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MIXED MODULES IN L*

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ABSTRACT. By assuming the set-theoretic hypothesis $V = L$ we show that, for a large class of rings R, there exist, for any regular not weakly compact cardinals κ, strongly κ-cyclic mixed R-modules having endomorphism algebra isomorphic to the split extension of the R-algebra A by the ideal of bounded endomorphisms provided A is free quâ R-module and $\kappa > |A|$.

1. Introduction. In this paper we deal with endomorphism algebras $E_R(G)$ of certain mixed R-modules G in the universe $V = L$. We shall always assume that R is a non-zero commutative ring with 1, with a given countable multiplicatively closed subset S of non-zero divisors. Let A be any fixed R-algebra which is S-reduced and S-torsion-free. (These and related concepts are defined in §2.) It has been established, working only in ZFC, that inter alia the following realization theorem holds; see [**1**] and [**2**].

THEOREM. *If A is an S-reduced, S-torsion-free R-module then there exists a mixed R-module G with $E_R(G) = A \oplus \mathrm{Bd}(G)$. (Here and throughout the paper $\mathrm{Bd}(G)$ will denote the ideal of bounded endomorphisms of G; $\phi \in \mathrm{Bd}(G)$ if and only if there is an $s \in S$ with $(G\phi)s = 0$.)*

Indeed the results can be extended to derive arbitrarily large rigid systems and semi-rigid proper classes (i.e., classes which are not sets.) Assuming $V = L$ we can sharpen these results considerably by imposing only slightly stronger conditions on the algebra A. In this context cyclic A-modules will either be copies of A or torsion A-modules A/sA for some $s \in S$. Recall that, in general, a module is said to be Σ-cyclic if it is a direct sum of cyclic modules. Observe that Σ-cyclic modules are reduced. A module is κ-cyclic, for some cardinal κ, if any

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submodule of cardinality $< \kappa$ is contained in a Σ-cyclic submodule of cardinality $< \kappa$. Furthermore a module H is said to be strongly κ-cyclic if any submodule H_0 of H, with $|H_0| < \kappa$, is contained in a Σ-cyclic submodule H_1 of cardinality $|H_1| < \kappa$ with $H/H_1 \kappa$-cyclic.

By imposing the additional condition that A be cotorsion (see [**1**], [**3**], [**4**] or §2) we shall obtain the following strengthened version of the above quoted theorem.

THEOREM 1.1. *($V = L$) Let A be a cotorsion-free R-algebra and κ any regular not weakly compact cardinal $> |A|$. Then*

(i) *there exist strongly κ-cyclic A-modules $G^\alpha_\kappa (\alpha < 2^\kappa)$ of cardinality κ such that $E_R(G^\alpha_\kappa) = A \oplus \mathrm{Bd}(G^\alpha_\kappa)$ and $t(G^\alpha_\kappa)$, the torsion part of G^α_κ, is equal to T where T is any prescribed strongly κ-cyclic torsion A-module.*

(ii) *if $\phi : G^\alpha_\kappa \to G^{\alpha^1}_{\kappa^1}$ is a homomorphism and $(\alpha, \kappa) \neq (\alpha^1, \kappa^1)$, then ϕ is bounded.*

(iii) *if G is any strongly κ_0-cyclic A-module of cardinality $\kappa_0 < \kappa$, then $\phi : G^\alpha_\kappa \to G$ is bounded.*

We remark that if κ is singular or weakly compact then it is known that any strongly κ-cyclic module must be Σ-cyclic, and so the conditions imposed on κ in Theorem 1.1 are necessary. Observe that the modules constructed in Theorem 1.1 are only strongly κ-cyclic quâ A-modules. However it is easy to obtain a similar family of strongly κ-cyclic R-modules:

COROLLARY 1.2. *Let R be cotorsion-free and let A be any R-algebra with free underlying R-module structure and κ any regular not weakly compact cardinal $> |A|$. Then*

(i) *there exist strongly κ-cyclic R-modules $H^\alpha_\kappa (\alpha < 2^\kappa)$ of cardinality κ such that $E_R(H^\alpha_\kappa) = A \oplus \mathrm{Bd}(H^\alpha_\kappa)$ and $t(H^\alpha_\kappa)$, the torsion part of H^α_κ, may be any prescribed κ-cyclic torsion R-module.*

(ii) *if $\phi : H^\alpha_\kappa \to H^{\alpha^1}_{\kappa^1}$ is a homomorphism and $(\alpha, \kappa) \neq (\alpha^1, \kappa^1)$, then ϕ is bounded.*

(iii) *if H is any strongly κ_0-cyclic R-module of cardinality $\kappa_0 < \kappa$, then $\phi : H^\alpha_\kappa \to H$ is bounded.*

PROOF. This follows immediately from Theorem 1.1. given that A is free quâ R-module. □

It follows immediately from Corollary 1.2, that, in $V = L$, essentially-rigid families of maximal size exist (and are composed of "almost" free modules) for all regular not weakly compact cardinals. Similarly using Corollary 1.2(i), (ii) it follows that a proper essentially rigid class (i.e., not a set) of almost free R-modules exists. (We remark that part (iii) of Corollary 1.2 has been used in the proof to establish part (ii) of Corollary 1.2.) It also seems worth observing that when working in ZFC it is only possible to establish the existence of proper essentially semi-rigid classes of R-modules [**1**]. The existence of an essentially-rigid class would be in conflict with the Vopenka-principle (cf. [**12**]).

It is, by now, a well established procedure to use this type of realization result to show the existence of various pathological groups and modules. It is worth noting that the usual choices made for the algebra A to produce such results are such that A is a free quâ R-module. Hence the restriction in Corollary 1.2 is not serious if one wants to show the existence of such pathological modules (cf. [**1**], [**3**].)

As a final application of our result we derive a generalization of Griffith's solution of the Baer problem. Recall that a torsion-free abelian group G is a Baer-group (see [**8**; Vol II, p.189]) if every mixed group M splits when $M/tM \cong G$. Griffith [**13**] showed that Baer-groups are free. Extending the notion of Baer-group to Baer-modules in the obvious way we obtain

THEOREM 1.3. ($V = L$) *If G is a Baer-module which is contained in some $\oplus S^{-1}R$, then G is a submodule of a free R-module.*

PROOF. Suppose G is a Baer-module of cardinality κ. Since κ^+ is automatically regular we may find from (1.1) a strongly κ^+-cyclic R-module H with torsion part tH of cardinality κ. Moreover, $\oplus_\kappa S^{-1}R \subseteq H/tH$ follows from the construction of H and thus G may be embedded in H/tH. Let K denote the preimage of G in H. Since $|K| = \kappa < \kappa^+$ we conclude K is contained in a Σ-cyclic submodule of H containing tH. But, if G is a Baer-module then $K \cong tH \oplus G$, whence G is a torsion-free submodule of a free module. □

REMARK. If R is a domain with quotient field Q countably generated over R, then $S^{-1}R = Q$ for a suitable (countable) S. In this case all torsion-free modules are submodules of some $\oplus S^{-1}R$ and the implication of Theorem 1.3 holds for all S-torsion-free Baer-modules G.

We remark that Griffith's solution of the Baer problem has been recently extended to arbitrary torsion theories over Dedekind domains in [**10**].

Finally observe that the results obtained in this paper are independent of ZFC. This follows since the torsion part of our modules is prescribed; in particular if $T = 0$ then we reduce to the cotorsion-free torsion-free case and this is known to be independent [**3**].

2. Preliminaries. In this section we develop some notation and derive two simple results which are useful for the rest of the paper.

Throughout, R shall be a commutative ring with 1 having a countable multiplicatively closed subset S containing no zero divisors and satisfying the Hausdorff condition $\cap_{s\in S} Rs = 0$. We enumerate the non-units in S as $s_1, s_2, \ldots$ and define q_n in S by $q_n = \prod_{i\leq n} s_i$. Thus $q_n (n < \omega)$ is a null sequence of non-units in S, and if $m \leq n < \omega$ then the fraction q_n/q_m is a well-defined element of S. The S-topology on an R-module M has the countable set of submodules $Ms (s \in S)$ as a basis of neighbourhoods of 0; it is Hausdorff if and only if $\cap_{s\in S} Ms = 0$ or, as we shall also say, if and only if M is S-reduced. The notation $\hat{M}$ is reserved for the S-completion of M and the completion topology on $\hat{M}$ coincides with the S-topology. The notions of S-pure, S-divisible and S-torsion-free are defined in an analogous fashion to that used for Abelian groups. Note that if $\phi : M \to N$ is a homomorphism then ϕ extends uniquely to a homomorphism $\hat{\phi} : \hat{M} \to \hat{N}$. Where no confusion is likely we shall continue to call the unique extension ϕ. Throughout the rest of the paper A shall denote a fixed R-algebra which, quâ R-module, is cotorsion-free; recall that an R-module G is said to be cotorsion-free if it is S-reduced, S-torsion-free and $\mathrm{Hom}(\hat{R}, G) = 0$. (See [**4**] for further details.) Since the set S is fixed throughout we shall often omit the prefix S.

We shall say that B *is a standard mixed* A*-module with a chain of* *summands* $\{B_n\}$ if B can be written in the form

$$B = \oplus_{\substack{n<\omega \\ \alpha \in I_n}} (\alpha, n)A \oplus \oplus_{\substack{n<\omega \\ \beta \in I}} (\beta, n)A,$$

where $\mathrm{Ann}(\alpha, n) = q_n A, \mathrm{Ann}(\beta, n) = 0$ and I_n, I are non-empty index sets (usually infinite) and, moreover, B can be expressed as $\cup_{n<\omega} B_n$ where each B_n is a direct summand of B. We shall also write $B = T \oplus F$ where T is the torsion part of B and F is free. Similarly $B_n = T_n \oplus F_n$.

Let G be a free R-module, $G = \oplus_{i \in I} e_i R$ and $x \in G$. Then the support $[x]$ of x (with respect to the given decomposition for G) is defined by $[x] = \{i \in I | r_i \neq 0 \text{ where } x = \Sigma e_i r_i\}$. Clearly $[x]$ is a finite subset of I. If $y \in \hat{G}$ then it is well known that y may be represented as $y = \Sigma e_i r_i$ where $\{r_i\}$ is a null sequence, and so the support of y, $[y]$, may be similarly defined. In this case $[y]$ is a countable subset of I. More generally, if X is a subset of G we define $[X] = \cup_{x \in X} [x]$.

LEMMA 2.1. *If F is a free A-module with a strictly increasing chain of summands $\{F_n\}$ $(n < \omega), F = \cup_{n<\omega} F_n$ and $\phi : F \to G$ is an unbounded homomorphism from F into a torsion module G, then there are decompositions $F_{n+1} = F_n \oplus D_n$ and basis elements $d_n \in D_n$ $(n < \omega)$ such that $\{d_n \phi\}$ is unbounded.*

PROOF. Consider any decomposition $F_{n+1} = F_n \oplus C_n$ and, since $c_n \neq 0$, write $c_n = c_{no} A \oplus C_n^1$. If we can find elements $c_n \in C_n$ with the properties of the d_n in (2.1) then we select these. If not we conclude, without loss of generality, that $(\oplus C_n)\phi$ is bounded. Since $F = F_o \oplus \oplus_{n<\omega} C_n$ we conclude that $F_o \phi$ is unbounded. Choose basis elements e_n of F_o so that $\{e_n \phi\}$ is unbounded. Now set $d_n = e_n + c_{no}$ and take $D_n = d_n A \oplus C_n^1$. A routine check shows $F_{n+1} = F_n \oplus D_n$ and the elements d_n have the desired properties. □

In the sequel we shall several times use properties of strongly κ-cyclic modules, and so now give a characterization of such modules. Because of Shelah's singular compactness theorem there is nothing to show if κ is singular. Hence we restrict ourselves to regular cardinals.

LEMMA 2.2. *Let κ be a regular uncountable cardinal. Then G is a* *strongly κ-cyclic A-module if and only if $G = \cup_{\alpha<\kappa} G_\alpha$, a κ-filtration, where $|G_\alpha| < \kappa$ and G/G_β is κ-cyclic for all non-limit ordinals β.*

PROOF. Assume G is strongly κ-cyclic and let $G = \cup G^1_\alpha$ be any κ-filtration by sets G^1_α and suppose inductively that $G_\alpha (\alpha < \gamma)$ has been constructed. If $\gamma = \alpha+1$ then, since $|G_\alpha| < \kappa$ we can find $Y \supsetneq G_\alpha \cup G^1_\alpha$ with $|Y| < \kappa, Y$ Σ-cyclic and $G/Y \kappa$-cyclic. Set $G_\gamma = Y$. If γ is a limit, take $G_\gamma = \cup_{\alpha<\gamma} G_\alpha$. Conversely suppose $G = \cup_{\alpha<\kappa} G_\alpha, |G_\alpha| < \kappa, G_\alpha$ is Σ-cyclic and G/G_β is κ-cyclic for all non-limit ordinals β. Now if $X \subseteq G$ with $|X| < \kappa$ then $X \subseteq G_\alpha$, some α. Let $Y = G_{\alpha+1}$. Since $\alpha + 1$ is not a limit we have $X \subseteq Y, |Y| < \kappa, G/Y \kappa$-cyclic. Thus G is strongly κ-cyclic. □

We shall refer to the module G_α in a filtration of a strongly κ-cyclic module G as the α layer of G. Finally we make the following simple observation:

LEMMA 2.3. *If $G_\alpha (\alpha < \kappa)$ is any layer of the strongly κ-cyclic module G then there is a layer $G_\beta \supseteq G_\alpha$ such that G_β is closed in the S-topology on G.*

PROOF.: Since $\alpha+1$ is not a limit, if we set $\beta = \alpha+1$ then $G_\beta \supseteq G_\alpha$ and G/G_β is κ-cyclic. However since cyclic and κ-cyclic A-modules are necessarily reduced, this ensures G_β is closed.

To place the construction of strongly κ-cyclic groups in a more general setting we introduce the notion of almost cotorsion-free (cf. [**9**].)

DEFINITION. A reduced A-module G is said to be *almost cotorsion-free* if every homomorphism $\sigma : \hat{B} \to G$ is bounded where $B = \oplus_{n<\omega} t_n A$ with $\mathrm{Ann}(t_n A) = q_n A$. If we now assume that the base ring R is a countable Dedekind domain then we obtain the characterization

THEOREM 2.4. *Let R be a countable Dedekind domain which is not a field and let $S = R\backslash\{0\}$. If G is a separable R-module then the following are equivalent:*

(1) *G is almost cotorsion-free.*

(2) *Every homomorphism $\sigma : \hat{B} \oplus \hat{R} \to G$ is bounded where $B = \oplus_{n<\omega} t_n R$ with* $\operatorname{Ann} t_n R = q_n R$;

(3) *$\hat{R}_p \subsetneq G$ for any localization R_p* of R.

PROOF. (1) implies (2) since $\hat{R}$ is a direct summand of $\hat{B}$. (2) implies (3) since $\hat{R}_p$ is a complete discrete valuation ring and so we could construct an unbounded map $\hat{B} \oplus \hat{R} \to G$. Finally to see that (3) implies (1) consider any $\phi : \hat{B} \to \hat{G}$ with ϕ unbounded. Then $\hat{B}\phi$ is a cotorsion submodule of the reduced module G. It now follows, since R is Dedekind, that we have $\hat{B}\phi = A \oplus C$, where A is torsion-free algebraically compact and $C = T^{\cdot}$ is the cotorsion hull of $T = t(\hat{B}\phi)$. (cf. [**8**, Theorem 55.5].) If $A \neq 0$ is algebraically compact it contains a copy of $\hat{R}_p$, and so we must conclude that $A = 0$. It now follows, by an analogous argument to that given by Harrison (see [**8**, Theorem 56.5]), that if T is unbounded (and thus $\hat{T}/T \neq 0$) we have $(T^{\cdot})^1 = \operatorname{Hom}(Q/R, \hat{T}/T) \neq 0$ where Q is the field of fractions of R and $(\ )^1$ denotes the first Ulm submodule of $(\ )$. But this is clearly impossible since $T^{\cdot} \subseteq G$ and G is separable. Thus ϕ is bounded.

We note that if R is an incomplete Dedekind domain and G is a κ-cyclic R-module of cardinality $\kappa \geq |R|^{\aleph_0}$ then G is almost cotorsion-free. This follows, since if $\hat{R}_p \subseteq G$ then $|\hat{R}_p| \leq |R|^{\aleph_0} < \kappa$ which would imply that $\hat{R}_p$ is contained in a free R-module. But then $\operatorname{Hom}(\hat{R}, R) \neq 0$ and this is possible if and only if R is complete (cf. [**4**]. Since this was excluded the result follows. □

3. The main algebraic construction. When using the set-theoretic condition $V = L$ to realize endomorphism algebras it is by now standard (cf. [**3**], [**5**], [**11**]) to develop the necessary algebraic tools separately in a series of step-lemmas. In this section we develop two such step-lemmas which will be vital to our construction in the final section of this paper.

STEP-LEMMA A. *Let $\kappa > |A|$ be a regular uncountable cardinal, G a strongly κ-cyclic A-module, B a standard mixed A-module with* *a strictly increasing chain of summands* $\{B_n\}$ *and* $\phi : B \to G$ *an unbounded R-homomorphism. Then there exists an extension* B^1 *of* B *such that*

(i) B^1 *is a* Σ*-cyclic* A*-module;*

(ii) B^1/B *is* S*-divisible and* S*-torsion-free;*

(iii) B_n *is a direct summand of* B^1*;*

(iv) ϕ *does not extend to an* R*-homomorphism* $B^1 \to G$*.*

PROOF. The proof is split into two cases:

Case (i). $F\phi$ *is unbounded.*

(a) $F\phi \subsetneq tG$. In this case we can find a countable rank A-summand F^* of F (say $F = F^* \oplus F^{**}$) such that $F^*\phi \subsetneq tG$. Since $|F^*| < \kappa$ and κ is regular, $F^*\phi$ is contained in some layer G_j of G. As observed in §2 we can find a layer $G_i \supseteq G_j$ such that G_i is closed in the S-topology on G. Set $G_i = T_i \oplus H_i$ where T_i is torsion and H_i is torsion-free. Thus we have a sequence $F^* \xrightarrow{\phi} G_i \xrightarrow{\pi} H_i$. Then, by a similar argument to [**6**, Lemma 2.1], we can find an extension F^1 of F^* with F^1 a free A-module, F^1/F S-divisible S-torsion-free, $F_n^*(= F_n \cap F^*)$ a direct summand of F^1 and such that $\phi\pi$ does not lift to an R-homomorphism $F^1 \to H_i$.

Now set $B^1 = F^1 \oplus T \oplus F^{**}$ and we note that (i), (ii) and (iii) all clearly hold. However if ϕ extends to an R-homomorphism $\phi^1 : B^1 \to G$ then, since F^1/F^* is S-divisible, we can conclude that $B^1\phi^1 \subseteq \overline{G}_i = G_i$. This however is impossible since it then follows that $\phi^1\pi$ restricted to F^1 would extend $\phi\pi$. Thus (iv) also holds.

(b) $F\phi \subseteq tG$. Choose, as is permitted by Lemma 2.1, decompositions $F_{n+1} = F_n \oplus D_n$ and elements $d_n \in D_n$ such that $\{d_n\phi\}$ is unbounded. Set $F^* = \oplus_{n<\omega} d_n A$ and write $F = F^* \oplus F^{**}$. Since $|F^*\phi| < \kappa$ we can find, as in (a) above, a closed layer $G_i = \overline{G}_i$ of G such that $F^*\phi \subseteq G_i \cap tG = T_i$ say. Observe that $T_i \subseteq T_{i+1} = tG_{i+1}$. Since $F^*\phi$ is unbounded we can construct inductively elements $e_i = d_i q_{k_i}$, with k_i strictly increasing, $q_{k_n} = 1$, such that $\mathrm{Ann}\{e_i\phi\}$ form a strictly descending chain and the $e_i\phi$ have pairwise disjoint supports; these supports are calculated in J where $T_{i+1} = \oplus_{j\in J} t_j A$. There will be no loss in generality if we assume that the elements e_i are indexed by

$i \in \omega$.

Now consider $y = \sum_{i<\omega} e_i$, an element of $\hat{F}^*$. Observe that $y\phi \in \hat{T}_i$. However $[y\phi] = \cup[e_i\phi]$, and so $y\phi$ has infinite support (calculated in J.) Thus we conclude $y\phi \in T_{i+1}$. However, $y\phi \in \hat{T}_{i+1}$, so it follows that $y\phi \in G_{i+1}$. But if $y\phi \in G$ then $y\phi \in \hat{T}_i \cap G \subseteq \overline{G}_i = G_i$ by the choice of G_i. This is clearly a contradiction since $G_i \subseteq G_{i+1}$ and so we conclude $y\phi \notin G$.

Define a divisibility chain y_n $(n < \omega)$ for y by $y_0 = y$, $y_n = \sum_{i\geq n} d_i(q_{k_i}/q_{k_n})$ and set $B^1 = \langle F^*, y_nA(n < \omega)\rangle \oplus F^{**} \oplus T$. It is immediate that (ii) and (iv) hold. However by observing that $d_n = y_n - y_{n+1}q_{k_{n+1}}$ one easily obtains that $\langle F^*, y_nA(n < \omega)\rangle = \oplus_{n<\omega} y_nA$. Moreover (iii) will follow if $\oplus_{i\leq n} d_iA$ is a summand of $\langle F^*, y_nA(n < \omega)\rangle$. However a direct calculation shows that $\langle F^*, y_nA(n < \omega)\rangle = \oplus_{i\leq n} d_iA \oplus \oplus_{k>n} y_kA$.

Case (ii). *$F\phi$ is bounded.* Since ϕ is unbounded we deduce that $T\phi$ is unbounded and so there exists an unbounded A-module B^*, a direct summand of T with $B^*\phi \subseteq T_i$ unbounded where T_i is the Σ-cyclic torsion part of some closed layer G_i of G. But now it follows that we can find $b \in \hat{B}^*$ with $b\phi \notin G$; if $\hat{B}^*\phi \subseteq G$ then we would have $\hat{B}^*\phi \subseteq \hat{T}_i \cap G = \overline{T}_i = T_i$, and this is impossible since T_i is Σ-cyclic torsion and $\hat{B}^*\phi$ contains torsion-free elements.

Write $b = \sum_{i<\omega} a_i q_{n_i}$ where $\{a_i\}$ is a basis of B^* and define a divisibility chain for b by $b = b_0, b_k = \sum_{n_i\geq k} a_i(q_{n_i}/q_k)$. Note that $[b_k - b_{k+1}s_{k+1}] = \{a_i | n_i = k\}$ and this is a finite set since the sequence q_{n_i} is Cauchy. Let $N = \{k \in \omega | [b_k - b_{k+1}s_{k+1}] \neq \emptyset\}$ and note that this is an infinite subset of ω since the support of b is infinite. There will be no loss in generality in assuming $N = \omega$. Then, for $k < \omega$, write $e_k = \sum_{n_i=k} a_i$; observe that, by a suitable re-arrangement of the original basis of B^*, we can take e_k as a basis element of B^*. Set $E = \oplus_{k<\omega} e_kA$ and note that E is a summand of B^*, whence of T; $T = E \oplus D$ say. Then $b = \Sigma e_k q_k$ has a "normal" form and we have $b_m - b_{m+1}s_{m+1} = e_m(m < \omega)$.

Consider now $y = \sum_{k<\omega} f_k q_k \in \hat{F}$ where f_k is a free generator of F. Let $F^* = \oplus_{k<\omega} f_kA$ and let $F = F^* \oplus F^{**}$. Now define a divisibility chain y_k for y by $y_o = y, y_n = \sum_{k\geq n} f_k(q_k/q_n)$. If $y\phi \notin G$ set $B^1 = \langle F^*, y_nA(n < \omega)\rangle \oplus F^{**} \oplus T$, and a routine check shows (i) - (iv) hold. If, however, $y\phi \in G$ then set $x = b + y$ and observe $x\phi \notin G$

since $x\phi = y\phi + b\phi$ and $b\phi \notin G$. Let $x_m = b_m + y_m$ and observe that the sum of the modules $x_m A$ is direct since the elements f_m are independent. Now set $B^1 = \langle E \oplus F^*, x_n A(n < \omega)\rangle \oplus D \oplus F^{**}$. Since $\langle E \oplus F^*, x_n A(n < \omega)\rangle = \oplus_{n<\omega} e_n A \oplus \oplus_{n<\omega} x_n A$, B^1 is a Σ-cyclic A-module. Moreover B^1/B is S-divisible and S-torsion-free and ϕ does not extend to an R-homomorphism from B^1 to G since $x\phi \notin G$. Thus it only remains to verify (iii). But observe from the above argument giving B^1 to be Σ-cyclic that T is a direct summand of B^1 (and $tB^1 = T$.) Thus it suffices to verify that F_n is a summand of B^1. If $F_n \subseteq F^{**}$ this is immediate so we can reduce the argument to showing that $\oplus_{i<n} f_i A$ is a summand of $\langle E \oplus F^*, x_n A(n < \omega)\rangle$. However a direct calculation gives $\langle E \oplus F^*, x_n A(n < \omega)\rangle = \oplus_{i<n} e_i A \oplus \oplus_{i<n} f_i A \oplus \oplus_{i>n} x_i A$. □

STEP-LEMMA B. *Let B be a Σ-cyclic mixed A-module of infinite rank having a strictly increasing chain of summands $\{B_n\}$ and let $B^* = B \oplus D$, where D is a free A-module of countable rank or* 0. *Then, if ϕ is an endomorphism of B^* leaving B invariant and ϕ is not an element of $A \oplus BdB$, there exists an extension $B^1 \supseteq B^*$ such that*

(i) *B^1 is Σ-cyclic,*

(ii) *B^1/B^* is S-divisible and S-torsion-free,*

(iii) *B_n is a direct summand of B^1 for all n,*

(iv) $\phi \notin E(B^1)$.

PROOF. Suppose $\phi : B \to B$ is not an element of $A \oplus BdB$. Let $B = T \oplus F$. Denote by ϕ^* the homomorphism induced from ϕ by mapping modulo $T, \phi^* : F \to F$. If ϕ^* is not multiplication by $a \in A$ then, as in [**3**, Corollary 2.8.], there is an extension $F^1 \supseteq F$ which satisfies (i)-(iii), and ϕ^* does not extend to an endomorphism of F^1 since if it did extend to ϕ^1 then $(\phi^1)^*$ would extend ϕ^*. Suppose then, that ϕ^* is multiplication by an element of A. Thus there exists a unique $a \in A$ such that $\phi - a : B \to T$. (The uniqueness is immediate, for if we have a, a^1 then $a - a^1 : B \to T$ which is clearly impossible unless $a - a^1 = 0$.) Set $\psi = \phi - a : B \to T$. Clearly ψ is unbounded. Then, for all $n < \omega$ and $b \in A$, we have $q_n\psi - b \notin BdB$. (This follows immediately from Lemma 3.1. below.) But from Step-Lemma A we deduce that, for each pair (q_k, b), there exists an element $x^{kb} \in \hat{B}$ such that $x^{kb}(q_k\psi - b) \notin B$. Moreover x_n^{kb} is a divisibility chain of x^{kb} and B_n is a direct summand of $\langle B, x_n^{kb}A(n < \omega)\rangle$.

Now pick a free A-module D of countable rank, say $D = \oplus e_i A$, and let $d = \sum_{n<\omega} e_n q_n \in \hat{D}$. Let $\{d_n\}$ be a divisibility chain for d. If $d\psi \notin \langle B \oplus D, d_n A(n < \omega)\rangle$ then setting $B^* = B \oplus D$ and $B^1 = \langle B \oplus D, d_n A(n < \omega)\rangle$ will do since this B^1 will clearly satisfy (i)-(iv).

If $d\psi \in \langle B \oplus D, d_n A(n < \omega)\rangle$ then there exists $a \in A, q_k \in S$ such that $d(q_k\psi - a) \in B \oplus D$. Set $w = d + x^{ka}$ and put $w_n = d_n + x_n^{ka}$. If $w\psi \in \langle B \oplus D, w_n A(n < \omega)\rangle$ then there is $m \in \omega, b \in A$ such that $w(q_m\psi - b) \in B \oplus D$. Without loss of generality we may take $q_m = q_k$ and so obtain $w(q_k\psi - b) \in B \oplus D$. But then we have $w(q_k\psi - b) = (d+x^{ka})(q_k\psi - b) \in B \oplus D$ implying that $d(a-b)+x^{ka}(q_k\psi - b) \in B \oplus D$. Since $d \in \hat{D}$ and $x^{ka}(q_k\psi - b) \in \hat{B}$ this forces $a = b$, and so $x^{ka}(q_k\psi - a) \in B$ - contradiction. Set $B^1 = \langle B \oplus D, w_n A(n < \omega)\rangle$; (ii) and (iv) are then immediate. Moreover (i) follows exactly as in Step-Lemma A while (iii) follows as in Step-Lemma A once we observe that $\langle B \oplus D, w_n A(n < \omega)\rangle = \langle B, x_n^{ka} A(n > \omega) < \oplus < D, d_n A(n < \omega)\rangle$. This completes the proof of Step-Lemma B. □

If the implication of Step-Lemma B holds for $D = 0$ we say that Option I holds; otherwise we will use Step-Lemma B as stated and we say that Option II holds.

LEMMA 3.1. *If $B = T \oplus F$ is a Σ-cyclic A-module and $F \neq 0$ then $A \oplus \mathrm{Bd}\, B$ is S-pure in $E(B)$.*

PROOF. Suppose $s\phi \in A \oplus \mathrm{Bd}$ B where $\phi \in E(B)$. Letting * denote homomorphisms induced modulo $T = tB$ we have that there is an $a \in A$ such that $(s\phi - a)^* = 0$. Thus $s\phi^* = a^* = a$ and if we consider any basis element f of F we get $f(s\phi^*) = fa$ implying $a = sb$ for some $b \in A$. Thus $s(\phi - b) \in \mathrm{Bd}$ B, and since this latter is clearly pure in $E(B)$ we conclude that $\phi - b \in \mathrm{Bd}$ B, whence $\phi \in A \oplus \mathrm{Bd}$ B. Thus $A \oplus \mathrm{Bd}$ B is S-pure in $E(B)$. □

4. Proof of the main theorem. In this section we construct A-

modules with the properties claimed in Theorem 1.1. The construction proceeds inductively with the underlying sets being prepared in a suitable fashion for the $\diamond$-machinery to yield the desired type of A-module. The set-theoretic preliminaries are standard (see [**14**]) and similar constructions have been used previously in [**3**], [**5**], [**7**] and [**11**]. Let κ be the given regular not weakly compact cardinal $> |A|$. Choose any sparse stationary subset $E \subseteq \{\lambda < \kappa \mid cf(\lambda) = \omega\}$. Using Solovay's decomposition theorem we get a partition $E = E_e \cup E_k \cup \cup_{\alpha<\kappa} E_\alpha$ into pairwise disjoint stationary sets $E_\alpha (\alpha \in \Gamma = \kappa \cup \{e, k\})$. Let $H = \cup_{\alpha<\kappa} H_\alpha$ be any κ-filtration of some set H of cardinal κ and let $\mathcal{X}$ denote a family of 2^κ incomparable subsets of κ. Since we are assuming $V = L$ holds we may assume $\diamond_\kappa(E_\alpha)$ for all $\alpha \in \Gamma$ and derive Jensen functions $\{\phi_\alpha : H_\alpha \to H_\alpha, \alpha \in E_e\}$ guessing endomorphisms of H and Jensen sets $\{U_\alpha \subseteq H_\alpha : \alpha \in E_k\}$ guessing kernels of homomorphisms into A-modules of cardinality $\kappa_0 < \kappa$. In addition, we obtain, for each $\gamma < \kappa$, Jensen sets of the form $\{(\phi_\alpha, +_\alpha, \cdot_\alpha) \subseteq H_\alpha^7 \times A : \alpha \in E_\gamma\}$. These latter sets are supposed to guess the additive and scalar multiplicative structure on the H_α and homomorphisms of these modules. (Thus $+_\alpha \subseteq H_\alpha^3, \cdot_\alpha \subseteq H_\alpha \times A \times H_\alpha, \phi_\alpha \subseteq H_\alpha^2$.) Further details of Jensen sets and functions may be found in [**14**, p. 226.]. Suppose that the torsion-module T has a filtration $T = \cup_{\alpha<\kappa} T_\alpha$ where T_α is Σ-cyclic. Such a filtration exists since T is assumed to be strongly κ-cyclic. For each $X \in \mathcal{X}$ and $\alpha < \kappa$ we define inductively an A-module H_α^X with domain H_α, and the desired A-modules will be obtained as $H^X = \cup_{\alpha<\kappa} H_\alpha^X$. The filtration of T may be so arranged that, for $\alpha < \beta < \kappa$, T_α is a direct summand of T_β if $\alpha \notin \cup\{E_\tau : \tau \in X \cup \{e, k\}\}$; let $T_{\beta\alpha}$ be some fixed complement of T_α in T_β. The induction depends on X and proceeds as follows:

(1) $H_0^X = T_0$ and each $H_\alpha^X (\alpha < \kappa)$ is a Σ-cyclic A-module.

(2) If α is a limit then $H_\alpha^X = \cup_{\beta<\alpha} H_\beta^X$.

(3) If $\alpha < \beta < \kappa$ and $\alpha \notin \cup\{E_\tau : \tau \in X \cup \{e'k\}\}$ then H_α^X is a direct summand of H_β^X.

(4) If H_α^X has been defined let (4.0) $H_{\alpha+1}^X = H_\alpha^X \oplus A \oplus T_{\alpha+1,\alpha}$ except in the following cases:

(4.1) If $\alpha \in E_e$ and $\phi_\alpha : H_\alpha^X \to H_\alpha^X$ is a homomorphism not in $A \oplus \mathrm{Bd} H_\alpha^X$ and Option I holds then we choose a sequence α_n in $\alpha \backslash E$ strictly increasing to α. Since $\alpha_n \notin E$ we have by (3) that $\{H_{\alpha_n}^X\}$

forms a chain of direct summands of H_α^X. Now apply Step-Lemma B to obtain an R-module $B^1 = H_{\alpha+1}^X$ extending H_α^X such that ϕ_α does not lift to an endomorphism of $H_{\alpha+1}^X$. Step-Lemma B (iii) ensures that (3) remains satisfied at this new stage, since if $\gamma < \alpha$ and $\gamma \notin \cup E_\tau$ then there exists α_n with $\gamma < \alpha_n$. But then H_γ^X is a summand of $H_{\alpha_n}^X$ which is, in turn, a summand of $H_{\alpha+1}^X$.

(4.2) If $\alpha \in E_k$ and $U_\alpha \subseteq H_\alpha^X$ is a submodule then let $\pi : H_\alpha^X \to H_\alpha^X/U_\alpha$ be the canonical projection. If π is an unbounded homomorphism into a strongly κ_0-cyclic A-module H_α^X/U_α (where $\kappa_0 < \kappa$) then we apply Step-Lemma A to obtain an extension $H_{\alpha+1}^X \supseteq H_\alpha^X$. Moreover if α_n is a strictly increasing sequence with limit α then $H_{\alpha_n}^X$ is a summand of $H_{\alpha+1}^X$ and π does not lift to $H_{\alpha+1}^X$. Step-Lemma A (iii) ensures, in a fashion similar to (4.1) above, that (3) remains satisfied at this stage.

(4.3) If $\alpha \in E_\gamma$ for some $\gamma \in X$ and $(H_\alpha, +_\alpha, \cdot_\alpha)$ is a κ_0-cyclic A-module with $\phi_\alpha : H_\alpha^X \to (H_\alpha, +_\alpha, \cdot_\alpha)$ an unbounded homomorphism, then we construct $H_{\alpha+1}^X$ via Step-Lemma A as in (4.2). Once again (3) is preserved.

As we noted at each stage this inductive construction is consistent, and so we obtain an A-module $H^X = \cup_{\alpha<\kappa} H_\alpha^X$. Since the filtration of T is smooth we also obtain $T = tH^X$. A routine check shows that since T is strongly κ-cyclic so also is H^X. Before establishing the main result we derive the following simple result which is clearly analogous to corresponding results in [**5**] and [**11**].

LEMMA 4.1. *Let $B = \cup_{\alpha<\kappa} B_\alpha$ be a κ-filtration of the mixed A-module B and $\phi : B \to H$ a homomorphism into the A-module H. If $V = \{\nu \in \kappa \mid \phi \restriction B_\nu$ is bounded $\}$ is unbounded in κ and $cf(\kappa) > \omega$ then ϕ is bounded.*

PROOF. For each $\nu \in V$ let $q_{n(\nu)} \in S$ be such that $B_\nu\ \phi\ q_{n(\nu)} = 0$. Clearly if $\{n(\nu) \mid \nu \in V\}$ is bounded then ϕ is bounded. Suppose $\{n(\nu) \mid \nu \in V\}$ is unbounded and choose a strictly increasing sequence $n(\nu_1) < \cdots < n(\nu_r) \cdots$. Since V is unbounded and $cf(\kappa) > \omega$, we can find a $\nu \in V$ with $\nu_i < \nu$ for all i. But $\nu \in V$ implies $\phi \restriction B_\nu$ is bounded, and since $B_{\nu_i} \subseteq B_\nu$ we have $n(\nu_i) \leq n(\nu)$ contrary to the unboundedness of the sequence $n(\nu_i)$. This establishes the lemma. □

Since $\mathcal{X} = 2^\kappa$ and $T = tH^X$ for each $X \in \mathcal{X}$, it suffices to derive the following four results to derive the main Theorem 1.1: For any $X \neq Y \in \mathcal{X}$ we have

(a) $E(H^X) = A \oplus \mathrm{Bd}H^X$.

(b) *If* $\phi : H^X \to H^Y$ *is a homomorphism then* ϕ *is bounded.*

(c) *If* $\phi : H^X \to G$ *is a homomorphism into a strongly* κ_0*-cyclic* A*-module of cardinality* $\kappa_0 < \kappa$ *then* ϕ *is bounded.*

(d) *If* $\kappa < \kappa^1$ *are regular not weakly compact cardinals* $> |A|$ *and* Y^1 *is the indexing set associated with the construction at the cardinal* κ^1, *then any homomorphism* $\phi : H^X \to H^{Y^1}$ *is bounded.*

PROOF. (a). Clearly $A \oplus \mathrm{Bd}H^X \subseteq E(H^X)$. So suppose that there is a $\phi \in E(H^X)\backslash A \oplus \mathrm{Bd}H^X$. Let $C = \{\alpha < \kappa | H^X_\alpha \phi \subseteq H^X_\alpha\}$ and observe that C is a cub in κ. Denote by C_0 and C_1 respectively the sets $\{\alpha \in \kappa \mid \phi \restriction H^X_\alpha \in A \oplus \mathrm{Bd}H^X_\alpha\}$ and $\{\alpha \in C \mid \phi \restriction H^X_\alpha \notin A \oplus \mathrm{Bd}H^X_\alpha$ and Option II holds$\}$. We claim C_0 and C_1 are bounded. C_1 is bounded since the construction in 4.0. ensures that there exists a $\beta > \alpha$ with $\beta \leq |A| < \kappa$ at which Option I may be chosen. If C_0 were unbounded in κ then, for all $\nu \in C_0$, there exists $a_\nu \in A$ such that $(\phi - a_\nu) \restriction H^X_\nu$ is bounded. But then if $\nu < \mu \in C_0$ we would have that $(\phi - a_\nu) \restriction H^X_\nu$ and $(\phi - a_\mu) \restriction H^X_\mu$ are both bounded. However this implies that $a_\nu - a_\mu$ is bounded on H^X_ν which can only happen if $a_\nu = a_\mu$ ($= a$ say.) But then $(\phi - a) \restriction H^X_\nu$ is bounded for all $\nu \in C_0$. It now follows from Lemma 4.1. that $\phi - a$ is bounded, a contradiction. Thus C_0 is also bounded.

Let $C^* = C\backslash(C_0 \cup C_1)$; C^* is still a cub in κ. From $\diamond_\kappa(E_e)$ we have that $D_e = \{\nu \in D_e \mid \phi_\nu = \phi \restriction H^X_\nu\}$ is a stationary set so there exists $\alpha \in D_e \cap C^*$. Thus $\phi_\alpha = \phi \restriction H^X_\alpha : H^X_\alpha \to H^X_\alpha$. By the construction 4.1, ϕ_α does not lift to a homomorphism $\phi^1 : H^X_{\alpha+1} \to H^X_{\alpha+1}$. However Step-Lemma B (ii) gave that $H^X_{\alpha+1}\backslash H^X_\alpha$ is S-divisible, and since $H^X / H^X_{\alpha+1}$ is κ-cyclic we conclude that $H^X_{\alpha+1}$ is the completion of H^X_α in H^X. But this immediately tells us that ϕ extends to a homomorphism $H^X_{\alpha+1} \to H^X_{\alpha+1}$, or, equivalently, $\phi \restriction H^X_\alpha = \phi_\alpha$ extends, a contradiction. Thus no such ϕ exists and we have established that $E(H^X) = A \oplus \mathrm{Bd}H^X$.

(b). Suppose $\phi : H^X \to H^Y$ is unbounded. Then it follows, as in the proof of (a), from Lemma 4.1. that $C = \{\nu < \kappa \mid H_\nu^X \phi \subseteq H_\nu^X, \phi \restriction H_\nu^X$ unbounded$\}$ is a cub in κ. Note that $H_\nu^X = H_\nu^Y$ quâ sets. Let $(+_y, \cdot_y)$ from $H^Y \times H^Y \to H^Y$ and $H^Y \times A \to H^Y$ denote the module structure on H^Y. From $\diamond_\kappa(E_\gamma)$ $(\gamma \in \kappa \backslash Y)$ we see that $W_\gamma = \{\nu \in E_\gamma \mid +_y \restriction (H_\nu \times H_\nu) = +_\nu, \cdot_y \restriction (H_\nu \times A) = \cdot_\nu, \phi \restriction H_\nu = \phi_\nu\}$ is a stationary set in κ because of the choice of Jensen sets in the construction. Since C is a cub we can find $\alpha \in C \cap W_\gamma$. Step-Lemma A (ii) ensures $H_{\alpha+1}^X \backslash H_\alpha^X$ is S-divisible and, as in the proof of (a) above, we conclude that $H_{\alpha+1}^X$ is the closure of H_α^X. Observe that the construction 4.3. implies $\phi \restriction H_\alpha^X : H_\alpha^X \to H_\alpha^X$ does not lift to a homomorphism $H_{\alpha+1}^X \to H_\alpha^Y$. However since $\alpha \in E_\gamma$ and $\gamma \notin Y$ we have from the construction of H^Y that H_α^Y is closed. This immediately implies that $\phi \restriction H_\alpha^X = \phi_\alpha$ does extend to a homomorphism $H_{\alpha+1}^X \to H_\alpha^Y$, a contradiction. This establishes (b).

(c). This is identical to the proof of (b) but using the construction 4.2. rather than 4.3.

(d). If $\phi : H^X \to H^{Y^1}$ is a homomorphism then $|H^X\phi| \leq |H^X| = \kappa < \kappa^1 = |H^{Y^1}|$, we conclude that $H^X\phi$ is Σ-cyclic since H^{Y^1} is κ-cyclic. If $H^X\phi$ is not bounded then we can find a projection onto either a free module or an unbounded direct sum of cyclic modules. In either case these modules are strongly κ-cyclic which contradicts (c). Thus $H^X\phi$ is bounded. □

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