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BOUNDS ON CHARACTERISTIC NUMBERS BY CURVATURE AND RADIUS

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ABSTRACT. We obtain explicit bounds on the Euler characteristic and Pontryagin numbers of closed, connected, oriented Riemannian manifolds in terms of sectional curvature and radius.

1. Introduction. Let M^m be a closed, connected and oriented Riemannian manifold of dimension m with sectional curvature $k \leq \sec \leq K$. By definition, the *radius* of M^m is the number $r = \min_{p\in M^m} \max_{q\in M^m} d(p,q)$, where $d(\ ,\ )$ denotes the distance function on M^m. The aim of this note is to give explicit bounds, in terms of k, K and r, on the Euler-Poincaré characteristic of M^{2n} and the Pontryagin numbers of M^{4n}, when $k \leq 0$. The case $k > 0$ was first studied by Berger, cf. [**1**], considering the diameter instead of the radius. He proved that, if M is a complete Riemannian manifold of dimension $2n$ and $0 < k \leq \sec \leq K$, then

$$|\chi(M)| \leq \frac{K^n}{2^n k^n} \cdot (2n)!,$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of M. Tsagas obtained explicit bounds for the Pontryagin numbers when $k > 0$, cf. [**12**]. In these cases, since $0 < k \leq \sec \leq K$, it follows from Myers' theorem that M has diameter $d \leq \pi/\sqrt{k}$ (so M must be compact) and the bound on the Euler-Poincaré characteristic will only depend on the sectional curvature bounds. Bishop and Goldberg noted in [**2**] that, using what is now known as Bishop's theorem, it is possible to bound the Euler-Poincaré characteristic of an even dimensional compact Riemannian manifold with bounded sectional curvature and diameter d, generalizing Berger's result. However, they did not carry

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out any computations. Explicit bounds for the Pontryagin numbers in the case $k \leq 0$ were also missing.

Our proofs are a straightforward application of the Chern-Weil theory of characteristic classes and Bishop's theorem. Since the radius is bounded above by the diameter, we obtain better estimates by considering the first quantity instead of the second one. We will assume without further comment that our Riemannian manifolds are connected. We note that, by Gromov's Betti number theorem, cf. [**8**], when $k = 0$, the Euler-Poincaré characteristic of a closed connected Riemannian manifold M^m is uniformly bounded by a large universal constant $\mathcal{C}(m)$ depending only on the dimension m. Since the first Pontryagin number of a closed oriented 4-manifold is proportional to the signature of the manifold, when $\sec \geq 0$, the first Pontryagin number is also uniformly bounded. In contrast, in each dimension $4n$, $n \geq 2$, there exists an infinite sequence of simply connected manifolds with $\sec \geq 0$ and mutually distinct Pontryagin numbers, cf. [**5**]. In this case, our results show that these numbers grow at most polynomially in terms of K and r.

2. The Euler-Poincaré characteristic. Let M be a closed, oriented Riemannian manifold of dimension $2n$ with sectional curvature $k \leq \sec \leq K$ and radius r. In this section we will estimate the Euler-Poincaré characteristic of M in terms of k, K and $\mathrm{Vol}\,(M)$, the volume of M. We will denote the coefficients of the curvature tensor of M by R_{ijlm}. The Euler-Poincaré characteristic $\chi(M)$ of M is given by the generalized Gauss-Bonnet theorem, cf. [**9, 11**]:

$$\frac{\mathrm{Vol}\,(\mathbf{S}^{2n})}{2}\chi(M) = \int_M f \cdot dV, \tag{2.1}$$

where $\mathbf{S}^{2n}$ is the unit $2n$-sphere, dV is the volume element of M and

$$f(p) = \frac{(-1)^n}{2^n(2n)!} \sum_{\substack{i_1,\dots,i_{2n}\\ j_1,\dots,j_{2n}}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}} \varepsilon^{i_1 \cdots i_{2n}} \varepsilon^{j_1 \cdots j_{2n}} \tag{2.2}$$

in terms of any orthonormal basis of $T_p(M)$. The symbol $\varepsilon^{l_1 \cdots l_{2n}}$ denotes the sign of permutation $(l_1 \cdots l_{2n})$ of the indices $\{1, \dots, 2n\}$.

Now we estimate function f in terms of k and K. Since $k \leq \sec \leq K$, then cf. [**4**]

$$|R_{ijlm}| \leq C := \max(-k,\ K,\ (2/3)(K-k)). \tag{2.3}$$

In particular, for $k \geq 0$, we have $|R_{ijlm}| \leq K$. It follows from equation (2.2) that, for any $p \in M$,

$$|f(p)| \leq \frac{(2n)!C^n}{2^n}. \tag{2.4}$$

Taking the absolute values in equation (2.1) and using estimate (2.4), we obtain

$$\frac{\mathrm{Vol}\,(\mathbf{S}^{2n})}{2}\big|\chi(M)\big| \leq \frac{(2n)!C^n}{2^n}\ \mathrm{Vol}\,(M). \tag{2.5}$$

3. The Pontryagin numbers. Let N be a closed, oriented Riemannian manifold of dimension $4n$ with sectional curvature $k \leq \sec \leq K$ and radius r. In this section we will estimate the Pontryagin numbers of N in terms of k, K and $\mathrm{Vol}\,(N)$.

A *partition* of n is an unordered sequence of positive integers $I = k_1, \dots, k_s$ with sum n. The Ith Pontryagin number $p_I(N) = p_{k_1} \cdots p_{k_s}(N)$ is defined to be the integer

$$p_I(N) = \int_N p_{k_1} \wedge \cdots \wedge p_{k_s},$$

where p_k is an exterior $2k$-form representing the kth Pontryagin class of N, cf. [**10**]. Let Ω^i_j be curvature 2-forms of N. By the Weil theorem, we can take

$$p_k = \frac{1}{(2\pi)^{2k}(2k)!} \sum \delta^{j_1 \cdots j_{2k}}_{i_1 \cdots i_{2k}} \Omega^{i_1}_{j_1} \wedge \cdots \wedge \Omega^{i_{2k}}_{j_{2k}},$$

where the summation runs over all ordered subsets $(i_1, \dots, i_{2k})$ of $2k$ elements of $(1, \dots, 4n)$ and all permutations $(j_1, \dots, j_{2k})$ of $(i_1, \dots, i_{2k})$. The symbol $\delta^{j_1 \cdots j_{2k}}_{i_1 \cdots i_{2k}}$ denotes the sign of the permutation $(j_1, \dots, j_{2k})$ of $(i_1, \dots, i_{2k})$, cf. [**9**].

Given a partition $I = k_1, \dots, k_s$ of n and using the definition of the wedge product, we can write

$$p_I(N) = \int_N p_{k_1} \wedge \cdots \wedge p_{k_s} = \int_N g \cdot dV, \tag{3.1}$$

where dV is the volume element of N. The function g can be written as

$$g(p) = \frac{1}{A} \sum_{l_1, \dots, l_{4n}} \varepsilon^{l_1 \cdots l_{4n}} F_{l_1 \cdots l_{4n}}, \tag{3.2}$$

where

$$A = 2^{4n} \pi^{2n} \prod_{i=1}^{s} (2k_i)!(4k_i)!$$

and $F_{l_1 \cdots l_{4n}}$ is the product of the s functions

$$\sum \delta^{j_1 \dots j_{2k_1}}_{i_1 \dots i_2 k_1} \sum_{\sigma} \operatorname{sgn}(\sigma) R_{l_{\sigma(1)} l_{\sigma(2)} i_1 j_1} \cdots R_{l_{\sigma(4k_1-1)} l_{\sigma(4k_1)} i_{2k_1} j_{2k_1}},$$

$$\vdots$$

$$\sum \delta^{j_1 \dots j_{2k_s}}_{i_1 \dots i_2 k_s} \sum_{\sigma} \operatorname{sgn}(\sigma) R_{l_{\sigma(4n-k_s+1)} l_{\sigma(4n-k_s+2)} i_1 j_1} \cdots R_{l_{\sigma(4n-1)} l_{\sigma(4n)} i_{2k_s} j_{2k_s}},$$

in terms of any orthonormal basis of $T_p(N)$. Here σ denotes a permutation of elements in the sets of indices $\{1, \dots, 4k_1\}$, $\{4k_1 + 1, \dots, 4(k_1 + k_2)\}, \dots, \{4n - k_s + 1, \dots, 4n\}$ determined by the partition of n.

Using (3.2) and estimate (2.3), it follows that, for any $p \in N$,

$$|g(p)| \leq BC^{2n}, \tag{3.3}$$

where

$$B = \frac{(4n)!^{s+1}}{2^{4n} \pi^{2n} \prod_{i=1}^{s} (4n - 2k_i)!}. \tag{3.4}$$

Taking the absolute values in equation (3.1) and using (3.3), we obtain

$$\left| p_I(N) \right| \leq BC^{2n} \operatorname{Vol}(N), \tag{3.5}$$

where B is the constant in (3.4).

4. Main results.

Theorem 4.1. *Let M be a closed, oriented Riemannian manifold of dimension $2n$ with sectional curvature $k \leq \sec \leq K$ and radius r. If $k = 0$, then*

$$|\chi(M)| \leq \frac{(2n)!^2}{n!^2} \cdot \frac{(Kr^2)^n}{2^{3n}}.$$

If $k < 0$, then

$$\begin{aligned}&\big|\chi(M)\big| \\ &\leq \frac{(2n)!^2 C^n (-1)^{n-1}}{2^{3n-1} n!(n-1)!(-k)^n} \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} \left(\frac{\cosh^{2m+1}(r\sqrt{-k}) - 1}{2m+1} \right),\end{aligned}$$

where C is given by (2.3).

Proof. Since $\sec \geq k$, it follows from Bishop's theorem that

$$\operatorname{Vol}(M) \leq v_k^{2n}(r), \tag{4.1}$$

where $v_k^{2n}(r)$ is the volume of the ball of radius r in the model space of constant sectional curvature k in dimension $2n$. It follows from inequalities (2.5) and (4.1) that

$$\frac{\operatorname{Vol}(\mathbf{S}^{2n})}{2} \big|\chi(M)\big| \leq \frac{(2n)! C^n}{2^n} v_k^{2n}(r). \tag{4.2}$$

For $k = 0$, we have $C = K$, cf. equation (2.3). The desired estimate follows from (4.2) by using the well-known facts

$$\operatorname{Vol}(\mathbf{S}^{2n}) = \frac{\pi^n 2^{2n+1} n!}{(2n)!}$$

and

$$v_0^{2n}(r) = \frac{\pi^n r^{2n}}{n!}.$$

Suppose now that $k < 0$. It is well known that

$$v_k^{2n}(r) = (-k)^{-n} \operatorname{Vol}(\mathbf{S}^{2n-1}) \int_0^{r\sqrt{-k}} \sinh^{2n-1}(x)\, dx. \tag{4.3}$$

The integral above has the explicit expression, cf. [**7**],

$$\int_0^{r\sqrt{-k}} \sinh^{2n-1}(x)\,dx = (-1)^{n-1}\sum_{m=0}^{n-1}(-1)^m\binom{n-1}{m}\left(\frac{\cosh^{2m+1}(r\sqrt{-k})-1}{2m+1}\right).$$

(A formula for this integral also appears in [**6**], although with a typo: the first exponent in the last part of the formula in [**6**] should be m instead of n.)

Since

$$\mathrm{Vol}\,(\mathbf{S}^{2n-1}) = \frac{2\pi^n}{(n-1)!},$$

straightforward computations using (4.2) and (4.3) yield the inequality in the statement of the theorem. □

The same argument used to prove Theorem 4.1, now using equation (3.5), yields bounds on the Pontryagin numbers.

Theorem 4.2. *Let N be a closed, oriented Riemannian manifold of dimension $4n$ with sectional curvature $k \leq \sec \leq K$ and radius r, and let $I = k_1, \ldots, k_s$ be a partition of n. If $k = 0$, then*

$$|p_I(N)| \leq \frac{(4n)!^{s+1}}{(2n)!\prod_{i=1}^s(4n-2k_i)!}\cdot\frac{(Kr^2)^{2n}}{2^{4n}}.$$

If $k < 0$, then

$$|p_I(N)| \leq A(n)\frac{C^{2n}}{(-k)^{2n}}\cdot\sum_{m=0}^{2n-1}(-1)^m\binom{2n-1}{m}\left(\frac{\cosh^{2m+1}(r\sqrt{-k})-1}{2m+1}\right),$$

where

$$A(n) = \frac{(4n)!^{s+1}(-1)^{2n-1}}{(2n-1)!2^{4n-1}\prod_{i=1}^s(4n-2k_i)!}.$$

Corollary 4.3. *Let N be a compact Riemannian manifold of dimension 4 with sectional curvature $0 \leq \sec \leq K$, radius r and let σ be its signature. Then*

$$|\sigma| \leq 3(Kr^2)^2.$$

Proof. By the signature theorem $\sigma = (1/3)p_1$. Combining this with Theorem 4.2, we obtain the desired estimate. □

Remark. By Theorem 4.1, a 4-manifold M with $0 \leq \sec \leq 1$ and radius r must have $|\chi(M)| \leq 9/4 \cdot r^4$. In dimension 4, a manifold with nonnegative sectional curvature must have nonnegative Euler characteristic, cf. [**3**]. Hence, $0 \leq \chi(M) \leq 9/4 \cdot r^4$.

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