries we have

$$D_1 D_2 \hat{L}_n(x) = D_1 D_3 \Big(a_{j_1}(x'') \hat{L}_n(x'') \Big).$$

Omitting the argument x'', we have

$$\begin{aligned}
|D_1 D_3 (a_{j_1} \hat{L}_n)| &= |(D_1 D_3 a_{j_1}) \hat{L}_n + (D_1 a_{j_1}) D_3 \hat{L}_n \\
&\quad + (D_3 a_{j_1}) D_1 \hat{L}_n + a_{j_1} D_1 D_3 \hat{L}_n| \\
&\le c_5 \bigg[\Big(\frac{n^2 d_1 + n}{(1+cd_1)^n} \Big) + \Big(\frac{n d_1}{(1+cd_1)^n} n \Big) \\
&\quad + \Big(\frac{n}{(1+cd_1)^n} \Big(\frac{n d_1}{(1+cd_1)^n} + \frac{n}{(1+cd_2)^n} \Big) \Big) \\
&\quad + \frac{1}{(1+cd_1)^n} \Big(\frac{n^2}{(1+cd_2)^n} + (n^2 d_1 + n) \Big) \bigg]
\end{aligned}$$

so that the contribution to $\int_{R_\varepsilon} |D_1 D_2 \hat{L}_n|$ from outside Ω is also finite. □

PROOF OF THEOREM 1. Let $\{y_i\} = y \in l^p(\mathbf{Z}^2)$ be a finite sequence and g be a compactly supported function in $L^q(\mathbf{R}^2)$, $1/p + 1/q = 1$. For $w \in \mathbf{R}^2$, let $j(w)$ be uniquely defined by $w - j(w) \in [-1/2, 1/2)^2$.

Then

$$\begin{aligned}
&\left|\int_{R^2} g(w) \sum_{j\in Z^2} y_j L_n(w-j)dw\right| \\
&\leq \left|\int_{R^2} g(w) y_{j(w)} L_n(w-j(w))dw\right| + \left|\int_{R^2} g(w) \sum_{j\neq j(w)} y_i L_n(w-j)dw\right| \\
&\leq ||y||_{l^p(Z^2)} ||g||_{L^q(R^2)} + \left|\int_{R^2} g(w) \sum_{j\neq j(w)} y_j L_n(w-j)dw\right|.
\end{aligned}$$

To estimate the second quantity we pass to the transform space. Let $\{\phi_\sigma\}$ be a smooth partition of unity for $\mathbf{R}^2$ subordinate to $\{R_{\sigma,\varepsilon}\}$. Then, in view of the decay of $\hat{L}_n$ and its derivatives at infinity, we have

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{R^2} \hat{L}_n(x) e^{i(w-j)x}dx \\
&\qquad = \frac{1}{(2\pi)^2}\sum_\sigma \int_{R_{\sigma,\varepsilon}} \phi_\sigma(x)\hat{L}_n(x)e^{i(w-j)x}dx \\
&\qquad = \frac{c}{(2\pi)^2}\sum_\sigma \int_{R_{\sigma,\varepsilon}} D_{\sigma,1}D_{\sigma,2}(\phi_\sigma(x)\hat{L}_n(x))\frac{e^{i(w-j)x}}{\prod_{\nu=1}^2 \gamma_{\sigma,\nu}(w-j)}dx
\end{aligned}$$

where $\gamma_{\sigma,\nu} := A\sigma\gamma_\nu$. Consequently, by Fubini's Theorem,

$$\begin{aligned}
&\left|\int_{R^2} g(w) \sum_{j\neq j(w)} y_i L_n(w-j)dw\right| \\
&\quad = \left|\int_{R^2} \sum_{j\neq j(w)} y_i\Big(\frac{1}{(2\pi)^2}\int_{R^2}\hat{L}_n(x)e^{i(w-j)x}dx\Big)g(w)dw\right| \\
&\quad \leq \frac{1}{(2\pi)^2}\sum_\sigma \int_{R_{\sigma,\varepsilon}} |D_{\sigma,1}D_{\sigma,2}(\phi_\sigma(x)\hat{L}_n(x))| \\
&\qquad \times \left|\int_{R^2} \sum_{j\neq j(w)} \frac{y_j e^{-ijx}}{\prod_{\nu=1}^2 \gamma_{\sigma,\nu}(w-j)} e^{iwx} g(w)dw\right| dx.
\end{aligned}$$

Let

$$H_\sigma y(w) = \sum_{j\neq j(w)} \frac{y_j}{\prod_{\nu=1}^2 \gamma_{\sigma,\nu}(w-j)}$$

denote the mixed bivariate Hilbert transform corresponding to independent directions $\gamma_{\sigma,1}, \gamma_{\sigma,2}$ in $\mathbf{R}^2$. Then H_σ is a bounded linear transformation from $l^p(\mathbf{Z}^2)$ to $L^p(\mathbf{R}^2)$ with norm $||H_\sigma||_p, 1 < p < +\infty$. Therefore, in view of Theorems 2 and 3 we have

$$\left|\int_{R^2} g(w) I_n y(w)dw\right| = \left|\int_{R^2} g(w) \sum_{j\in Z^2} y_j L_n(w-j)dw\right|$$
$$\leq C\Big(1+\sum_\sigma ||H_\sigma||_p\Big)||y||_{l^p(Z^2)}||g||_{L^q(R^2)}.$$

To show that $||I_n y - Wy||_p \to 0$ as $n \to +\infty$, it is enough to show this for the sequences $y = \delta_i$, $i \in \mathbf{Z}^2$, where $\delta_i(j) = 1$ if $j = i$ and is zero otherwise. Now

$$||I_n\delta_i - W\delta_i||_{L^p(R^2)} = ||I_n\delta_0 - W\delta_0||_{L^p(R^2)} = ||L_n - \hat{\chi}_\Omega||_p.$$

Theorem 3 implies that $L_n \to \hat{\chi}_\Omega$ uniformly in $\mathbf{R}^2$. Finally

$$|L_n(w)| \leq \frac{1}{(2\pi)^2}\sum\left|\int_{R_{\sigma,\varepsilon}} \phi_\sigma(x)\hat{L}_n(x)e^{iwx}dx\right|$$
$$\leq \frac{1}{(2\pi)^2}\sum_\sigma \frac{1}{|\prod_{\nu=1}^2 \gamma_{\sigma,\nu} w|}$$
$$\times \int_{R_{\sigma,\varepsilon}} |D_{\sigma,1}D_{\sigma,2}\hat{L}_n(x)|dx = O\Big(\frac{1}{|w|^2}\Big) \quad \text{for large } |w|$$

allows the estimate

$$|L_n(w) - \hat{\chi}_\Omega(w)| = O(\min(1, 1/|w|^2)).$$

Hence, $||L_n - \hat{\chi}_\Omega||_p \to 0$ by the dominated convergence theorem.

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