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# CONVERGENCE FOR ESSENTIALLY STRONGLY INCREASING DISCRETE TIME SEMI-FLOWS

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ABSTRACT. This paper introduces a class of essentially strongly increasing discrete time semi-flows, for which several principles of convergence of every precompact orbit to cycles or fixed points are established. Our results with weak monotonicity improve the classical ones in the literature. In particular, we present two examples illustrating that our main results overcome the drawbacks of the classical ones which require the delicate choice of state space and the technical *ignition* assumption in the applications to periodic quasi-monotone systems of delay differential equations.

**1. Introduction.** Hirsch [**9**] established that the generic precompact orbit of a cooperative and irreducible system of ordinary differential equations converges to the set of equilibria, and Manato [**17**] announced similar results. The results of Hirsch and Matano have greatly attracted the interest of many authors in the dynamics of strongly increasing discrete and continuous time semi-flows. Hirsch [**10**] showed that most orbits of a strongly increasing continuous time semi-flow on a strongly ordered space converge to the set of equilibria, which was later improved by many authors, see for example, [**11, 15, 18, 24, 25, 29, 34**]. The results of Poláčik and Tereščák [**19, 20**] and later improvements by Hess and Poláčik [**8**] established that the generic orbit of a smooth strongly increasing discrete time semi-flow converges to a cycle. The above-mentioned generic properties imply that precompact orbits have a strong tendency to converge to fixed points or cycles. In fact, if

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some additional conditions are imposed, then every precompact orbit converges to fixed points or cycles. For example, under the condition of orbital stability, Alikakos et al. [**1**] first established the convergence of every precompact orbit to a fixed point for strongly increasing discrete time semi-flows. Dancer and Hess [**5**] and Takáč [**28**] later improved the results of [**1**] by adding the condition of fixed point stability. Many other hypotheses, such as the sublinearity and the first integral, etc., have also been utilized in [**13, 14, 21, 27, 32, 35**] to guarantee the similar convergent properties. Other work related to this topic can also be found in the monographs [**4, 7, 12, 23, 36**].

The main purpose of the present paper is to investigate the convergence of essentially strongly increasing discrete time semi-flows. Our goal is to show that every precompact orbit of a conditionally $\alpha$-condensing, essentially strongly increasing discrete time semi-flow converges to fixed points or cycles if every fixed point is stable or the existence of a totally ordered arc (contained in the set of periodic points) holds. The stimulation of interest in our study is justified by several facts. Firstly, since the conditions such as the sublinearity and the first integral guarantee the fixed point stability or the existence of a totally ordered arc (contained in the set of fixed points), we will devote our attention to establishing the convergence principle provided that every fixed point is stable or the existence of a totally ordered arc (contained in the set of fixed points or periodic points) holds. Secondly, when applied to periodic quasi-monotone systems of delay differential equations and reaction-diffusion equations with delay, the classical convergence principles for strongly increasing discrete time semi-flows suffer from some drawbacks such as: the requirements of the delicate choice of state space and the technical *ignition* assumption, see [**16, 22**] for more details about this assumption. Motivated by Yi and Huang [**34**], we introduce the notion of an essentially strongly increasing discrete time semi-flow (see Section 2 for definition), and this allows us to overcome the above-mentioned drawbacks. Finally, the periodic mappings of periodic systems of functional differential equations are generally only conditionally $\alpha$-condensing or conditionally completely continuous, thereby impelling us to improve the compactness assumptions used in Dancer and Hess [**5**] and Takáč [**28, 30, 31**].

The organization of the rest of this paper is as follows. In Section 2, we give the definition of essentially strongly increasing discrete time semi-flows and establish several preliminary results which include one on the existence of the third fixed point. In Section 3, we state and prove our main results, which not only improve the classical ones by weakening the compactness and monotonicity assumptions but also correct some mistakes occurring in the proofs of Takáč [**28, 30, 31**]. Additionally, unlike the results of [**28, 30, 31**], our results do not require the state space to be strongly ordered. In Section 4, we provide examples to show that our results can successfully overcome the drawbacks of the classical results of [**5, 28, 30, 31**] in applications to periodic systems of delay differential equations.

**2. Preliminary results.** We start with some notations and definitions.

The space $X$ is called an ordered metric space if it is a metric space with metric $d$ and a closed partial order relation $R \subseteq X \times X$. For any $x, y \in X$, we write $x \leq y$ if $(x, y) \in R$, $x < y$ if $x \leq y$ and $x \neq y$, and $x \ll y$ if $(x, y) \in \operatorname{Int} R$. Given $a, b \in X$, we denote a closed order interval in $X$ by the set $[a, b] = \{x \in X : a \leq x \leq b\}$, $(a, b) = [a, b] \setminus \{a, b\}$, and an open order interval in $X$ by $[[a, b]] = \{x \in X : a \ll x \ll b\}$. We write $[a, \infty]] = \{x \in X : x \geq a\}$, and similarly for $[[-\infty, b]$, etc. A subset $Y \subseteq X$ is said to be order-convex in $X$ if $[a, b] \subseteq Y$, whenever $a, b \in Y$ and $a < b$; lower closed if $[[-\infty, b] \subseteq Y$, whenever $b \in Y$; and upper closed if $[a, \infty]] \subseteq Y$, whenever $a \in Y$. A subset $Y$ of $X$ is called order-bounded if it is contained in a finite union of order intervals in $X$, i.e., there exist $a_i, b_i \in X$ with $a_i \leq b_i$, $i = 1, 2, \dots, n$ such that $Y \subseteq \cup_{i=1}^{n} [a_i, b_i]$.

Let $V$ be a strongly ordered Banach space, that is, $V$ is a Banach space and $V_+ = \{x \in V : x \geq 0\}$ is an order cone with nonempty interior $\operatorname{Int}(V_+)$. For any $a, b \in V_+$, we write $a \leq b$ if $b - a \in V_+$, $a < b$ if $a \leq b$ and $a \neq b$, and $a \ll b$ if $b - a \in \operatorname{Int} V_+$. We denote closed order intervals in $V$ by $[a, b]_V = (a + V_+) \cap (b - V_+)$, $(a, b)_V = [a, b]_V \setminus \{a, b\}$, and open order intervals in $V$ by $[[a, b]]_V = (a + \operatorname{Int} V_+) \cap (b - \operatorname{Int} V_+)$. We write

$$[a, \infty]]_V = a + V_+, \qquad [[a, \infty]]_V = a + \operatorname{Int} V_+,$$

and similarly for $[[-\infty, b]_V$, $[[-\infty, b]]_V$. Observe that an order-bounded subset of $V$ is a bounded set.

Throughout this paper, we will always assume that $X$ is an ordered metric space, $V$ is a strongly ordered Banach space and $T : X \to X$ is a continuous mapping. However, we do not assume that $X$ is strongly ordered in sense of Hirsch [**10**].

**Definition 2.1.** The mapping $T$ is called increasing (strictly increasing, strongly increasing, respectively) if $x, y \in X$ and $x < y$ implies $Tx \leq Ty$ ($Tx < Ty$, $Tx \ll Ty$, respectively). $T$ is called essentially strongly increasing if $T$ is an increasing mapping which has the following properties:

(i) For every $x, y \in X$ with $x \leq y$, we have $Tx = Ty$ or $Tx \ll Ty$;

(ii) For every $x, y \in X$ with $x \ll y$, we have $Tx \ll Ty$.

Two points $x, y \in X$ are called essentially ordered if $x \leq y$ and $Tx < Ty$. In particular, $x$ is not essentially ordered with itself. A subset $Y$ of $X$ is called essentially ordered if $Y$ contains at least two essentially ordered points. Otherwise, $Y$ is called essentially unordered.

According to Definition 2.1, the essentially strongly increasing property is weaker than the strongly increasing property. As stated in the introduction, the theory of essentially strongly increasing mappings has great advantage when applied to periodic quasi-monotone systems of *delay* differential equations and reaction-diffusion equations with delay. In the following, we shall investigate the asymptotic behavior of an essentially strongly increasing mapping.

Let $O(x, T) = \{T^n(x) : n \geq 0\}$ be the positive semi-orbit through the point $x \in X$. If $O(x, T)$ is precompact, we define the omega limit set of $x$ by

$$\omega(x, T) = \{y \in X : T^{n_k} x \longrightarrow y \ (k \to \infty) \text{ for some sequence } n_k \to \infty\}.$$

One can observe that $\omega(x, T)$ is nonempty, compact and invariant. In what follows, we will write $O(x)$ and $\omega(x)$ for $O(x, T)$ and $\omega(x, T)$, respectively, so that no confusion might arise. A subset $Y$ of $X$ is called invariant if $T(Y) \subset Y$, and totally invariant if $T(Y) = Y$. We denote by $\mathcal{E}(T) = \{x \in X : T(x) = x\}$ the set of fixed points of $T$. Given a positive integer $k$, the elements of $\mathcal{E}(T^k) \setminus \cup_{i=1}^{k-1} \mathcal{E}(T^i)$ are called $k$-periodic points of $T$. We denote by $\mathcal{P}(T) = \cup_{k=1}^{\infty} \mathcal{E}(T^k)$ the set of all periodic points of $T$. The orbit $O(x)$ of a $k$-periodic point $x$ is called a $k$-cycle. We say that a subset $K$ of $X$ attracts another set $Y \subseteq X$ if $\overline{O(x)}$ is compact in $X$ and $\omega(x) \subseteq K$ for all $x \in Y$. A continuous mapping $T : X \to X$ is conditionally completely continuous if $A \subseteq X$ is bounded, and $TA$ bounded implies $\overline{TA}$ is compact. The mapping $T$ is completely continuous if it is conditionally completely continuous and also a bounded mapping. The Kuratowski measure of noncompactness, $\alpha$, is defined by

$$\alpha(A) = \inf\{r : A \text{ has a finite cover of diameter less than } r\}.$$

A continuous mapping $T : X \to X$ is conditionally $\alpha$-condensing if $\alpha(TA) < \alpha(A)$ for bounded sets $A \subseteq X$ for which $TA$ is bounded and $\alpha(A) > 0$. The mapping $T$ is $\alpha$-condensing if it is a conditionally $\alpha$-condensing mapping which is also a bounded mapping. We refer to Hale [**6**] for a study of the dynamics of (conditionally) $\alpha$-condensing mappings.

The following convergence criterion comes from [**12, 28, 30**].

**Proposition 2.1** (Monotone convergence criterion). *Let $T : X \to X$ be an increasing mapping. Assume that $x \in X$, $\overline{O(x)}$ is compact, and let $x \leq T^k x$ $(x \geq T^k x)$ for some integer $k > 0$. Then $T^{nk+l}x \to T^l p$ as $n \to \infty$, $l = 0, 1, \ldots, k-1$, for some $p \in \mathcal{E}(T^k)$, and $p \geq x$ $(p \leq x)$.*

The following nonordering principle has been proved by Takáč [**30**] under the assumption that $T$ is a strongly increasing mapping.

**Proposition 2.2** (Nonordering of limit sets). *Let $T : X \to X$ be an essentially strongly increasing mapping. Assume that $x \in X$ and $\overline{O(x)}$ is compact. Then $\omega(x)$ is essentially unordered.*

*Proof.* Assume on the contrary, that $\omega(x)$ is essentially ordered. Then there exist $p, q \in \omega(x)$ such that $p \leq q$ and $Tp < Tq$. Since $T$ is essentially strongly increasing, it follows that $Tp \ll Tq$. The invariance of $\omega(x)$ implies that $Tp, Tq \in \omega(x)$. By definition of $\omega(x)$, there exist $m_1, m_2 \geq 1$ such that $T^{m_1}x \ll T^{m_2}x$. Without loss of generality, we may assume $m_2 > m_1$; then $T^{m_1}x \ll T^{m_2-m_1}(T^{m_1}x)$, and hence $\omega(x) = \omega(T^{m_1}x)$ is a cycle by Proposition 2.1. Set $\omega(x)$ by $k = (\omega(x))^\sharp$. Since $\omega(x)$ is essentially ordered, it is easily seen from the above discussion that there exist $p \in \omega(x)$ and $1 \leq i \leq k-1$ such that $p \ll T^i p$. Using Proposition 2.1 again, we obtain $p \in \mathcal{E}(T^i)$. Therefore, we have $k = (\omega(x))^\sharp \leq i$, which yields a contradiction. This completes the proof.

The following result is the analogy to the continuous case in [**23**, Corollary 1.2.4].

**Proposition 2.3.** *Let $T : X \to X$ be an essentially strongly increasing mapping. Assume that $x \in X$ and $\overline{O(x)}$ is compact. If $p \in \omega(x)$ satisfying either $p \leq \omega(x)$ or $p \geq \omega(x)$, then $\omega(x) = \{p\}$.*

*Proof.* Without loss of generality, we may assume $p \leq \omega(x)$. If $\omega(x) = \{p\}$, then the proof is complete. Otherwise, we have $\omega(x)\backslash\{p\} \neq \varnothing$. By invariance of $\omega(x)$, we obtain $T\omega(x) = \omega(x) \geq Tp$. Thus, there exists a $q \in \omega(x)$ such that $q \geq p$ and $Tq > Tp$. Therefore, $p, q \in \omega(x)$ are essentially ordered, which contradicts Proposition 2.2. This completes the proof. $\square$

We will establish the following result by utilizing the theory of the fixed-point index in Amann [**2**]. For more details on applications of the theory of the fixed-point index to prove the existence of the third fixed point, we refer to [**5, 24**].

**Lemma 2.1.** *Let $K \subseteq [p, q]_V$ be a compact convex subset where $p \ll q$ and $p, q \in K$. Let $S : K \to K$ be a continuous and increasing mapping where $p, q \in \mathcal{E}(S)$. Assume that $A \subseteq K \cap [[p, q]]_V$ is a totally invariant, compact subset of $S$. Then $\mathcal{E}(S) \setminus \{p, q\} \neq \varnothing$.*

*Proof.* Set $K^0 = K \cap [[p, q]]_V$. Since $A$ is compact, there exists a $\delta > 0$ such that

$$V(p) = (K \cap O(p, \delta)) \ll A, \qquad V(q) = (K \cap O(q, \delta)) \gg A.$$

Define $F : [0, 1] \times \overline{V(p)} \to K$ and $H : [0, 1] \times \overline{V(q)} \to X$ by $F(\lambda, x) = \lambda p + (1-\lambda)S(x)$ and $H(\lambda, x) = \lambda q + (1-\lambda)S(x)$, respectively. We next distinguish several cases to finish the proof.

**Case 1.** $F(\lambda, x) = x$ for some $(\lambda, x) \in [0, 1] \times \partial\overline{V(p)}|_K$. In this case, we have $x \neq p$ and $x - S(x) = \lambda(p - S(x)) \leq 0$. By Proposition 2.1, there exists $r \in \mathcal{E}(S)$ such that $\lim_{n\to\infty} S^n(x) = r$ and $r \geq x > p$, and hence $r \gg p$. Again ,since $x \leq A$ and $A$ is totally invariant, we have $r \leq A \ll q$, completing the proof in this case.

**Case 2.** $H(\lambda, x) = x$ for some $(\lambda, x) \in [0,1] \times \partial\overline{V(q)}|_K$. Using a similar argument to that in the proof of Case 1, we can show this is also true.

**Case 3.** $F$ and $H$ are compact homeomorphisms. For this case, we have $i(S, V(p), K) = i(H(0,\cdot), V(p), K) = i(H(1,\cdot), V(p), K) = 1$ by invariance of the homotopy property of the fixed-point index in Amann [**2**]. Similarly, we have $i(S, V(q), K) = 1$. The additivity property of the fixed-point index implies that

$$\begin{aligned} i(S, K \setminus \overline{V(p) \cup V(q)}, K) &= i(S, K, K) - i(S, V(q), K) - i(S, V(q), K) \\ &= 1 - 1 - 1 \\ &= -1. \end{aligned}$$

Hence, it follows from the solution property of the fixed-point index that there exists an $r \in K \setminus \overline{V(p) \cup V(q)}$ such that $Sr = r$. This completes the proof. $\square$

**Proposition 2.4.** *Let the mapping* $T : [p,q]_V \to [p,q]_V$ *be* $\alpha$*-condensing and increasing where* $p \ll q$ *and* $p, q \in \mathcal{E}(T)$*. Assume that* $A \subseteq [[p,q]]_V$ *is a totally invariant, compact subset. Then* $\mathcal{E}(T) \cap ([p,q]_V \setminus \{p,q\}) \neq \varnothing$*.*

*Proof.* Let $\mathcal{A} = \{K \subseteq [p,q]_V : p, q \in K,\ A \subseteq K,\ K$ closed convex, $TK \subseteq K\}$, $K^* = \cap_{K \in \mathcal{A}} K$ and $K^{**} = \overline{\mathrm{co}\,(T(K^*))}$. It is easily seen that $K^* = K^{**} \in \mathcal{A}$. As $T$ is $\alpha$-condensing, the measure of noncompactness of $K^*$ is zero. Hence, $K^*$ is a compact convex subset. Applying Lemma 2.1 to $T|_{K^*}$, we obtain $(K^* \setminus \{p,q\}) \cap \mathcal{E}(T) \neq \varnothing$. Therefore, we have $\mathcal{E}(T) \cap ([p,q]_V \setminus \{p,q\}) \neq \varnothing$. This completes the proof. $\square$

*Remark* 2.1. It has been pointed out in Takáč [**28, 30, 31**] that if $T$ is a strongly increasing mapping and $H$ is a $d$-hypersurface for some invariant order decomposition of $T$, see [**28, 30**], then the conclusion of the above proposition can be replaced by $\mathcal{E}(T) \cap H \neq \varnothing$. But unfortunately, Takáč's proof seems to be incorrect, since Takáč [**28, 30**] incorrectly asserts that the star-shaped set $P(H)$, see [**30**, Proposition 1.2] for a definition, is convex. In fact, Example 1.3 of Takáč [**31**] shows that $P(H)$ is not necessarily convex. Also, Takáč [**31**] incorrectly asserts that the star-shaped set $P(H)$ is homeomorphic to convex sets. Moreover, we should point out that the conclusion $\mathcal{E}(T) \cap H \neq \varnothing$ is an important tool in the proof of the main results of Takáč [**28, 30, 31**]. It is therefore necessary to make corrections to Takáč's results.

**3. Main results.** The main goal of this section is to investigate the asymptotic behavior of an essentially strongly increasing mapping.

As an application of Proposition 2.4, we obtain the following result, which corrects and improves Theorem 1.4 of Takáč [**28**] and Theorem 2.4 of Takáč [**30**].

**Theorem 3.1.** *Let the mapping $T : [p,q]_V \to [p,q]_V$ be $\alpha$-condensing and essentially strongly increasing where $p \ll q$ and $[p,q]_V \cap \mathcal{E}(T) = \{p,q\}$. Then $[[p,q]]_V$ is attracted by either $p$ or $q$.*

*Proof.* As $T$ is $\alpha$-condensing, every orbit is precompact. By way of contradiction, suppose that neither $p$ nor $q$ attracts $[[p,q]]_V$. Let $L^0 = \{x \in [p,q]_V : x = tp + (1-t)q \text{ for some } t \in (0,1)\}$. We claim that there exists a point $c \in L^0$ such that $p, q \notin \omega(c)$. Otherwise, we have either $p \in \omega(x)$ or $q \in \omega(x)$, for every $x \in L^0$. It follows from Proposition 2.3 that either $p = \omega(x)$ or $q = \omega(x)$, for every $x \in L^0$. Consequently, by hypotheses and the fact that $T$ is an increasing mapping, there exists $c \in L^0$ such that, for each $x \in L^0$,

$$\omega(x) = p \quad \text{if } x < c, \quad \text{and } \omega(x) = q \quad \text{if } x > c.$$

Without loss of generality, we may assume $\omega(c) = p$, then $\omega(c) \ll c$. Hence, there exists a positive integer $k$ such that $T^k c \ll c$. Continuity of $T$ implies that there exist $c < c' \in L^0$ such that $T^k c' \ll c < c'$. Since $T$ is increasing, it follows from $\omega(c) = p$ that $\omega(c') = \omega(T^k c') \leq p$, which contradicts the choice of $c$ and establishes the above claim.

Since $\omega(c)$ is compact and invariant, $p, q \notin \omega(c)$ and $T$ is an essentially strongly increasing mapping, we have $\omega(c) \subseteq [[p,q]]_V$. Therefore, by Proposition 2.4, we obtain $([p,q]_V \cap \mathcal{E}(T)) \setminus \{p,q\} \neq \varnothing$, which is a contradiction. This completes the proof. $\square$

We now define the following ordering "$\preceq_T$." If $A, B \subseteq X$ and the mapping $T : X \to X$ is increasing, we denote $A_+^T = \cup_{i\geq 0,\ a\in A}[T^i a, \infty]]$ and $A_-^T = \cup_{i\geq 0,\ a\in A}[[-\infty, T^i a]$. Similarly, the notations $B_+^T$ and $B_-^T$ can be defined. We write $A \preceq_T B$ if $A \subseteq B_-^T$ and $B \subseteq A_+^T$. If $T = id_X$, then the ordering "$\preceq_{id_X}$" has been introduced by Takáč [**30**], where $id_X : X \to X$ is an identity mapping.

**Lemma 3.1.** *Let $T$ be an increasing mapping. Assume that $x \in \mathcal{P}(T)$, $y \in X$ and $\overline{O(y)}$ is compact. If the integer $k \geq 1$, then $O(x) \preceq_T \omega(y)$ if and only if $O(x, T^k) \preceq_T \omega(y, T^k)$.*

*Proof.* If $k = 1$, then the conclusion is obvious. It remains to prove the case $k \geq 2$. Clearly, $(O(x))_+^T = \cup_{i\geq 0}[T^i x, \infty]]$ and $(O(x, T^k))_+^T = \cup_{i\geq 0}[T^i x, \infty]]$, that is, $(O(x))_+^T = (O(x, T^k))_+^T$. Also, $\omega(y, T^k) \subseteq \omega(y)$ implies that $(\omega(y, T^k))_-^T \subseteq (\omega(y))_-^T$. We show next that $(\omega(y))_-^T \subseteq (\omega(y, T^k))_-^T$. Assume that $p \in (\omega(y))_-^T$. Then there exist $i \geq 0$ and $n_l \to \infty$ such that $\lim_{n_l\to\infty} T^{n_l} y \in \omega(y)$ and $p \leq T^i(\lim_{n_l\to\infty} T^{n_l} y)$. Since $n_l = m_l k + \delta_l$ and $0 \leq \delta_l \leq k-1$, we may assume, without loss of generality, that $\delta_l = \delta$ and $0 \leq \delta \leq k-1$. Then $p \leq T^{i+\delta}(\lim_{m_l\to\infty} T^{m_l k} y)$ and hence $p \in (\omega(y, T^k))_-^T$. Therefore, we have $(\omega(y))_-^T = (\omega(y, T^k))_-^T$. We prove only the sufficiency, the proof of the necessity following easily from the above discussion. Assume that $p \in O(x)$; then there exists $i \geq 0$ such that $p = T^i x$. It follows from $x \in O(x, T^k) \subseteq (\omega(y, T^k))_-^T$ that there exist $q \in \omega(y, T^k) \subseteq \omega(y)$ and $l \geq 0$ such that $x \leq T^l q$. Since $T$ is strictly increasing, we have $p = T^i x \leq T^{i+l} q$, that is, $O(x) \subseteq (\omega(y))_-^T$. Assume that $q \in \omega(y)$; then there exists $n_l = m_l k + \delta_l$ with $0 \leq \delta_l < k-1$ such that $q = \lim_{n_l\to\infty} T^{m_l k+\delta_l} y$. Without loss of generality, we may assume that $\delta_l = \delta$, $0 \leq \delta < k-1$, and $\lim_{m_l\to\infty} T^{m_l k} y$ exists. The compactness of $\overline{O(y)}$ implies $q = T^\delta(\lim_{m_l\to\infty} T^{m_l k} y)$. It follows that $\lim_{m_l\to\infty} T^{m_l k} y \in \omega(y, T^k) \subseteq (O(x, T^k))_+^T$, and hence $q \in (O(x))_+^T$ and $\omega(y) \subseteq (O(x))_+^T$. Therefore, we have $O(x) \preceq_T \omega(y)$. This completes the proof. $\square$

It should be emphasized that if "$\preceq_T$" in Lemma 3.1 is replaced by "$\preceq_{id}$," then the result of Lemma 3.1 does not necessarily remain valid. For example, suppose that $X = R^2$, $X_+ = R_+^2$ and define $T : R^2 \to R^2$ by $T(x, y) = (y, x)$. Let $\xi = (1, 3)$ and $\eta = (4, 2)$.

Then $O(\xi) = \{(1,3),(3,1)\}$, $\omega(\eta) = \{(4,2),(2,4)\}$, $O(\xi, T^2) = \{(1,3)\}$, $\omega(\eta, T^2) = \{(4,2)\}$. It follows that $O(\xi) \preceq_{id} \omega(\eta)$. But $O(\xi, T^2) \preceq_{id} \omega(\eta, T^2)$ does not hold. In the proof of Theorem 2.5 of Takáč [**30**], it was wrongly thought that Lemma 3.1 still holds if "$\preceq_T$" is replaced by "$\preceq_{id}$." Observe that the proofs in Theorem 2.5 of [**30**] are heavily dependent on this incorrect proposition.

As an improvement of Theorem 2.5 in Takáč [**30**], the following theorem is more effective than the previous one in applications to periodic delay differential equations. Our proof presented below, whose idea essentially originates from Takáč [**30**], also provides a correction of Takáč [**30**].

**Theorem 3.2.** *Let $T : X \to X$ be an essentially strongly increasing mapping. Suppose that $J : [0,1] \to X$ is a strictly increasing continuous path* (*with its image*) *contained in $\mathcal{P}(T)$, i.e., $\tau_1 < \tau_2$ implies $J(\tau_1) < J(\tau_2)$, with endpoints $a = J(0)$ and $b = J(1)$. Then we have the following statements*:

(i) *Let $\overline{O(J([0,1]))} = \overline{\cup_{\alpha\in[0,1]} O(J(\alpha))}$ be compact in $X$. If $x \in [a,b]$ and $\overline{O(x)}$ is compact in $X$, then $\omega(x) = O^+(J(\tau))$, where $\tau = \tau(x) : [a,b] \to [0,1]$ is an increasing mapping.*

(ii) *Let $X \subseteq V$ be open and order-convex. Denote $Y \equiv \cup_{l=0}^{\infty}[T^l a, T^l b]$ is an invariant, order-bounded set. Assume also that $T|_Y : Y \to Y$ is $\alpha$-condensing. Then there exists an integer $k \geq 1$ such that $O(J(\alpha))$ is a $k-$cycle for every $\alpha \in [0,1]$ and the order intervals $[T^l a, T^l b]$, $l = 0, 1, \ldots, k-1$, are pairwise disjoint. In particular, $O(J([0,1]))$ is the union of pairwise disjoint, simply ordered, closed arcs $T^l(J([0,1])) \subseteq \mathcal{E}(T^k) \setminus \cup_{i=1}^{k-1}\mathcal{E}(T^i)$, $0 \leq l \leq k-1$.*

*Proof of* (i). Assume that $x \in [a,b]$. Let $\mathcal{A} \equiv \{\alpha \in [0,1] : O^+(J(\alpha)) \preceq_T \omega(x)\}$. From $O^+(a) \preceq_T \omega(x)$, it follows that $\mathcal{A} \neq \varnothing$. Let $\alpha_0 = \sup \mathcal{A}$. We first show that $\alpha_0 \in \mathcal{A}$. Set $a_0 = J(\alpha_0)$. By $J(\alpha_0) \in \mathcal{P}(T)$, there exists an integer $k' \geq 1$ such that $J(\alpha_0) \in \mathcal{E}(T^{k'})$. If $\alpha_0 = 0$, then $\alpha_0 \in \mathcal{A}$ since $\mathcal{A} \neq \varnothing$. If $\alpha_0 > 0$, then the definition of $\alpha_0$ implies that there exists $\alpha_n \in \mathcal{A}$ such that $\alpha_n < \alpha_{n+1} < \alpha_0$ and $\alpha_n \to \alpha_0$ as $n \to \infty$. Set $a_n = J(\alpha_n)$ for $n \geq 1$. Then $a_n < a_{n+1} < a_0$ and $a_n \to a_0$ as $n \to \infty$. Fix any $y \in \omega(x)$. From $\alpha_n \in \mathcal{A}$, we have $T^{k_n} a_n \leq y$ for some sequence $k_n \geq 0$. Since $\overline{O(J[0,1])}$ is compact, we may assume, without loss of generality, that $T^{k_n}a_n \to z \in X$, then $z \le y$. As $T$ is increasing and $a_n < a_0$, it follows that there exists $k' > r \ge 0$ such that $z \le T^r a_0$. Thus, we have either $Tz = T^{r+1}a_0$ or $Tz \ll T^{r+1}a_0$, since $T$ is an essentially strongly increasing mapping. If the former one holds, then $T^{r+1}a_0 \le Ty$. If the latter one holds, then we can take a large $n \ge 0$ so that $TT^{k_n}a_n \ll T^{r+1}a_n$, which contradicts that $\omega(a_n) = O(a_n)$ is essentially unordered. Thus, for any $y \in \omega(x)$, there exists $0 \le r < k'$ such that $T^{r+1}a_0 \le Ty$. Again since $\omega(x)$ is totally invariant, it follows that, for every $y \in \omega(x)$, there exists $0 \le r < k'$ such that $T^{1+r}a_0 \le y$. Hence, $\omega(x) \subseteq (O(a_0))_+^T$. Choose an integer $l \ge 1$ such that $k'l > 1 + k'$. Then for any $y \in \omega(x)$, there exists $0 \le r < k'$ such that $T^{k'l-(1+r)}(T^{1+r}a_0) \le T^{k'l-(1+r)}y$ since $T$ is increasing. Thus, we have $a_0 \le T^{k'l-(1+r)}y \in \omega(x)$. This implies that $O(a_0) \subseteq (\omega(x))_-^T$, and hence $\alpha_0 \in \mathcal{A}$. Since $\alpha_0 \in \mathcal{A}$ and $T$ is an essentially strongly increasing mapping, it follows that for any $y \in \omega(x)$, there exists $k' > r = r_y \ge 0$ such that either $T^{r+1}(a_0) = Ty$ or $T^{r+1}(a_0) \ll Ty$.

If $a_0 \in \omega(x)$, since $\omega(x)$ is essentially unordered, it follows that for any $y \in \omega(x)$, there exists $r \ge 0$ such that $Ty = T^{r+1}(a_0) \in O(a_0)$. Hence, we obtain $\omega(x) = T\omega(x) \subseteq O(a_0)$. Therefore, we have $\omega(x) = O(a_0)$ and the proof is complete.

If $a_0 \notin \omega(x)$, then $a_0 \in \mathcal{P}(T)$ implies that, for any $y \in \omega(x)$, there exists $k' > r \ge 0$ such that $T^{r+1}(a_0) \ll Ty$. Thus, for any $y \in \omega(x)$, there exist $k' > r = r_y \ge 0$ and an open neighborhood $O_y$ of $y$ such that $T^{r+1}(a_0) \ll T(\overline{O_y})$. By the compactness of $\omega(x)$, we obtain that there exist $y_i \in \omega(x)$, $0 \le r_i < k'$, $1 \le i \le n_0$, such that $\cup_{i=1}^{n_0} O_{y_i} \supseteq \omega(x)$ and $T^{r_i+1}(a_0) \ll T(\overline{O_{y_i}} \cap \omega(x))$. As $\overline{O_{y_i}} \cap \omega(x)$ is compact, there exists $\alpha_i' > \alpha_0$ such that $T^{r_i+1}a_0 \ll T^{r_i+1}J(\alpha_i') \ll T(\overline{O_{y_i}} \cap \omega(x))$. Set $\alpha' = \min_{1\le i\le n_0} \alpha_i > \alpha_0$. Then, for $i = 1, 2, \dots, n_0$, we have $T^{r_i+1}a_0 \ll T^{r_i+1}J(\alpha') \ll T(\overline{O_{y_i}} \cap \omega(x))$. Again, since $(\cup_{i=1}^{n_0} \overline{O_{y_i}}) \cap \omega(x) = \omega(x)$ and $\omega(x)$ is totally invariant, we get $\omega(x) \subseteq (O(J(\alpha')))_+^T$. It follows from $T^{r_1+1}J(\alpha') \ll T(y_1)$ and $J(\alpha') \in \mathcal{P}(T)$ that $O(J(\alpha')) \subseteq (\omega(x))_-^T$. This means that $\alpha' \in \mathcal{A}$, a contradiction to the choice of $\alpha_0$.

To sum up, we have $\omega(x) = O^+(J(\tau))$, where $\tau = \tau(\cdot) \in [0, 1]$ depends on $x$.

Next we show that $\tau(\cdot)$ is increasing. If this is not the case, then there exist $x, y \in [a, b]$ with $x < y$ such that $\tau(x) > \tau(y)$. From the above discussion, we see that there exist two sequences $n_k$ and $m_k$ such that $T^{n_k}x \to J(\tau(x))$ as $n_k \to \infty$ and $T^{m_k}y \to J(\tau(y))$ as $m_k \to \infty$. Without loss of generality, we may assume that $p \equiv \lim_{m_k\to\infty} T^{m_k}x \in O(J(\tau(x)))$. Since $T$ is increasing, we get $p \le J(\tau(y))$. Hence, we have $p \ll J(\tau(x))$, which contradicts Proposition 2.2.

*Proof of* (ii). Let $k'$ be the smallest positive integer satisfying $a, b \in \mathcal{E}(T^{k'})$. Denote

$$Y_l = [T^l a, T^l b], \quad J_l = T^l(J([0,1])), \quad 0 \le l \le k'-1.$$

Then $Y \equiv \cup_{l=0}^{k'-1} Y_l$ is an invariant, order-bounded set. As $T|_Y : Y \to Y$ is $\alpha$-condensing and $\cup_{l=0}^{k'-1} J_l \subseteq Y$ is a totally invariant and bounded set, it follows that the set $\cup_{l=0}^{k'-1} J_l$ is precompact. Similarly, every orbit has compact closure. Hence, the hypotheses of Part (i) are all satisfied.

Set $I = \{\alpha \in [0,1] : J(\alpha) \in \mathcal{E}(T^{k'})\}$. Then $0, 1 \in I$. We claim that if $\alpha, \beta \in I$ with $\alpha < \beta$, then there exists $\gamma \in (\alpha, \beta) \cap I$. Denote $p = J(\alpha)$, $q = J(\beta) \in \mathcal{E}(T^{k'})$. Choose $\delta \in (\alpha, \beta)$. Then $J(\delta) \in \mathcal{P}(T^{k'})$. It follows from $p < J(\delta) < q$ and Proposition 2.3 that $p \ll \omega(J(\delta), T^{k'}) \ll q$. By Proposition 2.4, there exists $r \in \mathcal{E}(T^{k'}) \cap (p, q)$. Applying Theorem 3.2 (i), we obtain that there exists $\gamma \in [0,1]$ such that $\omega(r, T^{k'}) = \{r\} = O(J(\gamma), T^{k'}) = \{J(\gamma)\}$. Hence, $\gamma \in (\alpha, \beta) \cap I$. From the above claim and $0, 1 \in I$, it follows that $\overline{I} = [0,1]$. Again, since $I$ is a closed set, we obtain $I = \overline{I} = [0,1]$, from which we can conclude that $J([0,1]) \subseteq \mathcal{E}(T^{k'})$.

Hence $J_l \subseteq \mathcal{E}(T^{k'})$, and $J_l$ is a simply ordered, closed arc. By Theorem 3.2 (i), the arc $J_l$ attracts the set $Y_l$ under the mapping $T^{k'}$. Now suppose that the set $Y_l$, $0 \le l \le k-1$, are not pairwise disjoint. Otherwise, we may assume, without loss of generality, that $Y_0 \cap Y_m \ne \varnothing$ for some $m \in \{1, 2, \dots, k-1\}$. Choose $y \in Y_0 \cap Y_m$. Then $T^{nk'}y \to p$ as $n \to \infty$ for some $p \in J_0 \cap J_m$. Next we prove that $J_0^0 \cap J_m^0 \ne \varnothing$, where $J_l^0 = J_l \setminus \{T^l a, T^l b\}$ for $0 \le l \le k'-1$. If $p \in J_0^0 \cap J_m^0 \ne \varnothing$, then $J_0^0 \cap J_m^0 \ne \varnothing$. Otherwise, we have $p \in \{a, b, T^m a, T^m b\}$. Without loss of generality, we may assume that $p = a \in J_0 \cap J_m$. Since $X$ is open in $V$ and $J_m \subseteq \mathcal{E}(T^{k'})$ is a totally ordered arc, we obtain $J_m \cap [[a,b]] \ne \varnothing$, that is, there exists $\alpha^* \in (0,1)$ such that $J_m(\alpha^*) \in [[a,b]]$. By Theorem 3.2 (i), we have $\omega(J_m(\alpha^*), T^{k'}) = J_m(\alpha^*) = J_0(\alpha^{**})$ for some $\alpha^{**} \in [0,1]$. It is easily seen that $\alpha^{**} \in (0,1)$, and thus $L \equiv J_0^0 \cap J_m^0 \ne \varnothing$. Clearly, $L \subseteq \operatorname{Int}(Y_0)$

and $J_m^0 \setminus L \subseteq X \setminus Y_0$ which shows that $L = J_m^0$ because $J_m^0$ is connected. Again, since $\omega(J(\alpha))$ is essentially unordered for $\alpha \in [0,1]$, it follows that $J_m^0 \subseteq \mathcal{E}(T^m)$, and hence $a, b \in \mathcal{E}(T^m)$, which contradicts the definition of $k'$. Therefore, $Y_l$, $0 \le l \le k-1$, are pairwise disjoint sets and we have proved (ii). □

The following result improves Theorem 2 of Dancer and Hess [**5**].

**Theorem 3.3.** *Assume that $a^*, b^* \in V$ satisfy $a^* \ll b^*$, and let the mapping $T : [a^*, b^*]_V \to [a^*, b^*]_V$ be essentially strongly increasing and $\alpha$-condensing. Then we have the following statements*:

(i) *If $\mathcal{E}(T)$ is a singleton, then $\omega(x) = \mathcal{E}(T)$ for all $x \in [a^*, b^*]_V$.*

(ii) *If $\mathcal{E}(T)$ is not a singleton and every equilibrium of $T$ is Liapunov stable, then there exists a strictly increasing continuous mapping $J : [0,1] \to [a^*, b^*]_V$ such that $\mathcal{E}(T) = J([0,1])$ and, for any $x \in [a^*, b^*]_V$, there exists $\tau \in [0,1]$ such that $\omega(x) = \{J(\tau)\}$.*

*Proof.* Since $T$ is $\alpha$-condensing, $\overline{O(x)}$ is compact for all $x \in [a^*, b^*]_V$. By $a^* \le Ta^*$, $Tb^* \le b^*$ and Proposition 2.1, there exist $a, b \in \mathcal{E}(T)$ such that $T^n a^* \to a$ and $T^n b^* \to b$ as $n \to \infty$. Hence, $a \le b$. If $\mathcal{E}(T)$ is a singleton, then $a = b$. Conclusion (i) follows from the fact that $T$ is increasing. If $\mathcal{E}(T)$ is not a singleton, then $a \ll b$ since $T$ is essentially strongly increasing. Set $X = [a, b]_V$. Then $T : X \to X$ is essentially strongly increasing and $\alpha$-condensing. Next, we prove that $\mathcal{E}(T)$ is order-connected. Set $\mathcal{A} = \{A \subseteq \mathcal{E}(T) \cap [a, b] : A$ is a simply ordered subset$\}$. Now consider the partially ordered set $(\mathcal{A}, \subseteq)$. By Zorn's lemma, $\mathcal{A}$ has a maximal element $A^*$. We claim that $A^*$ has the following properties:

(i) $a, b \in A^*$.

(ii) $p, q \in A^*$ with $p < q$ implies that $p \ll c \ll q$ for some $c \in A^*$.

(iii) $A^*$ is compact.

(iv) $A^*$ is connected.

Property (i) is obvious. If property (ii) is not true, then $p \ll q$ since $T$ is essentially strongly increasing. By Theorem 3.1, $[[p, q]]$ is attracted by either $p$ or $q$, which contradicts that $p$ and $q$ are Liapunov stable. Therefore, property (ii) is true. Since $T$ is $\alpha$-condensing and $A^* \subseteq [a,b] \cap \mathcal{E}(T)$, it follows that $A^*$ is precompact. Next we show that $A^*$ is closed. Otherwise, there exist $x \in [a,b] \setminus A^*$ and $x_n \in A^*$ such that $\lim_{n\to\infty} x_n = x$. Without loss of generality, we may assume that $x_n \le x_{n+1} \le x$ for $n \ge 1$, since $A^*$ is totally ordered. By $x_n \in \mathcal{E}(T)$, we have $x \in \mathcal{E}(T)$. For any $y \in A^*$, we have either $y > x_n$ for all $n \ge 1$ or $y \le x_{n_0}$ for some $n_0 \ge 1$. If the former one holds, then $y \ge x$. If the latter one holds, then $x \ge y$. Therefore, $A^* \cup \{y\} \subseteq \mathcal{E}(T) \cap [a,b]$ is totally ordered, which contradicts the choice of $A^*$.

Now we are ready to prove property (iv). If $A^*$ is not connected, then there exist nonempty closed subsets $\widetilde{A}$ and $\widetilde{B}$ such that $\widetilde{A} \cap \widetilde{B} = \varnothing$ and $\widetilde{A} \cup \widetilde{B} = A^*$. Choose $p \in \widetilde{A}$ and $q \in \widetilde{B}$. Without loss of generality, we may assume that $p < q$. Set $A = \widetilde{A} \cap [p,q]$ and $B = \widetilde{B} \cap [p,q]$. Since $B$ is compact and totally ordered, there exists $q^* \in B$ such that $q^* = \inf B > p$. Again, since $A$ is compact and totally ordered, there exists $p^* \in A$ such that $q^* > p^* = \sup(A \cap [p, q^*])$. By property (ii), there exists $c \in A^*$ such that $p^* \ll c \ll q^*$. But $A \cup B = A^* \cap [p,q]$ implies that $c \in A \cup B$, a contradiction to the choices of $p^*$ and $q^*$. Hence, we have proved (iv).

Combining the properties (i), (iii) and (iv) with Theorem 11.12 in Chapter I of Wilder [**35**], we see that $Y^*$ is the image of the strictly increasing continuous path $J : [0,1] \to [a^*, b^*]_V$, where $J(0) = a$ and $J(1) = b$. Assume that $x \in [a^*, b^*]_V$. Then the choices of $a$ and $b$ imply that $a \le \omega(x) \le b$. If $a \in \omega(x)$ or $b \in \omega(x)$, then either $\omega(x) = \{a\}$ or $\omega(x) = \{b\}$. If $a, b \notin \omega(x)$, then $a \ll \omega(x) \ll b$ since $T$ is essentially strongly increasing. By the definition of $\omega(x)$, there exists $n' > 1$ such that $a \ll T^{n'}(x) \ll b$. Therefore, by Theorem 3.2 (i), for every $x \in [a^*, b^*]_V$ there exists $\tau_x \in (0,1)$ such that $\omega(x) = \{J(\tau_x)\}$. This completes the proof. $\square$

The following lemma is an improvement of a result of Dancer [**3**].

**Lemma 3.2.** *Let $X \subseteq V$ be closed, order-convex, and let the mapping $S : X \to X$ be continuous, increasing and conditionally $\alpha$-condensing. Assume also that every orbit is bounded in $X$. If every compact and totally invariant subset of $X$ has upper bound and lower bound in*

*$X$, then for any compact and totally invariant subset $K$, there exist $p, q \in \mathcal{E}(T)$ such that $p \leq K \leq q$.*

*Proof.* Let $K$ be a compact and totally invariant subset of $X$. Since $K$ has upper bound in $X$, there exists an $a \in X$ such that $K \leq a$. By the invariance of $K$, we have $K \leq \omega(a, S)$. Similarly, there exists a $b \in X$ such that $\omega(a, S) \leq \omega(b, S)$. Set $\mathcal{A} = \{A \subseteq X : \omega(a, S) \subseteq A,\ K \leq A \leq \omega(b, S),\ A$ is closed convex, $SA \subseteq A\}$, $A^* = \cap_{A \in \mathcal{A}} A$ and $A^{**} = \overline{co(S(A^*))}$. By the choices of $a$ and $b$, we obtain $\mathcal{A} \neq \varnothing$. It follows from the definitions of $A^*$ and $A^{**}$ that $A^* = A^{**} \in \mathcal{A}$ are closed convex. As $S$ is $\alpha$-condensing, we obtain that $A^*$ is a compact, totally invariant and convex subset. By Brouwer's fixed-point theorem, there exists a $q \in A^* \cap \mathcal{E}(S)$, and hence $K \leq q$. Similarly, there exists a $p \in \mathcal{E}(S)$ such that $p \leq K$. This completes the proof. $\square$

Our final theorem improves a result of Takáč [**28**] and also needs the following hypotheses:

(A) $X$ is a closed order-convex subset of $V$ and its every compact subset has upper and lower bound in $X$.

(B) $T : X \to X$ is conditionally $\alpha$-condensing and is an essentially strongly increasing mapping, and every equilibrium of $T$ is Liapunov stable.

(C) For any $x \in X$, $O(x)$ is a bounded set.

The following result improves Theorem 1.5 of Takáč [**28**] in several aspects. First, we weaken the monotonicity and compactness assumptions on the mappings. Second, we do not require the ordered space "$X$" to be strongly ordered. Finally, unlike in Theorem 1.5 of [**28**], we do not need any lattice structure and solely need that every compact subset of the ordered space $X$ has upper and lower bound. It should be noted that if $X$ is an order interval such as $[a, b]_V$ and $[a, \infty]]_V$, then assumption (A) follows naturally, but such an $X$ does not necessarily satisfy the lattice conditions as required in Theorem 1.5 of [**28**].

**Theorem 3.4.** *Let* (A), (B) *and* (C) *hold. Then $\mathcal{E}(T)$ is either a singleton or the image of a strictly increasing continuous path $J : I_J \to X$, $I_J \subseteq [0, 1]$. $I_J$ is a closed interval whenever $\mathcal{E}(T) \subseteq X$ is compact. Furthermore, if $x \in X$, then $\omega(x) = p$ for some $p \in \mathcal{E}(T)$.*

*Proof.* Suppose that $\mathcal{E}(T)$ is a singleton. Then Lemma 3.2 implies $\mathcal{E}(T) \le \omega(x) \le \mathcal{E}(T)$ for any $x \in X$ and hence $\omega(x) = \mathcal{E}(T)$. The proof is complete. Suppose that $\mathcal{E}(T)$ is not a singleton. Then, by Theorem 3.3, $\mathcal{E}(T)$ is order-connected. Denote by $\Omega$ the set of all maximal totally ordered subsets of $\mathcal{E}(T)$. By Zorn's lemma, we have $\Omega \ne \varnothing$. Choose $B \in \Omega$. It is easily seen that $B$ is either order-unbounded or compact. Next we show that $B = \mathcal{E}(T)$. Otherwise, there exists an $a \in \mathcal{E}(T) \setminus B$. Fix any $b \in B$. Set $K \equiv \{a, b\}$. Then $K$ is compact and totally invariant. It follows from Lemma 3.2 that there exist $p, q \in \mathcal{E}(T)$ such that $p \le K \le q$. By Theorem 3.3, $[p, q] \cap \mathcal{E}(T)$ is a totally ordered arc. Hence, $a, b \in [p, q] \cap \mathcal{E}(T)$ implies that either $a < b$ or $a > b$. Since $b$ is arbitrary, we see that $B' = B \cup \{a\}$ is still a totally ordered subsets of $\mathcal{E}(T)$, which contradicts $B \in \Omega$. Therefore, we obtain $B = \mathcal{E}(T)$. Using a similar argument as that in Theorem 3.3 (ii), we can show that $B$ is the image of the strictly increasing continuous mapping $J : I_J$ (e.g., $[0,1], (0,1), [0,1), (0,1]$) $\to X$ and if $\mathcal{E}(T)$ is compact then $I_J$ is a closed interval.

Fix any $x \in X$. Since $\omega(x)$ is compact and totally invariant, it follows from Lemma 3.2 that there exist $p, q \in \mathcal{E}(T)$ such that $p \le \omega(x) \le q$. If $p = q$, then $\omega(x) = \{p\} = \{q\}$ and the proof is complete. If $p < q$, then $p \ll q$ since $T$ is essentially strongly increasing. In this case, if $p \in \omega(x)$ or $q \in \omega(x)$, then Proposition 2.3 implies that $\omega(x) = p$ or $\omega(x) = q$; if $p, q \notin \omega(x)$, then $p \ll \omega(x) \ll q$. By the definition of $\omega(x)$, there exists an $n_0 \ge 1$ such that $p \ll T^{n_0}x \ll q$. It follows from Theorem 3.3 that there exists an $r \in \mathcal{E}(T)$ such that $\omega(x) = \omega(T^{n_0}x) = \{r\}$. This completes the proof. □

As an immediate consequence to Theorem 3.4, we obtain the following result, which improves Theorem $2'$ of [**5**] in many respects.

**Corollary 3.1.** *Let $T : X \to X$ be a continuous, essentially strongly order-preserving and conditionally $\alpha$-condensing mapping, where $X$ is $V$ or $V_+$. Assume further that every orbit is bounded and all fixed points are stable. Then $\omega(x) = \{q\} \subseteq \mathcal{E}(T)$, for any $x \in X$.*

**4. Examples.** Let $n$ be a given positive integer. Let $r > 0$ be a given real number, and let $C = C([-r, 0], R^n)$ be the Banach space of continuous mappings from $[-r,0]$ into $R^n$, equipped with the supremum norm. Define $C_+ = C([-r,0], R^n_+)$. It follows that $(C, C_+)$ is a strongly ordered Banach space. For any $\varphi, \psi \in C$, we write $\varphi \le \psi$ if $\psi - \varphi \in C_+$, $\varphi < \psi$ if $\varphi \le \psi$ and $\varphi \ne \psi$, $\varphi \ll \psi$ if $\psi - \varphi \in \operatorname{Int} C_+$. For any $A \subseteq C$, we write $\varphi \le A$ if $\varphi \le \psi$ for all $\psi \in A$, $\varphi < A$ if $\varphi < \psi$ for all $\psi \in A$, $\varphi \ll A$ if $\varphi \ll \psi$ for all $\psi \in A$. Similarly, we can define "$\ge$," "$>$" and "$\gg$." For instance, $\varphi \ge \psi$ if $\varphi - \psi \in C_+$. For any $x \in R^n$, we define $\widehat{x} \in C$ by $\widehat{x}(\theta) = x$, $\theta \in [-r,0]$.

Now, we consider applications of our main results to periodic systems of delay differential equations, which illustrate that our main results do not require the delicate choice of state space and the technical *ignition* assumption required by the classical results.

**Example 4.1.** Consider the following generalized gonorrhea model, see [**27**]:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = R_i(t,x_t) - C_i(t,x_t) \equiv F_i(t,x_t), \quad = 1,2,\dots,n. \tag{4.1}$$

Here $F_i : R^1_+ \times C_+ \to R^1$ is continuously differentiable and $\tau-$periodic in time $t \in R^1_+$. Assume that $t_0 \in R^1_+$ and $\varphi \in C_+$, and use $x_t(t_0,\varphi)$ to denote the solution of (4.1) with the initial data $x_{t_0}(t_0,\varphi) = \varphi$. Set $F = (F_1, F_2, \dots, F_n)^T$. It is also assumed that $F$ is completely continuous. In order to apply the main results developed above to system (4.1), we need the following several important hypotheses. Choose $P = (P_1, P_2, \dots, P_n)^T \in \operatorname{Int} R^n_+$.

(A1) $F(t,\widehat{0}) \ge 0$ and $F(t,\widehat{P}) \le 0$ for all $t \in R^1_+$.

(A2) Assume that $t \in R^1_+$ and $\varphi, \psi \in C_+$ with $\varphi \le \psi$. If $\varphi_i(0) = \psi_i(0)$, then $F_i(t,\varphi) \le F_i(t,\psi)$.

(A3) For any $\varphi \in [\widehat{0}, \widehat{P}]$ and $t \ge 0$, the $n \times n$ matrix $D_\varphi F(t,\varphi)(\widehat{e_1}, \widehat{e_2}, \dots, \widehat{e_n})$ is irreducible.

(A4) Whenever $\varphi \in [\widehat{0}, \widehat{P}]$, $t \in R^1_+$ and $\alpha \in [0,1]$, it follows that $\alpha F(t,\varphi) \le F(t, \alpha\varphi)$.

By the hypotheses (A1) and (A2), we conclude that for any $\varphi \in [\widehat{0}, \widehat{P}]$, $x_t(t_0,\varphi)$ is well defined for all $t \ge t_0$.

The following results can be proved similarly as [**22**, Theorem 2.5] and [**34**, Theorem 2.1].

**Proposition 4.1.** *Let* (A1), (A2) *and* (A3) *hold. If* $\varphi, \psi \in [\widehat{0}, \widehat{P}]$ *satisfy* $\varphi \le \psi$, *then either*

$$x_t(t_0, \varphi) = x_t(t_0, \psi) \quad \textit{for } t \ge (n+2)r + t_0, \quad \textit{or}$$
$$x_t(t_0, \varphi) \ll x_t(t_0, \psi) \quad \textit{for } t \ge (n+2)r + t_0.$$

**Theorem 4.1.** *Let* (A1), (A2), (A3) *and* (A4) *hold. Denote* $\mathcal{P} = \{\psi \in [\widehat{0}, \widehat{P}] : x_t(0, \psi)$ *is* $\tau$*-periodic in time* $t \in R^1_+\}$. *If* $\varphi \in [\widehat{0}, \widehat{P}]$, *then there exists* $\psi \in \mathcal{P}$ *such that* $\lim_{t\to 0} |x_t(0,\varphi) - x_t(0,\psi)| = 0$.

*Proof.* Define $T : [\widehat{0}, \widehat{P}] \to [\widehat{0}, \widehat{P}]$ by $T\varphi = x_{N\tau}(0, \varphi)$, where $N\tau \ge (n+2)r$. Then $T$ is a compact, essentially strongly increasing mapping. By (A4), we have $\alpha T\varphi \le T(\alpha\varphi)$ for all $\alpha \in [0,1]$ and $\varphi \in [\widehat{0}, \widehat{P}]$. This means that $T$ is sublinear. Arguing as in the proof of Lemma 2.3 in [**32**], we can show that every $x \in \mathcal{E}(T) \cap [[\widehat{0}, \widehat{P}]$ is Liapounov stable. Since $T$ is essentially strongly increasing and $T([\widehat{0}, \widehat{P}]) \subseteq [\widehat{0}, \widehat{P}]$, it follows that $\mathcal{E}(T) \subseteq (\mathcal{E}(T) \cap [[\widehat{0}, \widehat{P}]) \cup \{\widehat{0}\}$. If $\{\widehat{0}\} \notin \mathcal{E}(T)$ or $\widehat{0}$ is Liapounov stable, then Theorem 4.1 follows by applying Theorem 3.3. If $\{\widehat{0}\} \in \mathcal{E}(T)$ is Liapounov unstable, then $T\widehat{P} \le \widehat{P}$ implies that there exists $\widehat{p} \in \mathcal{E}(T) \cap [\widehat{0}, \widehat{P}]$. Choose $X_\varepsilon = [\varepsilon\widehat{p}, \widehat{P}]$. Then $TX_\varepsilon \subseteq X_\varepsilon$. As $T$ is essentially strongly increasing, for any $\varphi \in [\widehat{0}, \widehat{P}]$ we have either $\omega(\varphi) = \{\widehat{0}\}$ or there exist $\varepsilon_0 \in (0,1)$ such that $T\varphi \ge \varepsilon_0\widehat{p}$. Applying Theorem 3.3 to $T|_{X_{\varepsilon_0}}$, we obtain $\omega(\varphi) = \omega(T\varphi) = \{q\}$ for some $q \in \mathcal{E}(T)$. This completes the proof. $\square$

**Example 4.2.** Consider the following cyclic feedback system

$$\begin{cases} \mathrm{d}x^1/\mathrm{d}t = f(t, x^n_t) - \alpha_1(t)x^1(t), \\ \mathrm{d}x^i/\mathrm{d}t = x^{i-1}(t - r_{i-1}) - \alpha_i(t)x^i(t), \quad 2 \le i \le n, \end{cases} \tag{4.2}$$

where $r_i > 0$, $1 \le i \le n$, $\alpha_i(\cdot)$ and $f(\cdot, \varphi^n)$ ($\varphi^n \in C([-r_n, 0], R^1_+)$) are continuous and $\tau$-periodic in time $t \in R^1_+$, $\alpha_i = \min_{0\le t\le\tau} \alpha_i(t) > 0$ for $1 \le i \le n-1$, $f(t,\cdot)$ is a continuously differentiable and sublinear mapping on $C([-r_n, 0], R^1_+)$. System (4.2) has been used to model the control of protein systems in the cell, see e.g., [**22, 23, 26**] and the references therein.

To obtain boundedness of solution of system (4.2), Smith and Thieme [**26**] introduced the following assumption:

(B) There exist $a, b \geq 0$ such that $b < \alpha_1 \alpha_2 \cdots \alpha_n$ and $f(t, \varphi) \leq a + b\|\varphi\|$ for all $(t, \varphi) \in R^1_+ \times C_+$.

To establish the monotonicity properties of system (4.2), we also need the following assumptions:

(B1) Assume that $t \in R^1_+$ and $\varphi, \psi, \psi - \varphi \in C([-r_n, 0], R^1_+)$. If $\varphi(0) = \psi(0)$, then $f(t, \varphi) \leq f(t, \psi)$.

(B2) If $\varphi \in C([-r_n, 0], R^1_+)$ and $t \in R^1_+$, then $D_\varphi f(t, \varphi)\widehat{1} > 0$.

Set $r = \max_{1 \leq i \leq n} r_i$. Let

$$F(t, \varphi) = (f(t, \varphi^n) - \alpha_1(t)\varphi^1(0), \varphi^1(-r_1) - \alpha_2(t)\varphi^2(0), \ldots,\\ \varphi^{n-1}(r_{n-1}) - \alpha_n(t)\varphi^n(0)),$$

where $(t, \varphi) \in R^1_+ \times C_+$. Consider the following system of delay differential equations

$$x'(t) = F(t, x_t). \tag{4.3}$$

To study the asymptotic behavior of solutions of (4.2), we observe that it suffices to investigate the asymptotic behavior of solutions of (4.3). For $\varphi \in C_+$, we denote by $x_t(t_0, \varphi)$ the solution of (4.3) with the initial data $x_{t_0}(t_0, \varphi) = \varphi \in C_+$. Define $T(-\varphi) = x_t(0, \varphi)$ for all $\varphi \in C_+$. Then we have the following result.

**Lemma 4.1.** *Let* (B1) *and* (B2) *hold. Then $F$ satisfies assumptions* (A2) *and* (A4) *in Example* 4.1 *with* $[\widehat{0}, \widehat{P}]$ *replaced by* $C_+$*.*

*Let* $M_n = 1$ *and* $M_{i-1} = \alpha_i M_i$, $2 \leq i \leq n$. *Set*

$$B_r = \{\varphi \in C_+ : 0 \leq \varphi^i \leq r\widehat{M_i},\ 1 \leq i \leq n\}.$$

We can obtain the following lemma by arguing as in the proof of [**26**, Theorem 5.2].

**Lemma 4.2.** *Let* (B) *hold. Then, for sufficiently large positive integer* $r$, $x_t(0, \varphi) \in B_r$ *for all* $t \geq 0$ *and* $\varphi \in B_r$*.*

**Proposition 4.2.** *Let* (B1), (B2) *and* (B) *hold. Then $T$ is an essentially strongly positive sublinear operator.*

*Proof.* Proposition 4.2 follows easily from Lemma 4.1 and the fact that $f$ is sublinear.

Applying Corollary 3.1, we can prove the following result by arguing as in the proof of Theorem 4.1.

**Theorem 4.2.** *Let the hypotheses of Proposition* 4.2 *hold. Let $\mathcal{P} = \{\psi \in C_+ : x_t(0,\psi)$ is $\tau$-periodic in time $t \in R^1_+\}$. If $\varphi \in C_+$, then there exists a $\psi \in \mathcal{P}$ such that* $\lim_{t\to 0} |x_t(0,\varphi) - x_t(0,\psi)| = 0$.

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