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MULTIPLICITY RESULTS FOR A SEMI-LINEAR ELLIPTIC EQUATION INVOLVING SIGN-CHANGING WEIGHT FUNCTION

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ABSTRACT. In this paper, we study the combined effect of concave and convex nonlinearities on the number of solutions for a semi-linear elliptic equation with sign-changing weight functions. With the help of the Nehari manifold, we prove that there are at least two solutions for the equation $(E_{a,b})$.

1. Introduction. In this paper, we consider the multiplicity results of solutions of the following semi-linear elliptic equation:

$$(E_{a,b}) \qquad \begin{cases} -\Delta u = \lambda a(x)u^q + b(x)u^p & \text{in } \Omega, \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $0 \leq q < 1 < p < 2^* - 1$ ($2^* = (2N/N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 2$), $\lambda > 0$, and the weight functions a, b satisfy the following conditions:

(A) $a^+ = \max\{a, 0\} \not\equiv 0$ and $a \in L^{r_q}(\Omega)$ where $r_q = r/r - (q+1)$ for some $r \in (q+1, 2^*)$, with in addition $a(x) \geq 0$ almost everywhere in Ω in case $q = 0$;

(B) $b^+ = \max\{b, 0\} \not\equiv 0$ and $b \in L^{s_p}(\Omega)$ where $s_p = (s/s - (p+1))$ for some $s \in (p+1, 2^*)$.

The fact that the number of positive solutions of equation $(E_{a,b})$ is affected by the concave and convex nonlinearities has been the focus of a great deal of research in recent years. If the weight functions $a \equiv b \equiv 1$, the authors Ambrosetti, Brezis and Cerami [**1**] have investigated equation $(E_{1,1})$. They found that there exists $\lambda_0 > 0$ such that equation $(E_{1,1})$ admits at least two positive solutions for

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$\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Wu [**8**] proved that equation $(E_{a,b})$ has at least two positive solutions under the assumptions the weight functions a change sign in $\overline{\Omega}$, $b \equiv 1$, and λ is sufficiently small. For a more general result, de Figueiredo et al. [**4**] proved the following result:

Theorem 1.1. *Assume that the conditions* (A) *and* (B) *hold, and in addition*

(C1) *there exists a nonempty open subset* $\Omega_1 \subset \Omega$ *such that, on* Ω_1, $a(x) \geq \varepsilon_1$ *for some* $\varepsilon_1 > 0$ *and* $b(x)$ *is bounded from below;*

(C2) *there exists a nonempty open subset* $\Omega_2 \subset \Omega$ *such that, on* Ω_2, $b(x) \geq \varepsilon_2$ *for some* $\varepsilon_2 > 0$ *and* $a(x)$ *is bounded from below.*

Then there exists $\overline{\lambda} > 0$ *such that if* $\lambda \in (0, \overline{\lambda})$, *then equation* $(E_{a,b})$ *has at least two solutions.*

The main purpose of this paper is to use a new method to improve Theorem 1.1. In particular, we do this without assuming conditions (C1) and (C2). Our main result is the following.

Theorem 1.2. *Assume that the conditions* (A) *and* (B) *hold. Then there exists* $\Lambda_0 > 0$ *such that for* $\lambda \in (0, \Lambda_0)$, *equation* $(E_{a,b})$ *has at least two solutions.*

Among the other interesting problems which are similar to equation $(E_{a,b})$ for $q = 1$, Brown and Zhang [**2**] have investigated the following equation:

$$(1) \qquad \begin{cases} -\Delta u = \lambda a(x) u + b(x)|u|^{p-1}u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ and $a, b : \overline{\Omega} \to \mathbf{R}$ are smooth functions which change sign in $\overline{\Omega}$. They found existence and nonexistence results for positive solutions of equation (1) as λ changes.

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we prove that equation $(E_{a,b})$ has at least two solutions for λ sufficiently small.

2. Notations and preliminaries. Throughout this section, we denote by S_l the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^l(\Omega)$, where $1 < l \leq 2^*$. Associated with equation $(E_{a,b})$, we consider the energy functional J_λ, for each $u \in H_0^1(\Omega)$,

$$J_\lambda (u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{q+1} \int_\Omega a \, |u|^{q+1} \, dx - \frac{1}{p+1} \int_\Omega b \, |u|^{p+1} \, dx.$$

It is well known that the solutions of equation $(E_{a,b})$ are the critical points of the energy functional J_λ, see Rabinowitz [**6**]. Moreover, we consider the Nehari minimization problem: for $\lambda > 0$,

$$\alpha_\lambda (\Omega) = \inf \{J_\lambda (u) \ | \ u \in \mathbf{M}_\lambda (\Omega)\},$$

where $\mathbf{M}_\lambda(\Omega) = \{u \in H_0^1(\Omega) \backslash \{0\} \mid \langle J'_\lambda(u), u\rangle = 0\}$.

Define

$$\psi_\lambda (u) = \langle J'_\lambda (u), u\rangle = \|u\|^2_{H^1} - \lambda \int_\Omega a \, |u|^{q+1} \, dx - \int_\Omega b \, |u|^{p+1} \, dx.$$

Then for $u \in \mathbf{M}_\lambda(\Omega)$,

$$\langle \psi'_\lambda (u), u\rangle = 2 \, \|u\|^2_{H^1} - (q+1) \, \lambda \int_\Omega a \, |u|^{q+1} \, dx - (p+1) \int_\Omega b \, |u|^{p+1} \, dx.$$

Similarly to the method used in Tarantello [**7**], we split $\mathbf{M}_\lambda(\Omega)$ into three parts:

$$\begin{aligned} \mathbf{M}^+_\lambda (\Omega) &= \{u \in \mathbf{M}_\lambda (\Omega) \mid \langle \psi'_\lambda (u), u\rangle > 0\}, \\ \mathbf{M}^0_\lambda (\Omega) &= \{u \in \mathbf{M}_\lambda (\Omega) \mid \langle \psi'_\lambda (u), u\rangle = 0\}, \\ \mathbf{M}^-_\lambda (\Omega) &= \{u \in \mathbf{M}_\lambda (\Omega) \mid \langle \psi'_\lambda (u), u\rangle < 0\}. \end{aligned} \tag{2}$$

Then, we have the following results.

Lemma 2.1. *There exists* $\lambda_1 > 0$ *such that for each* $\lambda \in (0, \lambda_1)$ *we have* $\mathbf{M}^0_\lambda(\Omega) = \varnothing$.

Proof. Suppose otherwise, that is, $\mathbf{M}^0_\lambda(\Omega) \neq \varnothing$ for all $\lambda > 0$. Then, for $u_0 \in \mathbf{M}^0_\lambda(\Omega)$, we have

$$0 = \langle \psi'_\lambda (u), u\rangle = (1-q) \, \|u\|^2_{H^1} - (p-q) \int_\Omega b \, |u|^{p+1} \, dx \tag{3}$$

$$= (1-p) \, \|u\|^2_{H^1} - \lambda \, (q-p) \int_\Omega a \, |u|^{q+1} \, dx. \tag{4}$$

By the Sobolev imbedding theorem, there exist positive numbers C_1, C_2 such that

$$\|u\|_{H^1}^2 \le C_1 \|u\|_{H^1}^{p+1} \quad \text{and} \quad \|u\|_{H^1}^2 \le \lambda C_2 \|u\|_{H^1}^{q+1}$$

or

$$\|u\|_{H^1} \ge C_1^{1/1-p} \quad \text{and} \quad \|u\|_{H^1} \le (\lambda C_2)^{1/1-q}.$$

If λ is sufficiently small, this is impossible. Thus, we can conclude that there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, we have $\mathbf{M}_\lambda^0(\Omega) = \varnothing$. □

Lemma 2.2 (i) *If* $u \in \mathbf{M}_\lambda^+(\Omega)$, *then* $\int_\Omega a|u|^{q+1}\,dx > 0$;
(ii) *If* $u \in \mathbf{M}_\lambda^-(\Omega)$, *then* $\int_\Omega b|u|^{p+1}\,dx > 0$.

Proof. The proof is immediate from (3) and (4). □

By Lemma 2.1, for $\lambda \in (0, \lambda_1)$, we write $\mathbf{M}_\lambda(\Omega) = \mathbf{M}_\lambda^+(\Omega) \cup \mathbf{M}_\lambda^-(\Omega)$ and define

$$\alpha_\lambda^+(\Omega) = \inf_{u \in \mathbf{M}_\lambda^+(\Omega)} J_\lambda(u); \qquad \alpha_\lambda^-(\Omega) = \inf_{u \in \mathbf{M}_\lambda^-(\Omega)} J_\lambda(u).$$

The following lemma shows that the minimizers on $\mathbf{M}_\lambda(\Omega)$ are "usually" critical points for J_λ.

Lemma 2.3. *For* $\lambda \in (0, \lambda_1)$, *if* u_0 *is a local minimizer for* J_λ *on* $\mathbf{M}_\lambda(\Omega)$, *then* $J_\lambda'(u_0) = 0$ *in* $H^{-1}(\Omega)$.

Proof. Our proof is almost the same as that in [**2**, Theorem 2.3]. □

For each $u \in \mathbf{M}_\lambda^-(\Omega)$, we write

$$t_{\max} = \left(\frac{(1-q)\|u\|_{H^1}^2}{(p-q)\int_\Omega b\,|u|^{p+1}\,dx} \right)^{1/(p-1)} > 0.$$

Then, we have the following lemmas.

Lemma 2.4. *Let* $\lambda_2 = (p-1)/(p-q)(1-q)/(p-q)^{(1-q)/(p-1)} \times (1/(\|a\|_{L^{r_q}} S_r^{q+1}))(1/(S_s^{p+1}\|b\|_{L^{s_p}}))^{(1-q)/(p-1)}$. *Then for each* $u \in \mathbf{M}_\lambda^-(\Omega)$ *and* $\lambda \in (0, \lambda_2)$, *we have*

(i) *if* $\int_\Omega a|u|^{q+1}\,dx \le 0$, *then* $J_\lambda(u) = \sup_{t\ge 0} J_\lambda(tu) > 0$;

(ii) *if* $\int_\Omega a|u|^{q+1}\,dx > 0$, *then there is a unique* $0 < t^+ = t^+(u) < t_{\max}$ *such that* $t^+u \in \mathbf{M}_\lambda^+$ *and*

$$J_\lambda\left(t^+u\right) = \inf_{0\le t\le t_{\max}} J_\lambda\left(tu\right), \qquad J_\lambda\left(u\right) = \sup_{t\ge t_{\max}} J_\lambda\left(tu\right).$$

Proof. Fix $u \in \mathbf{M}_\lambda^-(\Omega)$. Let

$$h\left(t\right) = t^{1-q}\left\|u\right\|_{H^1}^2 - t^{p-q}\int_\Omega b\left|u\right|^{p+1}\,dx \quad \text{for } t \ge 0.$$

We have $h(0) = 0$, $h(t) \to -\infty$ as $t \to \infty$, $h(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$\begin{aligned} h\left(t_{\max}\right) &= \left(\frac{(1-q)\left\|u\right\|_{H^1}^2}{(p-q)\int_\Omega b\left|u\right|^{p+1}dx}\right)^{(1-q)/(p-1)}\left\|u\right\|_{H^1}^2 \\ &\quad - \left(\frac{(1-q)\left\|u\right\|_{H^1}^2}{(p-q)\int_\Omega b\left|u\right|^{p+1}dx}\right)^{(p-q)/(p-1)}\int_\Omega b\left|u\right|^{p+1}\,dx \\ &= \left\|u\right\|_{H^1}^{q+1}\left[\left(\frac{1-q}{p-q}\right)^{(1-q)/(p-1)} - \left(\frac{1-q}{p-q}\right)^{(p-q)/(p-1)}\right] \\ &\quad \times\left(\frac{\left\|u\right\|_{H^1}^{p+1}}{\int_\Omega b\left|u\right|^{p+1}dx}\right)^{(1-q)/(p-1)} \\ &\ge \left\|u\right\|_{H^1}^{q+1}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{(1-q)/(p-1)}\left(\frac{1}{S_s^{p+1}\left\|b\right\|_{L^{s_p}}}\right)^{(1-q)/(p-1)}, \end{aligned}$$

or

$$(5) \quad h\left(t_{\max}\right) \ge \left\|u\right\|_{H^1}^{q+1}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{(1-q)/(p-1)}\left(\frac{1}{S_s^{p+1}\left\|b\right\|_{L^{s_p}}}\right)^{(1-q)/(p-1)}.$$

(i) $\int_\Omega a|u|^{q+1}\,dx \le 0$. There is a unique $t^- > t_{\max}$ such that $h(t^-) = \int_\Omega a|u|^{q+1}\,dx$ and $h'(t^-) < 0$. Now,

$$\begin{aligned}
&(1-q)\left\|t^- u\right\|_{H^1}^2 - (p-q)\int_\Omega b\left|t^- u\right|^{p+1}\,dx \\
&= \left(t^-\right)^{2+q}\left[(1-q)\left(t^-\right)^{-q}\|u\|_{H^1}^2 - (p-q)\left(t^-\right)^{p-q-1}\int_\Omega b\,|u|^{p+1}\,dx\right] \\
&= \left(t^-\right)^{2+q} h'\left(t^-\right) < 0,
\end{aligned}$$

and

$$\begin{aligned}
\left\langle J_\lambda'\left(t^- u\right), t^- u\right\rangle &= \left(t^-\right)^2\|u\|_{H^1}^2 - \left(t^-\right)^{q+1}\lambda\int_\Omega a\,|u|^{q+1}\,dx \\
&\quad - \left(t^-\right)^{p+1}\int_\Omega b\,|u|^{p+1}\,dx \\
&= \left(t^-\right)^{q+1}\left[h\left(t^-\right) - \lambda\int_\Omega a\,|u|^{q+1}\,dx\right] = 0.
\end{aligned}$$

Thus, $t^- u \in \mathbf{M}_\lambda^-(\Omega)$ or $t^- = 1$. Since, for $t > t_{\max}$, we have

$$(1-q)\|tu\|_{H^1}^2 - (p-q)\int_\Omega b\,|tu|^{p+1}\,ds < 0, \qquad \frac{d^2}{dt^2}J_\lambda(tu) < 0$$

and

$$\frac{d}{dt}J_\lambda(tu) = t\|u\|_{H^1}^2 - \lambda t^q\int_\Omega a\,|u|^{q+1}\,dx - t^p\int_\Omega b\,|tu|^{p+1}\,dx = 0 \quad \text{for } t = t^-.$$

Thus, $J_\lambda(u) = \sup_{t\ge 0} J_\lambda(tu)$. Moreover,

$$J_\lambda(u) \ge J_\lambda(tu) \ge \frac{t^2}{2}\|u\|_{H^1}^2 - \frac{t^{p+1}}{p+1}\int_\Omega b\,|u|^{p+1}\,dx \quad \text{for all } t \ge 0.$$

Similar to the argument in the function $h(t)$, we obtain

$$J_\lambda(u) \ge \frac{p-1}{2(p+1)}\left(\frac{\|u\|_{H^1}^{p+1}}{\int_\Omega b\,|u|^{p+1}\,dx}\right)^{2/(p-1)} > 0.$$

(ii) $\int_\Omega a|u|^q\,dx > 0$. By (5) and

$$\begin{aligned}
h(0) = 0 &< \lambda \int_\Omega a\,|u|^{q+1}\ dx \le \lambda \|a\|_{L^{r_q}}\ S_r^{q+1}\ \|u\|_{H^1}^{q+1} \\
&< \|u\|_{H^1}^{q+1} \left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{(1-q)/(p-1)} \left(\frac{1}{S_s^{p+1}\,\|b\|_{L^{s_p}}}\right)^{(1-q)/(p-1)} \\
&\le h(t_{\max}) \quad \text{for } \lambda \in (0, \lambda_2),
\end{aligned}$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$h\left(t^+\right) = \lambda \int_\Omega a\,|u|^{q+1}\,dx = h\left(t^-\right)$$

and

$$h'\left(t^+\right) > 0 > h'\left(t^-\right).$$

We have $t^+u \in \mathbf{M}_\lambda^+(\Omega)$, $t^-u \in \mathbf{M}_\lambda^-(\Omega)$, and $J_\lambda(t^-u) \ge J_\lambda(tu) \ge J_\lambda(t^+u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \le J_\lambda(tu)$ for each $t \in [0, t^+]$. Thus, $t^- = 1$ and

$$J_\lambda(u) = \sup_{t\ge 0} J_\lambda(tu), \qquad J_\lambda\left(t^+u\right) = \inf_{0\le t\le t_{\max}} J_\lambda(tu).$$

This completes the proof. □

Lemma 2.5 (i) $\alpha_\lambda(\Omega) \le \alpha_\lambda^+(\Omega) < 0$;
(ii) J_λ *is coercive and bounded below on* $\mathbf{M}_\lambda(\Omega)$.

Proof. (i) Given $u \in \mathbf{M}_\lambda^+(\Omega)$, we have

$$\begin{aligned}
J_\lambda(u) &= \frac{p-1}{2(p+1)} \|u\|_{H^1}^2 \\
&\quad + \left(\frac{q-p}{(p+1)(q+1)}\right)\lambda \int_\Omega a\,|u|^{q+1}\ dx \\
&< \left[\frac{1}{2} - \frac{1}{q+1}\right]\frac{p-1}{p+1}\|u\|_{H^1}^2 < 0.
\end{aligned}$$

This yields $\alpha_\lambda(\Omega) \le \alpha_\lambda^+(\Omega) < 0$.

(ii) For $u \in \mathbf{M}_\lambda(\Omega)$, we have $\|u\|_{H^1}^2 = \lambda \int_\Omega a|u|^{q+1}\,dx + \int_\Omega b|u|^{p+1}\,dx$. Then by the Hölder and Young inequalities,

$$\begin{aligned} J_\lambda(u) &= \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{(p+1)(q+1)}\right)\int_\Omega a|u|^{q+1}\,dx \\ &\geq \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{(p+1)(q+1)}\right)\|a\|_{L^{r_q}} S_r^{q+1}\|u\|_{H^1}^{q+1}. \end{aligned}$$

Thus, J_λ is coercive and bounded below on $\mathbf{M}_\lambda(\Omega)$. □

3. Proof of Theorem 1. First, we will use the idea of Ni and Takagi [**5**] to get the following results.

Lemma 3.1. *For each $u \in \mathbf{M}_\lambda(\Omega)$, there exist $\varepsilon > 0$ and a differentiable function $\xi : B(0;\varepsilon) \subset H_0^1(\Omega) \to \mathbf{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u-v) \in \mathbf{M}_\lambda(\Omega)$ and*

$$(6)\quad \langle \xi'(0), v\rangle = \frac{2\int_\Omega \nabla u \nabla v\,dx - (q+1)\lambda\int_\Omega a|u|^{q-1}uv\,dx - (p+1)\int_\Omega b|u|^{p-1}uv\,dx}{(1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int_\Omega b|u|^{p+1}\,dx}$$

for all $v \in H_0^1(\Omega)$.

Proof. For $u \in \mathbf{M}_\lambda(\Omega)$, define a function $F : \mathbf{R} \times H_0^1(\Omega) \to \mathbf{R}$ by

$$\begin{aligned} F_u(\xi, w) &= \langle J_\lambda'(\xi(u-w)), \xi(u-w)\rangle \\ &= \xi^2\int_\Omega |\nabla(u-w)|^2\,dx - \xi^{q+1}\lambda\int_\Omega a|u-w|^{q+1}\,dx \\ &\quad - \xi^{p+1}\int_\Omega b|u-w|^{p+1}\,dx. \end{aligned}$$

Then $F_u(1,0) = \langle J_\lambda'(u), u\rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi}F_u(1,0) &= 2\int_\Omega |\nabla u|^2\,dx - (q+1)\lambda\int_\Omega a|u|^{q+1}\,dx \\ &\quad - (p+1)\int_\Omega b|u|^{p+1}\,dx \\ &= (1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int_\Omega b|u|^{p+1}\,dx \neq 0. \end{aligned}$$

According to the implicit function theorem, there exist $\varepsilon > 0$ and a differentiable function $\xi : B(0;\varepsilon) \subset H^1(\mathbf{R}^N) \to \mathbf{R}$ such that $\xi(0) = 1$,

$$\left\langle \xi'(0), v \right\rangle = \frac{2\int_\Omega \nabla u \nabla v\, dx - (q+1)\lambda \int_\Omega a|u|^{q-1}uv\, dx - (p+1)\int_\Omega b|u|^{p-1}uv\, dx}{(1-q)\int_\Omega |\nabla u|^2\, dx - (p-q)\int_\Omega b|u|^{p+1}\, dx}$$

and

$$F_u(\xi(v), v) = 0 \quad \text{for all } v \in B(0;\varepsilon)$$

which is equivalent to

$$\left\langle J'_\lambda(\xi(v)(u-v)), \xi(v)(u-v) \right\rangle = 0 \quad \text{for all } v \in B(0;\varepsilon),$$

that is, $\xi(v)(u-v) \in \mathbf{M}_\lambda(\Omega)$. □

Lemma 3.2. *For each $u \in \mathbf{M}_\lambda^-(\Omega)$, there exist $\varepsilon > 0$ and a differentiable function $\xi^- : B(0;\varepsilon) \subset H_0^1(\Omega) \to \mathbf{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u-v) \in \mathbf{M}_\lambda^-(\Omega)$ and*

(7)
$$\left\langle \left(\xi^-\right)'(0), v \right\rangle = \frac{2\int_\Omega \nabla u \nabla v\, dx - (q+1)\lambda \int_\Omega a|u|^{q-1}uv\, dx - (p+1)\int_\Omega b|u|^{p-1}uv\, dx}{(1-q)\int_\Omega |\nabla u|^2\, dx - (p-q)\int_\Omega b|u|^{p+1}\, dx}$$

for all $v \in H_0^1(\Omega)$.

Proof. Similar to the argument in Lemma 3.1, there exist $\varepsilon > 0$ and a differentiable function $\xi^- : B(0;\varepsilon) \subset H^1(\mathbf{R}^N) \to \mathbf{R}$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in \mathbf{M}_\lambda(\Omega)$ for all $v \in B(0;\varepsilon)$. Since

$$\left\langle \psi'_\lambda(u), u \right\rangle = (1-q)\|u\|_{H^1}^2 - (p-q)\int_\Omega b|u|^{p+1}\, dx < 0.$$

Thus, by the continuity of the function ξ^-, we have

$$\begin{aligned} &\left\langle \psi'_\lambda\left(\xi^-(v)(u-v)\right), \xi^-(v)(u-v) \right\rangle \\ &= (1-q)\left\|\xi^-(v)(u-v)\right\|_{H^1}^2 - (p-q)\int_\Omega b\left|\xi^-(v)(u-v)\right|^{p+1} dx < 0 \end{aligned}$$

if ε sufficiently small, this implies that $\xi^-(v)(u-v) \in \mathbf{M}_\lambda^-(\Omega)$. □

Proposition 3.3. *Let* $\Lambda_0 = \min\{\lambda_1, \lambda_2\}$*, then for* $\lambda \in (0, \Lambda_0)$*,*

(i) *there exists a minimizing sequence* $\{u_n\} \subset \mathbf{M}_\lambda(\Omega)$ *such that*

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1), \quad J'_\lambda(u_n) = o(1) \text{ in } H^{-1}(\Omega);$$

(ii) *there exists a minimizing sequence* $\{u_n\} \subset \mathbf{M}^-_\lambda(\Omega)$ *such that*

$$J_\lambda(u_n) = \alpha^-_\lambda(\Omega) + o(1), \quad J'_\lambda(u_n) = o(1) \quad \textit{in } H^{-1}(\Omega).$$

Proof. (i) By Lemma 2.5 (ii) and the Ekeland variational principle [**3**], there exists a minimizing sequence $\{u_n\} \subset \mathbf{M}_\lambda(\Omega)$ such that

$$J_\lambda(u_n) < \alpha_\lambda(\Omega) + \frac{1}{n} \tag{8}$$

and

$$J_\lambda(u_n) < J_\lambda(w) + \frac{1}{n}\|w - u_n\|_{H^1} \quad \text{for each } w \in \mathbf{M}_\lambda(\Omega). \tag{9}$$

By taking n large, from Lemma 2.5 (i) we have

$$\begin{aligned} J_\lambda(u_n) = {} & \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2_{H^1} \\ & - \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\lambda \int_\Omega a|u_n|^{q+1}\,dx \\ & < \alpha_\lambda(\Omega) + \frac{1}{n} < \frac{\alpha_\lambda(\Omega)}{2}. \end{aligned} \tag{10}$$

This implies

$$\|a\|_{L^{r_q}} S_r^{q+1}\|u_n\|^{q+1}_{H^1} \geq \int_\Omega a|u_n|^{q+1}\,dx > -\frac{(q+1)(p+1)}{\lambda(p-q)}\frac{\alpha_\lambda(\Omega)}{2} > 0. \tag{11}$$

Consequently, $u_n \neq 0$ and, putting together (10), (11) and the Hölder inequality, we obtain

$$\|u_n\|_{H^1} > \left[\frac{(q+1)(p+1)}{\lambda(p-q)}\frac{\alpha_\lambda(\Omega)}{2} S_r^{-(q+1)}\|a\|^{-1}_{L^{r_q}}\right]^{1/(q+1)} \tag{12}$$

and

$$\|u_n\|_{H^1} < \left[\frac{2\lambda\,(p-q)}{(p-1)\,(q+1)}\,\|a\|_{L^{r_q}}\,S_r^{q+1}\right]^{1/(1-q)}. \tag{13}$$

Now, we will show that

$$\|J'_\lambda\,(u_n)\|_{H^{-1}} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Applying Lemma 3.1 with u_n to obtain the functions $\xi_n : B(0;\varepsilon_n) \to \mathbf{R}^+$ for some $\varepsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathbf{M}_\lambda(\Omega)$. With $n \in \mathbf{N}$ fixed, we choose $0 < \rho < \varepsilon_n$. Let $u \in H_0^1(\Omega)$ with $u \not\equiv 0$, and let $w_\rho = \rho u/\|u\|_{H^1}$. We set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$. Since $\eta_\rho \in \mathbf{M}_\lambda(\Omega)$, we deduce from (9) that

$$J_\lambda\,(\eta_\rho) - J_\lambda\,(u_n) \geq -\frac{1}{n}\,\|\eta_\rho - u_n\|_{H^1},$$

and, by the mean value theorem we have

$$\langle J'_\lambda\,(u_n)\,, \eta_\rho - u_n\rangle + o\left(\|\eta_\rho - u_n\|_{H^1}\right) \geq -\frac{1}{n}\,\|\eta_\rho - u_n\|_{H^1}.$$

Thus,

$$\begin{aligned} &\langle J'_\lambda\,(u_n)\,, -w_\rho\rangle + (\xi_n\,(w_\rho) - 1)\,\langle J'_\lambda\,(u_n)\,, (u_n - w_\rho)\rangle \\ &\qquad\qquad \geq -\frac{1}{n}\,\|\eta_\rho - u_n\|_{H^1} + o\left(\|\eta_\rho - u_n\|_{H^1}\right). \end{aligned} \tag{14}$$

From $\xi_n(w_\rho)(u_n - w_\rho) \in \mathbf{M}_\lambda(\Omega)$ and (14), it follows that

$$\begin{aligned} &-\rho\left\langle J'_\lambda\,(u_n)\,, \frac{u}{\|u\|_{H^1}}\right\rangle \\ &\qquad + (\xi_n\,(w_\rho) - 1)\,\langle J'_\lambda\,(u_n) - J'_\lambda\,(\eta_\rho)\,, (u_n - w_\rho)\rangle \\ &\qquad\qquad \geq -\frac{1}{n}\,\|\eta_\rho - u_n\|_{H^1} + o\left(\|\eta_\rho - u_n\|_{H^1}\right). \end{aligned}$$

Thus,

$$\begin{aligned} \left\langle J'_\lambda\,(u_n)\,, \frac{u}{\|u\|_{H^1}}\right\rangle &\leq \frac{\|\eta_\rho - u_n\|_{H^1}}{n\rho} + \frac{o\left(\|\eta_\rho - u_n\|_{H^1}\right)}{\rho} + \frac{(\xi_n\,(w_\rho) - 1)}{\rho} \\ &\quad \times \langle J'_\lambda\,(u_n) - J'_\lambda\,(\eta_\rho)\,, (u_n - w_\rho)\rangle. \end{aligned} \tag{15}$$

Since

$$\|\eta_\rho - u_n\|_{H^1} \le \rho\,|\xi_n(w_\rho)| + |\xi_n(w_\rho) - 1|\,\|u_n\|_{H^1}$$

and

$$\lim_{n\to\infty} \frac{|\xi_n(w_\rho) - 1|}{\rho} \le \|\xi_n'(0)\|.$$

If we let $\rho \to 0$ in (15), then by (13) we can find a constant $C > 0$, independent of ρ, such that

$$\left\langle J_\lambda'(u_n), \frac{u}{\|u\|_{H^1}} \right\rangle \le \frac{C}{n}\left(1 + \|\xi_n'(0)\|\right).$$

We are done once we show that $\|\xi_n'(0)\|$ is uniformly bounded in n. By (6), (13) and the Hölder inequality, we have

$$\langle \xi_n'(0), v\rangle \le \frac{d\,\|v\|_{H^1}}{\left|(1-q)\int_\Omega |\nabla u_n|^2\,dx - (p-q)\int_\Omega b\,|u_n|^{p+1}\,dx\right|}$$

for some $d > 0$.

We only need to show that

$$\left|(1-q)\int_\Omega |\nabla u_n|^2\,dx - (p-q)\int_\Omega b\,|u_n|^{p+1}\,dx\right| > c \tag{16}$$

for some $c > 0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$. We have

$$(1-q)\int_\Omega |\nabla u_n|^2\,dx - (p-q)\int_\Omega b\,|u_n|^{p+1}\,dx = o(1). \tag{17}$$

Combining (17) with (12), we can find a suitable constant $k > 0$ such that

$$\int_\Omega b\,|u_n|^{p+1}\,dx \ge k \quad \text{for } n \text{ sufficiently large.} \tag{18}$$

In addition, (17), and the fact that $u_n \in \mathbf{M}_\lambda(\Omega)$ also give

$$\begin{aligned}\lambda\int_\Omega a\,|u_n|^{q+1}\,dx &= \|u_n\|_{H^1}^2 - \int_\Omega b\,|u_n|^{p+1}\,dx\\ &= \frac{p-1}{1-q}\int_\Omega b\,|u_n|^{p+1}\,dx + o(1)\end{aligned}$$

and

$$\|u_n\|_{H^1} \leq \left[\lambda\left(\frac{p-q}{p-1}\right)\|a\|_{L^{p^*}}\, S_r^{q+1}\right]^{1/(1-q)} + o(1). \tag{19}$$

This implies

$$\begin{aligned}
(20)\quad & I_\lambda(u_n) \\
&= K(p,q)\left(\frac{\|u_n\|_{H^1}^{2p}}{\int_\Omega b\,|u_n|^{p+1}\,dx}\right)^{1/(p-1)} - \lambda\int_\Omega a\,|u_n|^{q+1}\,dx \\
&= \left(\frac{1-q}{p-q}\right)^{p/(p-1)}\left(\frac{p-1}{1-q}\right)\left(\frac{(p-q/1-q)^p\left(\int_\Omega b\,|u_n|^{p+1}\,dx\right)^p}{\int_\Omega b\,|u_n|^{p+1}\,dx}\right)^{1/(p-1)} \\
&\quad - \frac{p-1}{1-q}\int_\Omega b\,|u_n|^{p+1}\,dx \\
&= o(1).
\end{aligned}$$

However, by (18), (19) and $\lambda \in (0,\Lambda_0)$,

$$\begin{aligned}
& I_\lambda(u_n) \\
&\geq K(p,q)\left(\frac{\|u_n\|_{H^1}^{2p}}{S_s^{p+1}\,\|b\|_{L^{s_p}}\,\|u_n\|_{H^1}^{p+1}}\right)^{1/(p-1)} - \lambda S_r^{q+1}\,\|a\|_{L^{r_q}}\,\|u_n\|_{H^1}^{q+1} \\
&= \|u_n\|_{H^1}^{q+1}\left(K(p,q)\,S_s^{(p+1)/(1-p)}\,\|b\|_{L^{s_p}}^{1/(1-p)}\,\|u_n\|_{H^1}^{-q} - \lambda S_r^{q+1}\,\|a\|_{L^{r_q}}\right) \\
&\geq \|u_n\|_{H^1}^{q+1}\Bigg\{K(p,q)\,S_s^{(p+1)/(1-p)}\,\|b\|_{L^{s_p}}^{1/(1-p)}\,\lambda^{-q/(1-q)} \\
&\qquad\times\left[\left(\frac{p-q}{p-1}\right)\|a\|_{L^{r_q}}\,S_s^{q+1}\right]^{-q/(1-q)} - \lambda S_r^{q+1}\,\|a\|_{L^{r_q}}\Bigg\}.
\end{aligned}$$

This contradicts (20). We get

$$\left\langle J_\lambda'(u_n),\, \frac{u}{\|u\|_{H^1}}\right\rangle \leq \frac{C}{n}.$$

This completes the proof of (i).

(ii) Similarly, by using Lemma 3.2, we can prove (ii). We will omit the detailed proof here. □

Now, we establish the existence of a local minimum for J_λ on $\mathbf{M}_\lambda^+(\Omega)$.

Theorem 3.4. *Let $\Lambda_0 > 0$ be as in Proposition* 3.3. *Then for $\lambda \in (0, \Lambda_0)$ the functional J_λ has a minimizer u_0^+ in $\mathbf{M}_\lambda^+(\Omega)$ and it satisfies*

(i) $J_\lambda(u_0^+) = \alpha_\lambda(\Omega) = \alpha_\lambda^+(\Omega)$;

(ii) u_0^+ *is a solution of equation* $(E_{a,b})$.

Proof. Let $\{u_n\} \subset \mathbf{M}_\lambda(\Omega)$ be a minimizing sequence for J_λ on $\mathbf{M}_\lambda(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1) \quad \text{and} \quad J'_\lambda(u_n) = o(1) \quad \text{in } H^{-1}(\Omega).$$

Then by Lemma 2.5 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^+ \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u_0^+ \quad \text{weakly in } H_0^1(\Omega)$$

and

$$u_n \longrightarrow u_0^+ \quad \text{strongly in } L^r(\Omega) \quad \text{for } 1 < r < 2^*. \tag{21}$$

First, we claim that $\int_\Omega a|u_0^+|^{q+1}\,dx \neq 0$. If not, by (21) and the Hölder inequality we can conclude that

$$\int_\Omega a\,|u_n|^{q+1}\,dx \longrightarrow \int_\Omega a\,\left|u_0^+\right|^{q+1}\,dx = 0 \quad \text{as } n \to \infty.$$

Thus,

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega p\,|u_n|^{p+1}\,dx + o(1)$$

and

$$J_\lambda(u_n) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla u_n|^2\,dx + o(1),$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda(\Omega) < 0$ as $n \to \infty$. Moreover,

$$o(1) = \langle J'_\lambda(u_n), \phi\rangle = \langle J'_\lambda(u_0), \phi\rangle + o(1) \quad \text{for all } \phi \in H_0^1(\Omega).$$

Thus, $u_0^+ \in \mathbf{M}_\lambda$ is a nonzero solution of equation $(E_{a,b})$ and $J_\lambda(u_0^+) \geq \alpha_\lambda(\Omega)$. We now prove that $J_\lambda(u_0^+) = \alpha_\lambda(\Omega)$. Since

$$\begin{aligned} J_\lambda\left(u_0^+\right) &= \frac{1}{2}\left\|u_0^+\right\|_{H^1}^2 - \frac{\lambda}{q+1}\int_\Omega a\left|u_0^+\right|^{q+1}\,dx \\ &\quad - \frac{1}{p+1}\int_\Omega b\left|u_0^+\right|^{p+1}\,dx \\ &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left\|u_0^+\right\|_{H^1}^2 \\ &\quad + \left(\frac{\lambda}{p+1}-\frac{\lambda}{q+1}\right)\int_\Omega a\left|u_0^+\right|^{q+1}\,dx \\ &\leq \liminf_{n\to\infty}\left(\left(\frac{1}{2}-\frac{1}{p+1}\right)\left\|u_n\right\|_{H^1}^2\right. \\ &\qquad\qquad \left. + \left(\frac{\lambda}{p+1}-\frac{\lambda}{q+1}\right)\int_\Omega a\left|u_n\right|^{q+1}\,dx\right) \\ &= \liminf_{n\to\infty} J_\lambda\left(u_n\right) = \alpha_\lambda\left(\Omega\right). \end{aligned}$$

Thus, $J_\lambda(u_0^+) = \alpha_\lambda(\Omega)$. Moreover, we have $u_0^+ \in \mathbf{M}_\lambda^+(\Omega)$. In fact, if $u_0^+ \in \mathbf{M}_\lambda^-(\Omega)$, by Lemma 2.4, there are unique t_0^+ and t_0^- such that $t_0^+ u_0^+ \in \mathbf{M}_\lambda^+(\Omega)$ and $t_0^- u_0^+ \in \mathbf{M}_\lambda^-(\Omega)$, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_\lambda\left(t_0^+u_0^+\right) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}J_\lambda\left(t_0^+u_0^+\right) > 0,$$

there exists $t_0^+ < \bar{t} \leq t_0^-$ such that $J_\lambda(t_0^+u_0^+) < J_\lambda(\bar{t}u_0^+)$. By Lemma 2.4,

$$J_\lambda\left(t_0^+u_0^+\right) < J_\lambda\left(\bar{t}u_0^+\right) \leq J_\lambda\left(t_0^-u_0^+\right) = J_\lambda\left(u_0^+\right),$$

which is a contradiction. Since $J_\lambda(u_0^+) = J_\lambda(|u_0^+|)$ and $|u_0^+| \in \mathbf{M}_\lambda^+(\Omega)$, by Lemma 2.3 we may assume that u_0^+ is a solution of equation $(E_{a,b})$. □

Next, we establish the existence of a local minimum for J_λ on $\mathbf{M}_\lambda^-(\Omega)$.

Theorem 3.5. *Let $\Lambda_0 > 0$ be as in Proposition* 3.3. *Then, for $\lambda \in (0, \Lambda_0)$, the functional J_λ has a minimizer u_0^- in $\mathbf{M}_\lambda^-(\Omega)$ and it satisfies* (i) $J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega)$;

(ii) *u_0^- is a solution of equation $(E_{a,b})$.*

Proof. By Proposition 3.3 (ii), there exists a minimizing sequence $\{u_n\}$ for J_λ on $\mathbf{M}_\lambda^-(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1) \quad \text{and} \quad J_\lambda'(u_n) = o(1) \quad \text{in } H^{-1}(\Omega).$$

By Lemma 2.5 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^- \in \mathbf{M}_\lambda^-(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^- \quad \text{weakly in } H_0^1(\Omega), \\ u_n &\longrightarrow u_0^- \quad \text{strongly in } L^s(\Omega) \end{aligned}$$

and

$$u_n \longrightarrow u_0^- \quad \text{strongly in } L^r(\Omega) \quad \text{for } 1 \leq r < 2^*.$$

Since

$$o(1) = \langle J_\lambda'(u_n), \phi \rangle = \langle J_\lambda'(u_0), \phi \rangle + o(1) \quad \text{for all } \phi \in H_0^1(\Omega)$$

and

$$\begin{aligned} 0 > \langle \psi_\lambda'(u_n), u_n \rangle &= (2-q)\|u_n\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u_n|^p \, ds \\ &\geq (2-q)\|u_0\|_{H^1}^2 - (p-q)\int_{\partial\Omega} g|u_0|^p \, ds. \end{aligned}$$

Thus, $u_0^- \in \mathbf{M}_\lambda^-(\Omega)$ is a nonzero solution of equation $(E_{a,b})$. We now prove that $u_n \to u_0^-$ strongly in $H_0^1(\Omega)$. Suppose otherwise; then $\|u_0^-\|_{H^1} < \liminf_{n\to\infty} \|u_n\|_{H^1}$ and so

$$\begin{aligned} &\left\|u_0^-\right\|_{H^1}^2 - \lambda \int_\Omega a\left|u_0^-\right|^{q+1} dx - \int_\Omega b\left|u_0^-\right|^{p+1} dx \\ &\quad < \liminf_{n\to\infty}\left(\|u_n\|_{H^1}^2 - \lambda\int_\Omega a|u_n|^{q+1}\, dx - \int_\Omega b|u_n|^{p+1}\, dx \right) = 0. \end{aligned}$$

This contradicts $u_0^- \in \mathbf{M}_\lambda^-(\Omega)$. Hence, $u_n \to u_0^-$ strongly in $H_0^1(\Omega)$. This implies

$$J_\lambda(u_n) \longrightarrow J_\lambda\left(u_0^-\right) = \alpha_\lambda^-(\Omega) \quad \text{as } n \to \infty.$$

Since $J_\lambda(u_0^-) = J_\lambda(|u_0^-|)$ and $|u_0^-| \in \mathbf{M}_\lambda^-(\Omega)$ by Lemma 2.3, we may assume that u_0^- is a solution of equation $(E_{a,b})$. □

Now, we complete the proof of Theorem 1.2: By Theorems 3.4, 3.5 and equation $(E_{a,b})$, there exist two solutions u_0^+ and u_0^- such that $u_0^+ \in \mathbf{M}_\lambda^+(\Omega)$, $u_0^- \in \mathbf{M}_\lambda^-(\Omega)$. Since $\mathbf{M}_\lambda^+(\Omega) \cap \mathbf{M}_\lambda^-(\Omega) = \varnothing$, this implies that u_0^+ and u_0^- are different.

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