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MORE ON HEIGHTS DEFINED OVER A FUNCTION FIELD

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1. Introduction. When confronted with a Diophantine equation (or set of equations) which has infinitely many solutions, one would like to estimate the number of such solutions with height less than a given bound. Examples of such estimates include quantitative versions of Northcott's theorem and Schanuel's result on the number of points in projective space over a number field.

In [**1**], the authors claim that, in many cases, the number of points of bounded height on certain varieties should grow at a prescribed rate. As evidence towards this, they prove such growth estimates for general flag manifolds. (See also [**4**].) Their method used deep results on the analytic continuation of certain Eisenstein series and dealt with an arbitrary number field. Independently, the author proved asymptotic estimates with explicit error terms for the number of points of bounded height on Grassmannians and flag varieties defined over a number field. The methods used were comparatively elementary, but were subsequently used in [**7**] to prove growth estimates for Schubert varieties–a case not dealt with in the results of [**1, 4**].

In a different but related vein, others, see e.g., [**2, 6**], have considered the number of points of bounded absolute height in projective space $\mathbf{P}^n(\overline{\mathbf{Q}})$ whose field of definition is a fixed (or bounded) degree over a given number field. Recently, Masser and Vaaler in [**3**] gave asymptotic estimates as $B \to \infty$ for the number of $(\rho, 1) \in \overline{\mathbf{Q}}^2$ with $[K(\rho)\colon K] = d$ and absolute height no more than B, where K is a fixed number field.

In this paper, we will use results and techniques developed by the author in [**8**] to prove function field analogs of the two results alluded to above. Specifically, we prove growth estimates and asymptotic estimates for Schubert varieties defined over function fields and also an analog of the result of Masser and Vaaler. In order to state our results, we need to briefly mention some notation. Let K be a finite algebraic extension of the field of rational functions $\mathbf{F}_q(X)$, where X

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is transcendental over the finite field $\mathbf{F}_q$ with q elements. We assume that $\mathbf{F}_q$ is the field of constants. Denote the genus by g and number of divisor classes of degree 0 by J. Let ζ_k denote the zeta function of K. For any $n > 1$, set

$$a(n) := \frac{Jq^{n(1-g)}}{(q-1)\zeta_k(n)}.$$

All implicit constants to follow depend only on K and either n or d, as appropriate. We define the heights used in Sections 1 and 3 below.

Theorem 1. *Set $\kappa = [K\colon \mathbf{F}_q(X)]$, and fix an algebraic closure $\overline{K}$ of K. For $d \geq 1$ and $m \geq 0$, let $\overline{N}(d,m)$ denote the number of $\rho \in \overline{K}$ of degree d over K with absolute height $\overline{h}(\rho, 1) = m$. Then*

$$\overline{N}(d,m) = da(d+1)q^{md\kappa(d+1)} + \begin{cases} O(q^{md^2\kappa}) & \text{if } d \neq 2, \\ O(m\kappa q^{m4\kappa}) & \text{if } d = 2. \end{cases}$$

For integers $0 < d < n$, let $c(n,d)$ denote the set of order increasing d-tuples of integers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_d \leq n$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the canonical basis vectors for K^n, and let K^d be the subspace spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_d$ for any $d \leq n$.

Definition. For $\boldsymbol{\alpha} \in c(n,d)$, the *Schubert variety* associated with $\boldsymbol{\alpha}$ is the set of d-dimensional subspaces $S \subseteq K^n$ which satisfy

$$\dim_K(S \cap K^{\alpha_i}) \geq i \quad \text{for } 1 \leq i \leq d.$$

We denote the number of such subspaces S with height $h(S) = m$ by $N(\boldsymbol{\alpha}, m)$.

Rather than give estimates for $N(\boldsymbol{\alpha}, m)$ directly, we deal with "cells" of the Schubert variety.

Definition. Let $N'(\boldsymbol{\alpha}, m)$ denote the number of subspaces S counted in $N(\boldsymbol{\alpha}, m)$ which also satisfy

$$\dim_K(S \cap K^{\alpha_i}) = i > \dim_K(S \cap K^{\alpha_i - 1}) \quad \text{for } 1 \leq i \leq d.$$

We write $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ for $\boldsymbol{\beta}, \boldsymbol{\alpha} \in c(n,d)$ to mean $\beta_i \leq \alpha_i$ for $1 \leq i \leq d$. One easily checks that

$$N(\boldsymbol{\alpha}, m) = \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} N'(\boldsymbol{\beta}, m). \tag{1}$$

For $\boldsymbol{\alpha} \in c(n,d)$ we set

$$\begin{aligned} c_1(\boldsymbol{\alpha}) &= \max_{1 \leq i \leq d} \{\alpha_i - 2i\} + d + 1 \\ c_2(\boldsymbol{\alpha}) &= \#\{i: \ c_1(\boldsymbol{\alpha}) = \alpha_i - 2i + d + 1\} \\ c_3(\boldsymbol{\alpha}) &= \max_{1 \leq i \leq d} \{i\colon c_1(\boldsymbol{\alpha}) = \alpha_i - 2i + d + 1\}. \end{aligned}$$

Conjecture. *Let $0 < d < n$ and $\boldsymbol{\alpha} \in c(n,d)$ with $\alpha_1 > 1$. Then, for some positive $a(\boldsymbol{\alpha})$ depending only on K and $\boldsymbol{\alpha}$,*

$$N'(\boldsymbol{\alpha}, m) = a(\boldsymbol{\alpha}) q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1} + \begin{cases} O\big(q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-2}\big) & \text{if } c_2(\boldsymbol{\alpha}) > 1, \\ O\big(q^{m(c_1(\boldsymbol{\alpha})-1)} m^{d-1}\big) & \text{otherwise.} \end{cases}$$

Note that if $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ then $c_1(\boldsymbol{\beta}) \leq c_1(\boldsymbol{\alpha})$, and if these two constants are equal, then $c_2(\boldsymbol{\beta}) \leq c_2(\boldsymbol{\alpha})$ as well. Thus, by (1), proving the conjecture would imply a similar asymptotic estimate for $N(\boldsymbol{\alpha}, m)$ as well. Also, if $\alpha_1 = 1$, it is a simple matter to see that $N(\boldsymbol{\alpha}, m) = N(\boldsymbol{\beta}, m)$, where $\boldsymbol{\beta} = (\alpha_2 - 1, \ldots, \alpha_d - 1)$ (see Lemma 0 below). Thus, the hypothesis $\alpha_1 > 1$ is harmless. We will prove the conjecture for a wide variety of $\boldsymbol{\alpha}$'s. For $d > 1$ and $\boldsymbol{\alpha} \in c(n,d)$, it will be convenient in what follows to denote $(\alpha_1, \ldots, \alpha_{d-1})$ by $\boldsymbol{\alpha}'$.

Theorem 2. *Let $0 < d < n$ and $\boldsymbol{\alpha} \in c(n,d)$ with $\alpha_1 > 1$. Suppose $c_3(\boldsymbol{\alpha}) = d$. If $c_2(\boldsymbol{\alpha}) = 1$, then the conjecture holds for $\boldsymbol{\alpha}$ with $a(\boldsymbol{\alpha}) = a(\alpha_1)$ if $d = 1$ and*

$$a(\boldsymbol{\alpha}) = a(\alpha_d - d + 1) \sum_{j=0}^{\infty} N'(\boldsymbol{\alpha}', j) q^{j(d-\alpha_d)}$$

if $d > 1$. If $c_2(\boldsymbol{\alpha}) > 1$ (so that $d > 1$) and the conjecture holds for $\boldsymbol{\alpha}'$, then it holds for $\boldsymbol{\alpha}$ with

$$a(\boldsymbol{\alpha}) = a(\alpha_d - d + 1)\frac{a(\boldsymbol{\alpha}')}{c_2(\boldsymbol{\alpha}')}.$$

Implicit in the statement of Theorem 2 is that the sum $\sum_{j=0}^{\infty} N'(\boldsymbol{\alpha}', j) q^{j(d-\alpha_d)}$ converges. That is indeed the case; it follows from one of the inequalities in Theorem 5 below (see Proposition 2 in Section 2).

Theorem 3. *Let $0 < d < n$ and $\boldsymbol{\alpha} \in c(n,d)$ with $\alpha_1 > 1$. Suppose $c_3(\boldsymbol{\alpha}) < d$ and the conjecture holds for $\boldsymbol{\alpha}'$. If the genus $g = 0$, then the conjecture holds for $\boldsymbol{\alpha}$. If $\alpha_{i+1} = \alpha_i + 1$ for all i with $d-1 > i \geq c_3(\boldsymbol{\alpha}')$, then the conjecture holds for $\boldsymbol{\alpha}$.*

Theorem 4. *If the genus of K is 0, then the conjecture holds.*

Theorem 5. *Let $0 < d < n$ and $\boldsymbol{\alpha} \in c(n,d)$ with $\alpha_1 > 1$. Then there is a $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ with $c_1(\boldsymbol{\alpha}) = c_1(\boldsymbol{\beta})$ and $c_2(\boldsymbol{\alpha}) = c_2(\boldsymbol{\beta})$ for which the conjecture holds, and*

$$N'(\boldsymbol{\alpha}, m) \gg\ll q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}.$$

I. Definitions and notation. Here we will define the heights to be used throughout and also introduce some auxiliary definitions and notation. (We postpone the definition of absolute height to Section 3, however.) In addition to the notation established above, we will write $M(K)$ for the set of places of K and $K_{\mathbf{A}}$ for the adele ring. For a place $v \in M(K)$, we let K_v denote the topological completion of K at v and let ord_v be the order function on K_v, normalized to have image $\mathbf{Z} \cup \{\infty\}$. We extend ord_v to K_v^n by defining

$$\mathrm{ord}_v(x_1, \ldots, x_n) = \min_{1 \leq i \leq n} \mathrm{ord}_v(x_i).$$

For any $\mathbf{x} = (\mathbf{x}_v) \in K_{\mathbf{A}}{}^n$ with $\mathrm{ord}_v(\mathbf{x}_v) \in \mathbf{Z}$ for all places v and with $\mathrm{ord}_v(\mathbf{x}_v) = 0$ for all but finitely many places, we get a divisor

$$\mathrm{div}\,(\mathbf{x}) := \sum_{v \in M(K)} \mathrm{ord}_v(\mathbf{x}_v) \cdot v.$$

Thus, for any nonzero $\mathbf{x} \in K^n$ and $A \in \mathrm{GL}_n(K_{\mathbf{A}})$ we have a divisor $\mathrm{div}\,(A\mathbf{x})$ and the additive and multiplicative heights

$$h_A(\mathbf{x}) := -\deg \mathrm{div}\,(A\mathbf{x}), \qquad H_A(\mathbf{x}) := q^{h_A(\mathbf{x})}.$$

Since the degree of a principal divisor is 0, one sees that these heights are actually functions on projective $(n-1)$-space.

These heights are extended to arbitrary subspaces of K^n via Grassmann coordinates. Specifically, suppose $1 \le d \le n$, and $S \subseteq K^n$ is a d-dimensional subspace with basis $\mathbf{x}_1, \dots, \mathbf{x}_d$. Then $\mathbf{X} = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d \in K^{\binom{n}{d}}$, and we define

$$h_A(S) := h_{\bigwedge^d A}(\mathbf{X}) = -\deg \mathrm{div}\,(A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_d), \quad H_A(S) := q^{h_A(S)}.$$

Note that $h_A(K^n) = -\deg \mathrm{div} \det(A)$. We define $h_A(\{\mathbf{0}\}) = 0$. The case where $A = I_n$, the identity element of $\mathrm{GL}_n(K_{\mathbf{A}})$, gives the usual "untwisted" heights, which will be simply denoted by h and H without the subscript.

For $A \in \mathrm{GL}_n(K_{\mathbf{A}})$ the successive minima $\mu_1(A) \le \cdots \le \mu_n(A)$ are

$$\mu_i(A) := \min\{m \colon K^n \text{ contains } i \text{ linearly independent } x \text{ with } h_A(\mathbf{x}) \le m\}$$

for $1 \le i \le n$.

Throughout this paper we will use capital script German letters to denote divisors: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc., and simply use 0 to denote the zero divisor. We say a divisor $\mathfrak{A}$ is nonnegative and write $\mathfrak{A} \ge 0$ if $\mathrm{ord}_v(\mathfrak{A})$ is nonnegative for all places $v \in M(K)$. More generally, we write $\mathfrak{A} \ge \mathfrak{B}$ if $\mathfrak{A} - \mathfrak{B} \ge 0$. We let μ denote the Möbius function on nonnegative divisors; μ is defined by $\mu(0) = 1$, $\mu(\mathfrak{P}) = -1$ for prime divisors $\mathfrak{P}$, $\mu(m\mathfrak{P}) = 0$ if $m > 1$, and $\mu(\mathfrak{A} + \mathfrak{B}) = \mu(\mathfrak{A}) \cdot \mu(\mathfrak{B})$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ have disjoint support.

2. Schubert cells. We start by recalling some pertinent results from [**8**] and proving a few preparatory lemmas.

For $A \in \mathrm{GL}_n(K_{\mathbf{A}})$, set $N(A,1,m)$ equal to the number of one-dimensional subspaces $K\mathbf{x} \in K^n$ with height $h_A(\mathbf{x}) = m$. The first result follows from [**8**, Theorem 1].

Theorem 0. *Suppose $n \geq 2$ and $A \in \mathrm{GL}_n(K_{\mathbf{A}})$. Then, for all $m \geq \mu_n(A)$,*

$$N(A,1,m) = a(n)\frac{q^{nm}}{H_A(K^n)} + O\bigg(\frac{q^{m-\mu_n(A)}}{q^{-n\mu_n(A)}H_A(K^n)}\bigg).$$

We will also use the following from [**8**, Lemmas 8a and 8b] which shows how our heights behave when restricting to subspaces and when looking at factor spaces.

Lemma 0. *Let $A \in \mathrm{GL}_n(K_{\mathbf{A}})$ and $1 \leq d < n$. Let $S \subset K^n$ be a d-dimensional subspace, and choose a basis $\mathbf{x}_1,\dots,\mathbf{x}_n$ of K^n such that $\mathbf{x}_1,\dots,\mathbf{x}_d$ is a basis for S. There are $A' \in \mathrm{GL}_d(K_{\mathbf{A}})$ and $A'' \in \mathrm{GL}_{n-d}(K_{\mathbf{A}})$ such that the height $h_{A'}$ on S with respect to the basis $\mathbf{x}_1,\dots,\mathbf{x}_d$ is equal to h_A and the height $h_{A''}$ on K^n/S with respect to the basis $\mathbf{x}_{d+1}+S,\dots,\mathbf{x}_n+S$ satisfies $h_{A''}(V/S) = h_A(V) - h_A(S)$ for all subspaces $V \supseteq S$.*

We will need the following three simple results.

Lemma 1. *Let $S \subset K^n$ be a d-dimensional subspace, where $0 \leq d < n$. Then, for any canonical basis vector $\mathbf{e} \notin S$, we have*

$$h(S \oplus K\mathbf{e}) \leq h(S).$$

Proof. This is obvious if $S = \{\mathbf{0}\}$, so assume $d > 0$ and let $\mathbf{x}_1,\dots,\mathbf{x}_d$ be a basis for S. Clearly the components of $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d \wedge \mathbf{e}$ are, up to sign, also components of $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d$. Thus, for any place $v \in M(K)$, we have

$$\mathrm{ord}_v(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d \wedge \mathbf{e}) \geq \mathrm{ord}_v(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d),$$

so that $h(S \oplus K\mathbf{e}) \leq h(S)$. □

Lemma 2. *Let S be as in Lemma 1, and let $B \in \mathrm{GL}_{n-d}(K_{\mathbf{A}})$ give the height on K^n/S as in Lemma 0. Then $\mu_{n-d}(B) \leq 0$.*

Proof. Let $\mathbf{x}_1, \ldots, \mathbf{x}_d$ be a basis for S, and let $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-d}}$ be canonical basis vectors such that $\mathbf{x}_1, \ldots, \mathbf{x}_d, \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-d}}$ is a basis for K^n. Then $\mathbf{e}_{i_j} + S \in K^n/S$ are linearly independent for $j = 1, \ldots, n-d$, and by Lemma 1,

$$h_B(\mathbf{e}_{i_j} + S) = h(S \oplus K\mathbf{e}_{i_j}) - h(S) \leq 0$$

for all j. □

Lemma 3. *Let $T \subseteq K^{n-1}$ be a $(d-1)$-dimensional subspace where $1 \leq d \leq n$. Then for any $\mathbf{x} \in K^n \setminus K^{n-1}$, we have*

$$h(T \oplus K\mathbf{x}) \geq h(T).$$

Proof. This is clear if $d = 1$, so assume $d > 1$, and let $\mathbf{x}_1, \ldots, \mathbf{x}_{d-1}$ be a basis for T. Let $\mathbf{x} \in K^n \setminus K^{n-1}$; without loss of generality, we may assume $\mathbf{x} = \mathbf{e}_n + \mathbf{v}$, where $\mathbf{v} \in K^{n-1}$. Then, up to sign, every component of $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_{d-1}$ is a component of $\mathbf{e}_n \wedge \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_{d-1}$, hence is also a component of $\mathbf{x} \wedge \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_{d-1}$. As in the proof of Lemma 1, this shows that $h(T \oplus K\mathbf{x}) \geq h(T)$. □

We now turn to estimating the number of subspaces of given height. We use just a little more notation. For $A \in \mathrm{GL}_n(K_{\mathbf{A}})$ and a divisor $\mathfrak{A}$, let

$$\begin{aligned} L(\mathfrak{A}, A) &:= \{\mathbf{x} \in K^n \colon \mathrm{ord}_v(A_v\mathbf{x}) \geq -\mathrm{ord}_v(\mathfrak{A}) \text{ for all } v \in M(K)\} \\ L^{\mathrm{inh}}(\mathfrak{A}, A) &:= \{\mathbf{x} \in K^n \colon \min\{\mathrm{ord}_v(A_v\mathbf{x}), 0\} \\ &\qquad\qquad \geq -\mathrm{ord}_v(\mathfrak{A}) \text{ for all } v \in M(K)\} \end{aligned}$$

$$L^{\mathrm{inh}'}(\mathfrak{A}, A) := \{\mathbf{x} \in K^n \colon \min\{\mathrm{ord}_v(A_v\mathbf{x}), 0\} \\ = -\mathrm{ord}_v(\mathfrak{A}) \text{ for all } v \in M(K)\}$$

and denote their cardinalities by $\lambda(\mathfrak{A}, A)$, $\lambda^{\mathrm{inh}}(\mathfrak{A}, A)$ and $\lambda^{\mathrm{inh}'}(\mathfrak{A}, A)$, respectively. As discussed in [**8**], $L(\mathfrak{A}, A)$ is a vector space of finite dimension $l(\mathfrak{A}, A)$ over $\mathbf{F}_q$, so that $\lambda(\mathfrak{A}, A) = q^{l(\mathfrak{A}, A)}$. Clearly,

$$q^{l(\mathfrak{A}, A)} = \lambda(\mathfrak{A}, A) = \lambda^{\mathrm{inh}}(\mathfrak{A}, A) \text{ for } \mathfrak{A} \geq 0. \tag{2}$$

Lemma 4 [**8**, Lemma 3]. *Let $A \in \mathrm{GL}_n(K_{\mathbf{A}})$, and let $\mathfrak{A}$ be a nonnegative divisor. Then*

$$\sum_{0 \leq \mathfrak{C} \leq \mathfrak{A}} \mu(\mathfrak{C}) \lambda^{\mathrm{inh}}(\mathfrak{A} - \mathfrak{C}, A) = \lambda^{\mathrm{inh}'}(\mathfrak{A}, A).$$

Lemma 5. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in c(n, d)$ with $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$. Then $N'(\boldsymbol{\alpha}, m) \geq N'(\boldsymbol{\beta}, m)$ for all m.*

Proof. For $1 < i < d$, let $\boldsymbol{\alpha}_i = (\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_d)$, let $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_d = \boldsymbol{\beta}$. Then $\boldsymbol{\alpha}_0 \geq \boldsymbol{\alpha}_1 \geq \cdots \geq \boldsymbol{\alpha}_d = \boldsymbol{\beta}$. For all $i = 1, \ldots, d$, the permutation of K^n given by $\mathbf{e}_{\alpha_i} \leftrightarrows \mathbf{e}_{\beta_i}$ takes all subspaces counted in $N'(\boldsymbol{\alpha}_i, m)$ to subspaces counted in $N'(\boldsymbol{\alpha}_{i-1}, m)$ in a one-to-one fashion. Thus,

$$N'(\boldsymbol{\alpha}, m) = N'(\boldsymbol{\alpha}_0, m) \geq N'(\boldsymbol{\alpha}_1, m) \geq \cdots \geq N'(\boldsymbol{\alpha}_d, m) = N'(\boldsymbol{\beta}, m). \quad \square$$

Proposition 1. *Let $T \subset K^{n-1}$ be a $(d-1)$-dimensional subspace, where $1 \leq d < n$, and let $m \geq 0$. Then the number N of d-dimensional subspaces $S \subset K^n \setminus K^{n-1}$ with $S \supset T$ and $h(S) = m + h(T)$ satisfies*

$$N = a(n-d+1) q^{m(n-d+1)} H(T) + O\big(q^{m(n-d)} H(T)\big).$$

Proof. Let $B \in \mathrm{GL}_{n-d+1}(K_{\mathbf{A}})$ and $B^- \in \mathrm{GL}_{n-d}(K_{\mathbf{A}})$ give the heights on K^n/T and K^{n-1}/T, respectively. Then $N = N(B, 1, m)$

$- N(B^-, 1, m)$. By Lemma 2, $\mu_{n-d+1}(B), \mu_{n-d}(B^-) \le 0$. Further, $h_B(K^{n-d+1}) = h(K^n) - h(T) = -h(T)$ and $h_{B^-}(K^{n-d}) = h(K^{n-1}) - h(T) = -h(T)$. The proposition thus follows from Theorem 0. □

Proposition 2. *Let $1 \le d < n$ and $m \ge 0$. Then, for all $\boldsymbol{\alpha} \in c(n,d)$ with $\alpha_1 > 1$, we have*

$$N'(\boldsymbol{\alpha}, m) \ll q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}.$$

Proof. We prove this by induction on d. In the case $d = 1$, we apply Proposition 1 with $n = \alpha_1$ and $T = \{\mathbf{0}\}$ to get

$$N'(\alpha_1, m) = a(\alpha_1) q^{m\alpha_1} + O\big(q^{m(\alpha_1 - 1)}\big). \tag{3}$$

Since $c_1(\boldsymbol{\alpha}) = \alpha_1$ and $c_2(\boldsymbol{\alpha}) = 1$ here, this proves the case $d = 1$.

Now assume $d > 1$ and the proposition is true for $d-1$. If S is counted in $N'(\boldsymbol{\alpha}, m)$, then $S = T \oplus K\mathbf{x}$ where $T = S \cap K^{\alpha_{d-1}}$ is counted in $N'(\boldsymbol{\alpha}', j)$ for some j and $\mathbf{x} \in K^{\alpha_d} \setminus K^{\alpha_d - 1}$ is unique modulo T. By Lemma 3, $h(S) = m \ge h(T)$. Hence, by Proposition 1,

$$\begin{aligned} &N'(\boldsymbol{\alpha}, m) \\ &= \sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j) \Big[a(\alpha_d - d + 1) q^{(m-j)(\alpha_d - d + 1)} q^j + O\big(q^{(m-j)(\alpha_d - d)} q^j\big) \Big]. \end{aligned} \tag{4}$$

Applying the induction hypothesis to $\boldsymbol{\alpha}'$, we get

$$N'(\boldsymbol{\alpha}, m) \ll q^{m(\alpha_d - d + 1)} \sum_{j=0}^{m} q^{j(c_1(\boldsymbol{\alpha}') - \alpha_d + d)} j^{c_2(\boldsymbol{\alpha}') - 1}. \tag{5}$$

There are three possibilities. First, if $\alpha_d - d > c_1(\boldsymbol{\alpha}')$, then $c_1(\boldsymbol{\alpha}) = \alpha_d - d + 1$ and $c_2(\boldsymbol{\alpha}) = 1$. In this case, using $c_1(\boldsymbol{\alpha}') - \alpha_d + d - 1 \le -1$ in (5) gives

$$\begin{aligned} N'(\boldsymbol{\alpha}, m) &\ll q^{mc_1(\boldsymbol{\alpha})} \sum_{j=0}^{m} q^{-j} j^{d-2} \\ &\ll q^{mc_1(\boldsymbol{\alpha})}. \end{aligned}$$

Second, if $\alpha_d - d = c_1(\boldsymbol{\alpha}')$, then $c_1(\boldsymbol{\alpha}) = \alpha_d - d + 1$, $c_2(\boldsymbol{\alpha}) = c_2(\boldsymbol{\alpha}') + 1$, and by (5),

$$\begin{aligned} N'(\boldsymbol{\alpha}, m) &\ll q^{mc_1(\boldsymbol{\alpha})} \sum_{j=0}^{m} j^{c_2(\boldsymbol{\alpha})-2} \\ &\ll q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}. \end{aligned}$$

Finally, if $\alpha_d - d < c_1(\boldsymbol{\alpha}')$, then $c_1(\boldsymbol{\alpha}) = c_1(\boldsymbol{\alpha}') + 1$, $c_2(\boldsymbol{\alpha}) = c_2(\boldsymbol{\alpha}')$ and by (5)

$$\begin{aligned} N'(\boldsymbol{\alpha}, m) &\ll q^{m(\alpha_d - d + 1)} \sum_{j=0}^{m} q^{j(c_1(\boldsymbol{\alpha}') - \alpha_d + d)} j^{c_2(\boldsymbol{\alpha}')-1} \\ &\le q^{m(\alpha_d - d + 1)} m^{c_2(\boldsymbol{\alpha})-1} \sum_{j=0}^{m} q^{j(c_1(\boldsymbol{\alpha}') - \alpha_d + d)} \\ &\ll q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}. \qquad \square \end{aligned}$$

With these results in hand, we can now prove one of our main results.

Proof of Theorem 2. The case $d = 1$ is (3), so assume that $d > 1$ and suppose first that $c_2(\alpha) = 1$. Then, since $c_3(\boldsymbol{\alpha}) = d$, we must have $c_1(\boldsymbol{\alpha}) = \alpha_d - d + 1 > c_1(\boldsymbol{\alpha}') + 1$. By Proposition 2,

$$\sum_{j=0}^{\infty} N'(\boldsymbol{\alpha}', j) q^{j(d-\alpha_d)}$$

converges; call this sum C. Also by Proposition 2,

$$\sum_{j>m} N'(\boldsymbol{\alpha}', j) q^{j(d-\alpha_d)} \ll q^{-m} m^{c_2(\boldsymbol{\alpha}')-1} < q^{-m} m^{d-1}$$

and

$$\sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j) q^{j(1+d-\alpha_d)} \ll \sum_{j=0}^{m} j^{c_2(\boldsymbol{\alpha}')-1} \ll m^{d-1}.$$

Theorem 2 follows from (4) and these estimates.

Now suppose $c_2(\boldsymbol{\alpha}) > 1$. Then $c_1(\boldsymbol{\alpha}) = \alpha_d - d + 1 = c_1(\boldsymbol{\alpha}') + 1$ and $c_2(\boldsymbol{\alpha}) = c_2(\boldsymbol{\alpha}') + 1$. Assuming the conjecture holds for $\boldsymbol{\alpha}'$, if $c_2(\boldsymbol{\alpha}') = 1$ we have

$$\begin{aligned}\sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j) q^{j(d-\alpha_d)} &\\ &= \sum_{j=0}^{m} a(\boldsymbol{\alpha}') q^{j(c_1(\boldsymbol{\alpha}')+d-\alpha_d)} + O\big(q^{j(c_1(\boldsymbol{\alpha}')+d-\alpha_d-1)} j^{d-2}\big) \\ &= \sum_{j=0}^{m} a(\boldsymbol{\alpha}') + O(q^{-j} j^{d-2}) \\ &= a(\boldsymbol{\alpha}') m + O(1)\end{aligned}$$

and

$$\sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j) q^{j(1+d-\alpha_d)} \ll \sum_{j=0}^{m} q^j \ll q^m.$$

If $c_2(\boldsymbol{\alpha}') > 1$, we have

$$\begin{aligned}\sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j) q^{j(d-\alpha_d)} &= \sum_{j=0}^{m} a(\boldsymbol{\alpha}') q^{j(c_1(\boldsymbol{\alpha}')+d-\alpha_d)} j^{c_2(\boldsymbol{\alpha}')-1} \\ &\quad + O\big(q^{j(c_1(\boldsymbol{\alpha}')+d-\alpha_d)} j^{c_2(\boldsymbol{\alpha}')-2}\big) \\ &= \sum_{j=0}^{m} a(\boldsymbol{\alpha}') j^{c_2(\boldsymbol{\alpha}')-1} + O(j^{c_2(\boldsymbol{\alpha}')-2}) \\ &= \frac{a(\boldsymbol{\alpha}')}{c_2(\boldsymbol{\alpha}')} m^{c_2(\boldsymbol{\alpha}')} + O\big(m^{c_2(\boldsymbol{\alpha}')-1}\big)\end{aligned}$$

and

$$\sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j) q^{j(1+d-\alpha_d)} \ll \sum_{j=0}^{m} q^j j^{c_2(\boldsymbol{\alpha}')-1} \ll q^m m^{c_2(\boldsymbol{\alpha}')-1}.$$

Theorem 2 in these cases follows from (4) and these estimates. □

To prove the remaining results dealing with Schubert cells, we need the following.

Proposition 3. *Let $\boldsymbol{\alpha} \in c(n,d)$ where $0 < d < n$. Suppose $\alpha_1 > 1$, and suppose further that $\alpha_{i+1} = \alpha_i + 1$ for all i with $d > i \geq c_3(\boldsymbol{\alpha})$. Let $V \subset K^{\alpha_d}$ be an $(\alpha_d - 1)$-dimensional subspace and denote by $N'(\boldsymbol{\alpha}, m, V)$ the number of subspaces $S \subseteq V$ which are counted in $N'(\boldsymbol{\alpha}, m)$. Then, for all $m \geq 0$,*

$$N'(\boldsymbol{\alpha}, m, V) \ll \begin{cases} q^{m(c_1(\boldsymbol{\alpha})-1)} m^{d-1} & \text{if } c_2(\boldsymbol{\alpha}) = 1, \\ q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-2} & \text{otherwise.} \end{cases}$$

Proof. Clearly $N'(\boldsymbol{\alpha}, m, V) \leq N'(\boldsymbol{\alpha}, m)$. Suppose $d = 1$, and let $A \in \mathrm{GL}_{\alpha_1 - 1}(K_{\mathbf{A}})$ give the height on V as in Lemma 0. Then $\mu_1(A) \geq 0$ and [**8**, Theorem 1] imply that

$$N'(\boldsymbol{\alpha}, m) \leq N(A, 1, m) \ll q^{m(\alpha_1 - 1)} = q^{m(c_1(\boldsymbol{\alpha})-1)}.$$

We will now assume $d > 1$. Arguing as in the proof of Proposition 2, every S counted in $N'(\boldsymbol{\alpha}, m, V)$ can be written $S = T \oplus K\mathbf{x}$, where T is counted in $N'(\boldsymbol{\alpha}', j, V \cap K^{\alpha_{d-1}})$ for some j and $\mathbf{x}$ is unique modulo T. By Lemma 3, $m = h(S) \geq h(T)$, so we have $j \leq m$. For such a subspace T, let $A(T) \in \mathrm{GL}_{\alpha_d - d}(K_{\mathbf{A}})$ give the height on V/T as in Lemma 0. We then have

$$N'(\boldsymbol{\alpha}, m, V) \leq \sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j, V \cap K^{\alpha_{d-1}}) N(A(T), 1, m - j).$$

For any subspace $W/T \subseteq V/T$, we have $h_{A(T)}(W/T) = h(W) - h(T) \geq -h(T)$. This together with [**8**, Theorem 1] shows that $N(A(T), 1, j) \ll q^{j(\alpha_d - d)} H(T)$. Thus,

$$N'(\boldsymbol{\alpha}, m, V) \leq \sum_{j=0}^{m} N'(\boldsymbol{\alpha}', j, V \cap K^{\alpha_{d-1}}) q^{(m-j)(\alpha_d - d) + j}. \tag{6}$$

We now proceed by induction on $d - c_3(\boldsymbol{\alpha}) + 1$. Suppose first that $d - c_3(\boldsymbol{\alpha}) + 1 = 1$, i.e., $c_1(\boldsymbol{\alpha}) = \alpha_d - d + 1$. Since $N'(\boldsymbol{\alpha}', j, V \cap K^{\alpha_{d-1}}) \leq N'(\boldsymbol{\alpha}', j)$, Proposition 2 and (6) give us

$$N'(\boldsymbol{\alpha}, m, V) \ll \sum_{j=0}^{m} q^{m(\alpha_d - d)} q^{j(c_1(\boldsymbol{\alpha}') - \alpha_d + d + 1)} j^{c_2(\boldsymbol{\alpha}') - 1}.$$

In the case $c_2(\boldsymbol{\alpha}) = 1$, we must have $\alpha_d - d > c_1(\boldsymbol{\alpha}')$, so

$$\begin{aligned} N'(\boldsymbol{\alpha}, m, V) &\ll q^{m(\alpha_d - d)} \sum_{j=0}^{m} j^{c_2(\boldsymbol{\alpha}')-1} \\ &\ll q^{m(\alpha_d - d)} m^{c_2(\boldsymbol{\alpha}')} \\ &\le q^{m(c_1(\boldsymbol{\alpha})-1)} m^{d-1}. \end{aligned}$$

In the case $c_2(\boldsymbol{\alpha}) > 1$ we must have $\alpha_d - d = c_1(\boldsymbol{\alpha})$, so

$$\begin{aligned} N'(\boldsymbol{\alpha}, m, V) &\ll q^{m(\alpha_d - d)} \sum_{j=0}^{m} q^j j^{c_2(\boldsymbol{\alpha}')-1} \\ &\ll q^{m(\alpha_d - d + 1)} m^{c_2(\boldsymbol{\alpha}')-1} \\ &= q^{m c_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-2}. \end{aligned}$$

Now assume $d - c_3(\boldsymbol{\alpha}) + 1 > 1$. Then we may use the induction hypothesis on $N'(\boldsymbol{\alpha}', j, V \cap K^{\alpha_{d-1}})$ provided that the dimension of $V \cap K^{\alpha_{d-1}}$ is $\alpha_{d-1} - 1$. Now we have $\alpha_d = \alpha_{d-1} + 1$, so either this is the case or $V = K^{\alpha_{d-1}}$. But $V = K^{\alpha_{d-1}}$ implies that $N'(\boldsymbol{\alpha}, m, V) = 0$, so we may apply the induction hypothesis in (6). Note that $c_1(\boldsymbol{\alpha}') = c_1(\boldsymbol{\alpha}) - 1 \ge \alpha_d - d$ and $c_2(\boldsymbol{\alpha}') = c_2(\boldsymbol{\alpha})$. If $c_2(\boldsymbol{\alpha}) = c_2(\boldsymbol{\alpha}') = 1$, we have

$$\begin{aligned} N'(\boldsymbol{\alpha}, m, V) &\le \sum_{j=0}^{m} q^{(m-j)(\alpha_d - d) + j} q^{j(c_1(\boldsymbol{\alpha}')-1)} j^{d-2} \\ &\le q^{m(\alpha_d - d)} m^{d-2} \sum_{j=0}^{m} q^{j(c_1(\boldsymbol{\alpha}') - \alpha_d + d)} \\ &\ll q^{m c_1(\boldsymbol{\alpha}')} m^{d-1} \\ &= q^{m(c_1(\boldsymbol{\alpha})-1)} m^{d-1}. \end{aligned}$$

If $c_2(\boldsymbol{\alpha}) = c_2(\boldsymbol{\alpha}') > 1$, then

$$\begin{aligned} N'(\boldsymbol{\alpha}, m, V) &\leq \sum_{j=0}^{m} q^{(m-j)(\alpha_d - d)+j} q^{jc_1(\boldsymbol{\alpha}')} j^{c_2(\boldsymbol{\alpha}')-2} \\ &\leq q^{m(\alpha_d-d)} m^{c_2(\boldsymbol{\alpha}')-2} \sum_{j=0}^{m} q^{j(c_1(\boldsymbol{\alpha}')-\alpha_d+d+1)} \\ &\ll q^{m(c_1(\boldsymbol{\alpha}')+1)} m^{c_2(\boldsymbol{\alpha}')-2} \\ &= q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-2}. \qquad \square \end{aligned}$$

We now prove the remaining results on Schubert cells.

Proof of Theorem 3. Every S counted in $N'(\boldsymbol{\alpha}, m)$ can be written $S = T \oplus K(\mathbf{x} + \mathbf{e}_{\alpha_d})$, where $T = S \cap K^{\alpha_{d-1}}$ is counted in $N'(\boldsymbol{\alpha}', j)$ for some j and $\mathbf{x} \in K^{\alpha_d - 1}$ is unique modulo T. By Lemma 3 once more, $h(T) \leq h(S) = m$. Given a T counted in $N'(\boldsymbol{\alpha}', j)$, let $B(T) \in \mathrm{GL}_{\alpha_d - d}(K_{\mathbf{A}})$ give the height on $K^{\alpha_d - 1}/T$. Then

$$\sum_{\deg(\mathfrak{A})=m-j} \lambda^{\mathrm{inh}'}(\mathfrak{A}, B(T))$$

will give the number of S counted in $N'(\boldsymbol{\alpha}, m)$ with $S \cap K^{\alpha_{d-1}} = T$. We thus have by Lemma 4

$$\begin{aligned} N'(\boldsymbol{\alpha}, m) &= \sum_{\substack{\mathfrak{A} \geq 0 \\ \deg(\mathfrak{A}) \leq m}} \sum_{T} \lambda^{\mathrm{inh}'}\big(\mathfrak{A}, B(T)\big) \\ &= \sum_{\substack{\mathfrak{A} \geq 0 \\ \deg(\mathfrak{A}) \leq m}} \sum_{0 \leq \mathfrak{C} \leq \mathfrak{A}} \mu(\mathfrak{C}) \sum_{T} \lambda^{\mathrm{inh}}\big(\mathfrak{A} - \mathfrak{C}, B(T)\big), \end{aligned} \tag{7}$$

where the inner sums are over subspaces T counted in $N'(\boldsymbol{\alpha}', m - \deg(\mathfrak{A}))$.

To ease the notation in what follows, let $a = \alpha_{d-1} - d + 1$ and $r = \alpha_d - \alpha_{d-1} - 1$. Now each $B(T)$ above is of the form

$$B(T) = \begin{pmatrix} I_r & 0 \\ 0 & B^-(T) \end{pmatrix},$$

where $B^-(T) \in \mathrm{GL}_a(K_{\mathbf{A}})$ gives the height on $K^{\alpha_{d-1}}/T$ and "0" denotes an appropriately sized matrix of all zeros. Combining (2) with [**8**, Lemma 16] yields

$$\lambda^{\mathrm{inh}}\big(\mathfrak{A}-\mathfrak{C}, B(T)\big) = \begin{cases} q^{a(\deg(\mathfrak{A}-\mathfrak{C})+1-g)+l(\mathfrak{A}-\mathfrak{C},I_r)+h(T)} \\ \qquad \text{if } \deg(\mathfrak{A}-\mathfrak{C}) \geq \mu_a(B^-(T)) + 2g-1, \\ O\big(q^{(a+r)\deg(\mathfrak{A}-\mathfrak{C})+h(T)}\big) \\ \qquad \text{otherwise.} \end{cases}$$

Since $\mathfrak{A} \geq \mathfrak{C}$ here, by Lemma 2 the latter case can only happen if $\deg(\mathfrak{A}-\mathfrak{C}) < 2g-1$ and $\mu_a(B^-(T)) > 1-2g$. Combining this with (7) gives

(8)

$$N'(\boldsymbol{\alpha}, m) = \sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})\leq m}} \sum_{0\leq\mathfrak{C}\leq\mathfrak{A}} \mu(\mathfrak{C}) q^{a(\deg(\mathfrak{A}-\mathfrak{C})+1-g)+l(\mathfrak{A},\mathfrak{C},I_r)+m-\deg(\mathfrak{A})} \cdot N'\big(\boldsymbol{\alpha}', m-\deg(\mathfrak{A})\big) + O\Bigg(\sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})\leq m}} \sum_{0\leq\mathfrak{C}\leq\mathfrak{A}} \sum_{T'} q^{m-\deg(\mathfrak{A})}\Bigg),$$

where the inner sum in the error term is over subspaces T' counted in $N'(\boldsymbol{\alpha}', m-\deg(\mathfrak{A}))$ with $\mu_a\big(B^-(T)\big) > 1-2g$.

From the Riemann-Roch theorem (since $\mathfrak{A}-\mathfrak{C}\geq 0$)

$$\begin{aligned} l(\mathfrak{A}-\mathfrak{C}, I_r) &= rl(\mathfrak{A}-\mathfrak{C}, I_1) \\ &= r\big[\deg(\mathfrak{A}-\mathfrak{C}) + 1 - g + l(\mathfrak{W}+\mathfrak{C}-\mathfrak{A}, I_1)\big] \\ &\leq r\big[\deg(\mathfrak{A}-\mathfrak{C})+1\big], \end{aligned}$$

where $\mathfrak{W}$ is any divisor in the canonical class. From this, we note that the double sum

$$\sum_{\mathfrak{A}\geq 0} \sum_{0\leq\mathfrak{C}\leq\mathfrak{A}} \mu(\mathfrak{C}) q^{a\deg(\mathfrak{A}-\mathfrak{C})+l(\mathfrak{A}-\mathfrak{C},I_r)-(c_1(\boldsymbol{\alpha}')+1)\deg(\mathfrak{A})} \big(\deg(\mathfrak{A})\big)^s$$

converges for all $s\geq 0$, since $a+r-c_1(\boldsymbol{\alpha}')-1 = \alpha_d - d - c_1(\boldsymbol{\alpha}')-1 \leq -2$ and $a+r = \alpha_d - d \geq 1$. Moreover,

$$\sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})> m}} \sum_{0\leq\mathfrak{C}\leq\mathfrak{A}} \mu(\mathfrak{C}) q^{a\deg(\mathfrak{A}-\mathfrak{C})+l(\mathfrak{A}-\mathfrak{C},I_r)-(c_1(\boldsymbol{\alpha}')+1)\deg(\mathfrak{A})} = O(q^{-m})$$

and

$$\sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})\leq m}} \sum_{0\leq \mathfrak{C}\leq \mathfrak{A}} \mu(\mathfrak{C}) q^{a\deg(\mathfrak{A}-\mathfrak{C})+l(\mathfrak{A}-\mathfrak{C},I_r)-c_1(\boldsymbol{\alpha}')\deg(\mathfrak{A})} \big(\deg(\mathfrak{A})\big)^s = O(m^{s+1})$$

for all $s \leq n$. Thus, assuming the conjecture holds for $\boldsymbol{\alpha}'$, we have

$$\begin{aligned}(9)\quad &\sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})\leq m}} \sum_{0\leq \mathfrak{C}\leq \mathfrak{A}} \mu(\mathfrak{C}) q^{a(\deg(\mathfrak{A}-\mathfrak{C})+1-g)+l(\mathfrak{A},\mathfrak{C},I_r)+m-\deg(\mathfrak{A})} \cdot N'\big(\boldsymbol{\alpha}', m-\deg(\mathfrak{A})\big)\\ &= a(\boldsymbol{\alpha}) q^{m(c_1(\boldsymbol{\alpha}')+1)} m^{c_2(\boldsymbol{\alpha}')-1} + \begin{cases} O\big(q^{mc_1(\boldsymbol{\alpha}')} m^{d-1}\big) & \text{if } c_2(\boldsymbol{\alpha}')=1,\\ O\big(q^{m(c_1(\boldsymbol{\alpha}')+1)} m^{c_2(\boldsymbol{\alpha}')-2}\big) & \text{otherwise,}\end{cases}\\ &= a(\boldsymbol{\alpha}) q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1} + \begin{cases} O\big(q^{m(c_1(\boldsymbol{\alpha})-1)} m^{d-1}\big) & \text{if } c_2(\boldsymbol{\alpha})=1,\\ O\big(q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-2}\big) & \text{otherwise,}\end{cases}\end{aligned}$$

where

$$a(\boldsymbol{\alpha}) = a(\boldsymbol{\alpha}') q^{a(1-g)} \sum_{\mathfrak{A}\geq 0} \sum_{0\leq \mathfrak{C}\leq \mathfrak{A}} \mu(\mathfrak{C}) q^{a\deg(\mathfrak{A}-\mathfrak{C})+l(\mathfrak{A}-\mathfrak{C},I_r)-(c_1(\boldsymbol{\alpha}')+1)\deg(\mathfrak{A})}.$$

In the case $g=0$, the sum in the error term of (8) is empty, so this case follows from (8) and (9).

Suppose that $\alpha_{i+1} = \alpha_i + 1$ for all i with $d-1 > i \geq c_3(\boldsymbol{\alpha}')$, and let T' be a subspace occurring in the error term of (8). Then by Lemma 2 and Minkowski's theorem, see [**8**], $T' \subseteq V$ for some $(\alpha_{d-1}-1)$-dimensional subspace $V \subset K^{\alpha_{d-1}}$ where

$$\begin{aligned} h(V) - h(T) &\leq \mu_1\big(B^-(T)\big) + \cdots + \mu_{a-1}\big(B^-(T)\big)\\ &= \mu_1\big(B^-(T)\big) + \cdots + \mu_a\big(B^-(T)\big) - \mu_a\big(B^-(T)\big)\\ &\leq ag - h(T) - \mu_a\big(B^-(T)\big)\\ &< (a+2)g - 1 - h(T).\end{aligned}$$

Thus,

$$\sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})\leq m}} \sum_{0\leq \mathfrak{C}\leq \mathfrak{A}} \sum_{T'} q^{m-\deg(\mathfrak{A})} \leq \sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})\leq m}} \sum_{0\leq \mathfrak{C}\leq \mathfrak{A}} \sum_{V} N'(\boldsymbol{\alpha}', m-\deg(\mathfrak{A}), V) q^{m-\deg(\mathfrak{A})}, \tag{10}$$

where the first inner sum is over subspaces T' counted in $N'(\boldsymbol{\alpha}', m - \deg(\mathfrak{A}))$ with $\mu_a(B^-(T)) > 1-2g$ and the second is over $(\alpha_{d-1}-1)$-dimensional subspaces $V \subset K^{\alpha_{d-1}}$ with $h(V) < (a+2)g-1$. Using Proposition 3, one readily verifies that

$$\sum_{\substack{\mathfrak{A}\geq 0\\ \deg(\mathfrak{A})\leq m}} \sum_{0\leq \mathfrak{C}\leq \mathfrak{A}} \sum_{V} N'(\boldsymbol{\alpha}', m-\deg(\mathfrak{A}), V) q^{m-\deg(\mathfrak{A})} \ll \begin{cases} O\big(q^{mc_1(\boldsymbol{\alpha}')} m^{d-1}\big) & \text{if } c_2(\boldsymbol{\alpha}')=1, \\ O\big(q^{m(c_1(\boldsymbol{\alpha}')+1)} m^{c_2(\boldsymbol{\alpha}')-2}\big) & \text{otherwise.} \end{cases} \tag{11}$$

(Here the implicit constant will depend on the number of such subspaces V, but that number depends only on α_{d-1} and K.) Since $c_1(\boldsymbol{\alpha}) = c_1(\boldsymbol{\alpha}')+1$ and $c_2(\boldsymbol{\alpha}) = c_2(\boldsymbol{\alpha}')$ here, the remaining case of Theorem 3 follows from (8)–(11). □

Proof of Theorem 4. We use induction on d. If $d=1$, then the conjecture holds by Theorem 2. Suppose $d>1$. The conjecture holds by the induction hypothesis and Theorem 3 if $c_3(\boldsymbol{\alpha}) < d$, and by the induction hypothesis and Theorem 2 if $c_3(\boldsymbol{\alpha}) = d$. □

Proof of Theorem 5. We use induction on d again. If $d=1$, then the conjecture holds by Theorem 2. Suppose $d>1$, and let $\boldsymbol{\alpha} \in c(n,d)$ with $\alpha_1 > 1$.

We first consider the case where $c_3(\boldsymbol{\alpha}) = d$. Get a $\boldsymbol{\beta}' \leq \boldsymbol{\alpha}'$ with $c_1(\boldsymbol{\beta}') = c_1(\boldsymbol{\alpha}')$ and $c_2(\boldsymbol{\beta}') = c_2(\boldsymbol{\alpha}')$ for which the conjecture holds. Let $\beta_d = \alpha_d$. Then $c_1(\boldsymbol{\beta}) = c_1(\boldsymbol{\alpha})$, $c_2(\boldsymbol{\beta}) = c_2(\boldsymbol{\alpha})$ and the conjecture holds for $\boldsymbol{\beta}$ by Theorem 2. By Lemma 5

$$N'(\boldsymbol{\alpha}, m) \geq N'(\boldsymbol{\beta}, m) \gg q^{mc_1(\boldsymbol{\beta})} m^{c_2(\boldsymbol{\beta})-1} = q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}$$

and $N'(\boldsymbol{\alpha}, m) \ll q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}$ by Proposition 2.

Next, consider the case where $c_3(\boldsymbol{\alpha}) < d$. Let $\boldsymbol{\alpha}'' = (\alpha_1, \dots, \alpha_{c_3(\boldsymbol{\alpha})})$. We apply the induction hypothesis, getting a $\boldsymbol{\beta}'' \leq \boldsymbol{\alpha}''$ with $c_1(\boldsymbol{\beta}'') = c_1(\boldsymbol{\alpha}'')$ and $c_2(\boldsymbol{\beta}'') = c_2(\boldsymbol{\alpha}'')$ for which the conjecture holds. Set

$$\boldsymbol{\beta} = (\beta_1, \dots, \beta_{c_3(\boldsymbol{\alpha})}, \beta_{c_3(\boldsymbol{\alpha})} + 1, \dots, \beta_{c_3(\boldsymbol{\alpha})} + d - c_3(\boldsymbol{\alpha})).$$

Then $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$, $c_1(\boldsymbol{\beta}) = c_1(\boldsymbol{\alpha})$, $c_2(\boldsymbol{\beta}) = c_2(\boldsymbol{\alpha})$, and the conjecture holds for $\boldsymbol{\beta}$ via repeated applications of Theorem 3. By Lemma 5

$$N'(\boldsymbol{\alpha}, m) \geq N'(\boldsymbol{\beta}, m) \gg q^{mc_1(\boldsymbol{\beta})} m^{c_2(\boldsymbol{\beta})-1} = q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}$$

and $N'(\boldsymbol{\alpha}, m) \ll q^{mc_1(\boldsymbol{\alpha})} m^{c_2(\boldsymbol{\alpha})-1}$ by Proposition 2. □

3. Theorem 1. We first define absolute height. Fix K and $\overline{K} = \overline{\mathbf{F}_q(X)}$. Suppose $\mathbf{x} \in \overline{K}^n$ is a nonzero vector in the algebraic closure. Then $\mathbf{x} \in L$ for some finite extension L of $\mathbf{F}_q(X)$. If we let h and H denote the additive and multiplicative heights on L, then the absolute additive and multiplicative heights are

$$\overline{h}(\mathbf{x}) := \frac{h(\mathbf{x})}{[L\colon \mathbf{F}_q(X)]}, \qquad \overline{H}(\mathbf{x}) := H(\mathbf{x})^{1/[L\colon \mathbf{F}_q(X)]}.$$

In what follows, Y is transcendental over K.

Lemma 6 [**5**, Lemma 4.9]. *Let $P(Y) \in K[Y]$ be a monic irreducible polynomial of degree d. Denote the coefficient vector of P by $\mathbf{P}$, and let $\rho \in \overline{K}$ be a root of P. Then*

$$d\overline{h}(\rho, 1) = \overline{h}(\mathbf{P}).$$

Proposition 4. *The number N of monic reducible polynomials $P(Y) \in K[Y]$ of degree d with $h(\mathbf{P}) = md\kappa$ satisfies $N \ll q^{md^2\kappa}$.*

Proof. For any $i \geq 1$ and $j \geq 0$, the number of reducible monic polynomials $Q(Y)$ of degree i and $h(\mathbf{Q}) = j$ is certainly no more than the total number of monic polynomials of degree i and height j, which is $N(I_{i+1}, 1, j)$. Now any reducible monic polynomial $P(Y)$ of degree d can be written as a product: $P(Y) = Q(Y)R(Y)$, where $Q(Y)$ and $R(Y)$ are monic and the degree of $Q(Y)$ is at least 1 and no greater than $d/2$. Moreover, by [**5**, Lemma 4.9] $h(\mathbf{P}) = h(\mathbf{Q}) + h(\mathbf{R})$. Letting c denote the greatest integer less than or equal to $d/2$, we have by Theorem 0

$$\begin{aligned} N &\leq \sum_{i=1}^{c} \sum_{j=0}^{md\kappa} N(I_{i+1}, 1, j) N(I_{d-i+1}, 1, md\kappa - j) \\ &\ll \sum_{i=1}^{c} \sum_{j=0}^{md\kappa} q^{(i+1)j} q^{(d-i+1)(md\kappa - j)} \\ &= \sum_{i=1}^{c} \sum_{j=0}^{md\kappa} q^{(d-i+1)md\kappa} q^{j(2i-d)}. \end{aligned}$$

Now if d is odd, then $2c \leq d-1$, and we have

$$\begin{aligned} N &\ll \sum_{i=1}^{c} \sum_{j=0}^{md\kappa} q^{(d-i+1)md\kappa} q^{j(2i-d)} \\ &\ll \sum_{i=1}^{c} q^{(d-i+1)md\kappa} \\ &\ll q^{md^2\kappa}. \end{aligned}$$

If d is even, then $2c = d$, and we have

$$\begin{aligned} N &\ll \sum_{i=1}^{c} \sum_{j=0}^{md\kappa} q^{(d-i+1)md\kappa} q^{j(2i-d)} \\ &\ll md\kappa q^{(c+1)md\kappa} + \sum_{i=1}^{c-1} q^{(d-i+1)md\kappa}. \qquad \square \end{aligned}$$

Proposition 5. *The number N of monic polynomials $P(Y) \in K[Y^p]$ of degree d with $h(\mathbf{P}) = \mathrm{md}\,\kappa$, where p is the characteristic of K, satisfies $N \ll q^{\mathrm{md}^2\kappa}$.*

Proof. This is clear if $d = 1$, so assume $d > 1$. Certainly such a polynomial P has no degree 1 term. Thus, $N \leq N(I_d, 1, \mathrm{md}\kappa) \ll q^{\mathrm{md}^2\kappa}$ by Theorem 0. □

Proof of Theorem 1. Clearly the number of monic polynomials $P(Y) \in K[Y]$ of degree d with $h(\mathbf{P}) = md\kappa$ is $N(I_{d+1}, 1, md\kappa)$. By Theorem 0 and the above two propositions, we see that there are $a(d+1)q^{md\kappa(d+1)} + O(q^{md^2\kappa})$ of these polynomials which are irreducible and have exactly d distinct roots. By Proposition 5 any possible remaining irreducible monic polynomials (which have less than d distinct roots each) are accounted for in the error term. Theorem 1 follows from these estimates and Lemma 6. □

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