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AN EMBEDDING THEOREM FOR MIXED NORMED SPACES

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1. Introduction. If $s, r, \beta > 0$, a function f analytic on the upper half-plane $U = \{z = x + iy,\ x \in R,\ y > 0\}$, is said to belong to the space A^{β}_{sr} if

$$||f||^{r}_{s,r,\beta} = \int_{0}^{\infty} y^{r\beta-1} M_s(y,f)^r dy < \infty,$$

where

$$M_s(y,f) = \Big(\int_{-\infty}^{\infty} |f(x+iy)|^s dx\Big)^{1/s}.$$

Let $0 < p < s < \infty$, $0 < q < r < \infty$ and μ be some positive finite Borel measure on U. In this paper we will find conditions on μ that are equivalent to the estimate: There is a constant C such that

$$\Big(\sum_{j}\Big(\sum_{k}\int_{Q_{jk}} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \le C||f||_{s,r,\beta}, \quad \text{for all } f \in A^{\beta}_{sr},$$

where Q_{jk} are squares $\{z = x + iy,\ \ k2^j \le x < (k+1)2^j,\ \ 2^j \le y < 2^{j+1}\}$, j and k are integers; i.e., the injection mapping from A^{β}_{sr} to the space $L^{p,q}_{\mu}$ (with the obvious definition) is bounded.

Our work was motivated by a recent paper of D. Luecking [**5**]. For $0 < p < s < \infty$, he has characterized the positive measures μ on the unit disc D for which there is a $C > 0$ such that

$$\Big(\int_{D} |f(z)|^p d\mu(z)\Big)^{1/p} \le C||f||_{s,s,\beta}, \quad \text{for all } f \in A^{\beta}_{ss},$$

the Bergman space of functions f which are analytic in D and for which

$$||f||^{s}_{s,s,\beta} = \int_{0}^{1}\int_{0}^{2\pi} |f(\rho e^{i\theta})|^s (1-\rho)^{s\beta-1} d\rho d\theta < \infty \quad (\beta > 0).$$

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His result is that this occurs if and only if

$$k(z) = \mu(D_\varepsilon(z))/m_{s,\beta}(D_\varepsilon(z)) \in L^q(m_{s,\beta}), \quad \text{for } 1/q + p/s = 1,$$

where $D_\varepsilon(z)$ is the pseudo-hyperbolic disc around $z \in D$ having radius ε and $m_{s,\beta}(z) = (1-|z|)^{s\beta-1} dm(z)$, for dm two-dimensional Lebesgue measure on D.

In this paper we extend Luecking's theorem to the mixed norm spaces $(0 < p < s,\ 0 < q < r)$. Actually, if $p = q$ and $r = s$ we have another equivalent condition on the measure μ. The precise statement and the proof will be given in §3. For technical reasons, our proof will be performed in the upper half plane.

The main ingredient of Luecking's theorem mentioned above is a theorem of E. Amar [**1**] which says that every separated sequence in D is a finite union of interpolation sequences for A^β_{ss} (relevant definitions will be given in §2). To make use of Luecking's idea, we require such a theorem for mixed norm spaces. This follows from our main theorem in [**3**]. For the sake of completeness we restate it in §2.

F. Ricci and M. Taibleson's decomposition theorem for mixed norm spaces [**6**] is another key theorem used in the proof of the main theorem.

We conclude the paper with an application of the method used in §3. Pointwise multipliers from A^β_{sr} to A^α_{pq} are characterized provided that $0 < p < s < \infty,\ \ 0 < q < r < \infty,\ \ 0 < \beta < \alpha$.

2. Interpolating sequences. Let $\rho(z,w)$ denote the pseudo hyperbolic metric

$$\rho(z,w) = \left|\frac{z-w}{z-\overline{w}}\right|, \quad z, w \in U.$$

For $a \in U$ and $0 < \delta < 1$ let

$$D_\delta(a) = \{z \in U : \ \rho(z,a) < \delta\}.$$

The following lemma is a simple consequence of the fact that $D_\delta(a)$ is a disc in the Euclidean metric with center $c = \operatorname{Re} a + i \operatorname{Im} a \, \frac{1+\delta^2}{1-\delta^2}$ and radius $R = \frac{2\delta}{1-\delta^2} \operatorname{Im} a$.

LEMMA 2.1. *For given $0 < \delta < 1$ there exist positive integers n_1, n_2, n_3, n_4, depending only on δ, such that if $a \in Q_{jk}$ then $D_\delta(a) \subset Q_{jk}^{(\delta)}$, where $Q_{jk}^{(\delta)}$ is a rectangle in U.*

$$\{z = x + iy : (k - n_1)2^j \le x < (k + 1 + n_2)2^j,\ 2^{j-n_3} \le y < 2^{j+1+n_4}\}.$$

Let ℓ_{sr} denote the mixed norm space of all double sequences $a = \{a_{jk}\}$, $j, k \in \mathbf{Z}$, for which

$$||a||_{s,r}^r = \sum_j \Big(\sum_k |a_{jk}|^s\Big)^{r/s} < \infty \quad (0 < s, r < \infty).$$

A sequence $\{z_{jk}\}$, $j, k \in \mathbf{Z}$, in U is said to be δ-separated if there exists a $\delta > 0$ such that if $(m, n) \ne (j, k)$ then

$$\rho(z_{mn}, z_{jk}) \ge \delta.$$

A sequence $\{z_{jk}\}$ in U is called an interpolation sequence for A_{sr}^β if whenever $\{x_{jk}\} \in \ell_{sr}$, then there exists $f \in A_{sr}^\beta$ satisfying $f(z_{jk})(\operatorname{Im} z_{jk})^{\beta+1/s} = x_{jk}$, i.e., if the operator R defined by $Rf = \{f(z_{jk})(\operatorname{Im} z_{jk})^{\beta+1/s}\}$ is a bounded map of A_{sr}^β onto ℓ_{sr}.

It follows from the open mapping theorem that a constant M may be associated with any given interpolation sequence $\{z_{jk}\}$ such that any $\{x_{jk}\}$ with $||\{x_{jk}\}||_{sr} \le 1$ is the image under R of a function $f \in A_{sr}^\beta$ with $||f||_{s,r,\beta} \le M$. This M will be referred to as the interpolation constant of $\{z_{jk}\}$.

THEOREM A. **[3]** *Let n, m be positive integers and $m_0 \in \{0, 1, 2, \dots, n-1\}$. Suppose that $\{z_{m_0+jn,k}\}, j, k \in \mathbf{Z}$, is a sequence in U which satisfies the following conditions*:

(i) $2^{m_0+(j+1)n-1} \le \operatorname{Im} z_{m_0+jn,k} < 2^{m_0+(j+1)n}$, *for all* $k \in \mathbf{Z}$;

(ii) $|\operatorname{Re} z_{m_0+jn,k_1} - \operatorname{Re} z_{m_0+jn,k_2}| \ge (m+1)2^{m_0+(j+1)n-1}$, *if* $k_1 \ne k_2$.

If n and m are large enough, then T_{sr}^β defined by

$$T_{sr}^\beta(f) = \{f(z_{m_0+jn,k})\operatorname{Im} z_{m_0+jn,k})^{\beta+1/s}\}, \quad f \in A_{sr}^\beta,$$

is a continuous linear map A_{sr}^β onto ℓ_{sr}. In fact, there is a continuous linear map V of ℓ_{sr} into A_{sr}^β so that $T_{sr}^\beta V$ is the identity mapping on ℓ_{sr}.

Throughout the paper we use $C, C_1, C_2, \ldots,$ to denote positive constants, depending on the particular parameters $r, s, \ldots, M, \delta, \eta, \beta, \ldots,$ concerned in the particular problem in which they appear. They are not necessarily the same on any two occurrences.

3. Embedding theorem for mixed norm spaces. Before stating the main result we establish two more preliminary lemmas.

LEMMA 3.1. *Let $\delta > 0$ and $s, r, \beta > 0$. If f is a holomorphic function in U, then the following statements are equivalent:*

i) $f \in A^{\beta}_{sr}$,

ii) $\{2^{j(\beta+1/s)} \sup_{z\in Q_{jk}} |f(z)|\} \in \ell_{sr}$,

iii) $\{2^{j(\beta+1/s)} \sup_{z\in Q^{(\delta)}_{jk}} |f(z)|\} \in \ell_{sr}$.

PROOF. From the decomposition theorem for the spaces A^{β}_{sr}, F. Ricci and M. Taibleson have obtained that $||f||_{s,r,\beta}$ is equivalent to $||\{2^{j(\beta+1/s)} \sup_{z\in Q_{jk}} |f(z)|\}||_{s,r}$ (Lemma 6.3 in [**6**]). Obviously, iii) implies ii). It remains to show that ii) implies iii). Recall that $Q^{(\delta)}_{jk}$ is the rectangle

$$\{z = x + iy : (k - n_1)2^j \le x < (k + 1 + n_2)2^j, 2^{j-n_3} \le y < 2^{j+1+n_4}\}$$

for some positive integers n_1, n_2, n_3 and n_4 depending only on δ. Thus,

$$
\begin{aligned}
&\sum_j 2^{j(\beta+1/s)r}\Big(\sum_k \sup_{z\in Q_{jk}^{(\delta)}} |f(z)|^s\Big)^{r/s} \\
&\le \sum_j 2^{j(\beta+1/s)r}\Big(\sum_k \sum_{m=j-n_3}^{j+1+n_4} \sum_{n=k-n_1}^{k+1+n_2} \sup_{z\in Q_{mn}} |f(z)|^s\Big)^{r/s} \\
&\le C\sum_j 2^{j(\beta+1/s)r} \sum_{m=j-n_3}^{j+n_4} \Big(\sum_k \sum_{n=k-n_1}^{k+1+n_2} \sup_{z\in Q_{mn}} |f(z)|^s\Big)^{r/s} \\
&\le C\sum_j 2^{j(\beta+1/s)r} \sum_{m=j-n_3}^{j+1+n_4} \Big(\sum_k \sup_{z\in Q_{mk}} |f(z)|^s\Big)^{r/s} \\
&\le C\sum_j 2^{j(\beta+1/s)r}\Big(\sum_k \sup_{z\in Q_{jk}} |f(z)|^s\Big)^{r/s}. \square
\end{aligned}
$$

LEMMA 3.2. [4] *Let* $0 < p < s < \infty$, $0 < q < r < \infty$, $1/u = 1/p - 1/s$, $1/v = 1/q - 1/r$. *Then, for any* $\{x_{jk}\} \in l_{uv}$, *we have*

$$||\{x_{jk}\}||_{u,v} = \sup_{||\{y_{jk}\}||_{sr}=1} ||\{x_{jk}y_{jk}\}||_{p,q}.$$

THEOREM 3.3. *Let* $0 < p < s < \infty$, $0 < q < r < \infty$, $1/u = 1/p - 1/s$, $1/v = 1/q - 1/r$. *There is a constant* C *such that*

$$\Big(\sum_j \Big(\sum_k \int_{Q_{jk}} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \le C||f||_{s,r,\beta}, \quad \textit{for all } f \in A^{\beta}_{sr},$$

if and only if

$$||\{2^{-j(\beta+1/s)}\mu(Q_{jk})^{1/p}\}||_{u,v} < \infty.$$

PROOF. Let $||\{2^{-j(\beta+1/s)}\mu(Q_{jk})^{1/p}\}||_{u,v} < \infty$ and $f \in A^{\beta}_{sr}$. By using

Hölder's inequality, with indices s/p and r/q respectively, we obtain

$$\begin{aligned}
&\Big(\sum_j\Big(\sum_k\int_{Q_{jk}}|f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q}\\
&\qquad\le\Big(\sum_j\Big(\sum_k\sup_{z\in Q_{jk}}|f(z)|^p)\mu(Q_{jk})\Big)^{q/p}\Big)^{1/q}\\
&\qquad\le\Big(\sum_j\Big(\sum_k(\sup_{z\in Q_{jk}}|f(z)|^s)^{q/s}\Big(\sum_k\mu(Q_{jk})^{u/p}\Big)^{q/u}\Big)^{1/q}\\
&\qquad\le||\{2^{j(\beta+1/s)}\sup_{z\in Q_{jk}}|f(z)|\}||_{s,r}||\{2^{-j(\beta+1/s)}\mu(Q_{jk})^{1/p}\}||_{u,v}\\
&\qquad\le C||f||_{s,r,\beta}||\{2^{-j(\beta+1/s)}\mu(Q_{jk})^{1/p}\}||_{u,v}
\end{aligned}$$

by Lemma 3.1.

To prove necessity, we fix n and m large enough, so that any sequence satisfying the conditions of Theorem A is an interpolating sequence for A^β_{sr}, and construct a δ-lattice, that is, a δ-separated sequence $\{w_{jk}\}$ such that discs $D_\delta(w_{jk})$ cover U. Without loss of generality we may suppose that $2^{n-1+j}\le\operatorname{Im}w_{jk}<2^{n+j}$, $j,\ k\in\mathbf{Z}$.

Since $\{w_{jk}\}$ is separated we can split it into n_0 sequences X_α such that each of them has at most one point in any square Q_{jk}. Now split each of the sequences X_α into m sequences $X_{\alpha\gamma}$, $\gamma=1,2,\dots,m$, defined by $w_{jk}\in X_{\alpha\gamma}$ if and only if $w_{jk}\in X_\alpha\cap Q_{lt}$, where $t=\gamma(\operatorname{mod}m)$. Finally, we split $X_{\alpha\gamma}$ into n sequences $X_{\alpha\gamma\nu},\nu=1,2,\dots,n$, so that $w_{jk}\in X_{\alpha\gamma\nu}$ if and only if $w_{jk}\in X_{\alpha\gamma}\cap Q_{lt}$, where $l=\nu(\operatorname{mod}n)$. To obtain $N=n_0mn$ interpolation sequences for A^β_{sr} we enumerate the sequences $X_{\alpha\gamma\nu}$, so that the conditions of Theorem A are satisfied. We may suppose that all are η-separated for some η. Let $\{a_{jk}\}=\{w_{m_0+jn},s_k\}$, $0\le m_0\le n-1$, be one of the sequences $X_{\alpha\gamma\nu}$. By Theorem A, any sequence $\{y_{jk}\}$ with $||\{y_{jk}\}||_{s,r}=1$ is of the form $\{f(a_{jk})(\operatorname{Im}a_{jk})^{\beta+1/s}\}$ for some $f\in A^\beta_{sr}$ with $||f||_{s,r,\beta}\le M$, where M is an interpolation constant associated with $\{a_{jk}\}$. (Note that M is also an interpolation constant for any sequence $X_{\alpha\gamma\nu}$.)

Thus, by Lemma 3.2,

$$\begin{aligned}
(3.1)\qquad &||\{2^{-jn(\beta+1/s)}\mu(D_\delta(a_{jk}))^{1/p}\}||_{u,v}\\
&\qquad\le C||\{f(a_{jk})\mu(D_\delta(a_{jk}))^{1/p}\}||_{p,q},
\end{aligned}$$

for some $f \in A^{\beta}_{sr}$ with $\|f\|_{s,r,\beta} \le M$. Let f satisfy (3.1). Then

$$\int_{D_\delta(a_{jk})} |f(z)|^p d\mu(z) \tag{3.2}$$

$$\ge C_1 |f(a_{jk})|^p \mu(D_\delta(a_{jk})) - C_2 \int_{D_\delta(a_{jk})} |f(z) - f(a_{jk})|^p d\mu(z),$$

where C_1 and C_2 are positive constants depending only on p. Summing over k, raising both sides of the inequality (3.2) to the power q/p, applying inequality of type (3.2) with p replaced by q/p and summing over j, we obtain

$$\begin{aligned} &\Big(\sum_j \Big(\sum_k \int_{D_\delta(a_{jk})} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \\ &\quad \ge C_3\Big(\sum_j \Big(\sum_k |f(a_{jk})|^p \mu(Q_\delta(a_{jk})\Big)^{q/p}\Big)^{1/q} \\ &\quad - C_4\Big(\sum_j \Big(\sum_k \int_{D_\delta(a_{jk})} |f(z) - f(a_{jk})|^p d\mu(z)\Big)^{q/p}\Big)^{1/q}. \end{aligned} \tag{3.3}$$

We may assume that $\delta < \eta/4$. So, if $z \in D_\delta(a_{jk})$, then

$$|f(z) - f(a_{jk})|^p \le C\delta^p \frac{1}{m(D_\eta(a_{jk}))} \int_{D_\eta(a_{jk})} |f(z)|^p dm(z)$$

where the constant C depends only on η and p (see [5] for details). Thus

$$\Big(\sum_j \Big(\sum_k \int_{D_\delta(a_{jk})} |f(z) - f(a_{jk})|^p d\mu(z)\Big)^{q/p}\Big)^{1/q}$$

$$\le C\delta\Big(\sum_j \Big(\sum_k \mu(D_\delta(a_{jk}) \sup_{z \in G^{(\eta)}_{jk}} |f(z)|^p\Big)^{q/p}\Big)^{1/q}.$$

Here, G_{jk} denotes the square $Q_{m_0+(j+1)n-1,s_k}$ which contains the point $w_{m_0+jn,s_k} = a_{jk}$ and $G^{(\eta)}_{jk}$ is the rectangle associated with G_{jk} (Lemma 2.1).

Proceed as in the proof of sufficiency to conclude that

$$
\begin{aligned}
&\Big(\sum_j \Big(\sum_k \int_{D_\delta(a_{jk})} |f(z) - f(a_{jk})|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
&\qquad \le C\delta ||\{2^{-jn(\beta+1/s)}\mu(D_\delta(a_{jk}))^{1/p}\}||_{u,v} \\
&\qquad\quad ||\{2^{jn(\beta+1/s)} \sup_{z\in Q^{(\eta)}_{m_0+(j+1)n-1,s_k}} |D(z)|\}||_{p,q} \\
&\qquad \le C\,\delta ||\{2^{-jn(\beta+1/s)}\mu(D_\delta(a_{jk}))^{1/p}\}||_{u,v} \\
&\qquad\quad ||\{2^{jn(\beta+1/s)} \sup_{z\in Q^{(\eta)}_{m_0+(j+1)n-1,k}} |f(z)|\}||_{p,q} \\
&\qquad \le C\,\delta ||\{2^{-jn(\beta+1/s)}\mu(D_\delta(a_{jk}))^{1/p}\}||_{u,v} \\
&\qquad\quad ||\{2^{j(\beta+1/s)} \sup_{z\in Q^{(\eta)}_{j,k}} |f(z)|\}||_{p,q} \\
&\qquad \le C\,\delta ||\{2^{-jn(\beta+1/s)}\mu(D_\delta(a_{jk}))^{1/p}\}||_{u,v} ||f||_{s,r,\beta}
\end{aligned}
\tag{3.4}
$$

by Lemma 3.1.

Since the discs $D_\delta(a_{jk})$ are disjoint, we have

$$
\begin{aligned}
&\Big(\sum_j \Big(\sum_k \int_{D_\delta(a_{jk})} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
&\qquad \le \Big(\sum_j \Big(\sum_k \int_{Q^{(\delta)}_{m_0+(j+1)n-1,s_k}} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
&\qquad \le \Big(\sum_j \Big(\sum_k \int_{Q^{(\delta)}_{m_0+(j+1)n-1,k}} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
&\qquad \le \Big(\sum_j \Big(\sum_k \int_{Q^{(\delta)}_{j,k}} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q}
\end{aligned}
$$

Now proceed as in the proof of Lemma 3.1 to conclude

$$
\begin{aligned}
&\Big(\sum_j \Big(\sum_k \int_{Q^{(\delta)}_{jk}} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \\
&\qquad \le c\Big(\sum_j \Big(\sum_k \int_{Q_{jk}} |f(z)|^p d\mu(z)\Big)^{q/p}\Big)^{1/q} \le C||f||_{s,r,\beta},
\end{aligned}
\tag{3.5}
$$

by assumption.

If we choose δ small enough, then from (3.1), (3.3), (3.4) and (3.5) we see that

$$||\{2^{-jn(\beta+1/s)}\mu(Q_\delta(a_{jk}))^{1/p}\}||_{u,v} \leq C.$$

Sum over the N sequences $X_{\alpha\gamma\nu}$ to get

$$||\{2^{-j(\beta+1/s)}\mu(Q_\delta(w_{jk}))^{1/p}\}||_{u,v} < \infty. \tag{3.6}$$

Recall that each square Q_{jk} contains at most n_0 points of the sequence $\{w_{jk}\}$. On the other hand, $\{w_{jk}\}$ is a δ-lattice. Thus, using these facts and (3.6), we conclude

$$||\{2^{-j(\beta+1/s)}\mu(Q_{jk})^{1/p}\}||_{u,v} < \infty.$$

4. Multipliers of mixed normed spaces. Let $M(A^\beta_{sr}, A^\alpha_{pq})$ be the collection of all functions which multiply A^β_{sr} into A^α_{pq}, i.e., fg is A^α_{pq} for all $g \in A^\beta_{sr}$. In [**2**], $M(A^{1/s}_{ss}, A^{1/p}_{pp})$ is characterized in the case $0 < p < s < \infty$. Using the same method as in §3 we can find multipliers in the following cases.

THEOREM 4.1. *Let* $0 < p < s < \infty$, $0 < q < r < \infty$, $0 < \beta < \alpha < \infty$, $1/u = 1/p - 1/s$, $1/v = 1/q - 1/r$. *Then*

$$M(A^\beta_{sr}, A^\alpha_{pq}) = A^{\alpha-\beta}_{uv}.$$

We omit details.

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