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INCLUSION RELATIONS OF CERTAIN GRAPH EIGENSPACES

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ABSTRACT. Motivated by the fact that an inclusion relation exists between the eigenspace for eigenvalue λ of a graph and the eigenspace for eigenvalue $-1-\lambda$ of its complement, one may ask if for some given λ there exist graph classes such that the direction of this inclusion is the same for all its members. The main result of this paper is that the eigenspace for eigenvalue 0 of a tree always contains the eigenspace for eigenvalue -1 of its complement.

1. Introduction. One of the many areas of algebraic graph theory is the study of graph eigenspaces. Many interesting results have been obtained so far, some of which are covered by [**4**].

In particular, trees have been well studied both with respect to spectrum and eigenvector structure [**7**]. Moreover, there exist several powerful algorithms for that purpose. It is possible to test for eigenvalues and derive the characteristic polynomial in linear time [**6, 10**]. It is even possible to efficiently compute the eigenvectors of trees [**11**].

Our motivation is a well-known result from [**3**] stating that the dimension of the eigenspace for eigenvalue λ of a given graph and the dimension of the eigenspace for eigenvalue $-1-\lambda$ of its complement cannot differ by more than one. We augment this result by showing that an inclusion relation exists between these eigenspaces. This serves as a motivation for our main theorem.

Namely, one may ask if graph classes exist with uniform direction of this inclusion. For example, for a path it is not difficult to show that the eigenspace for eigenvalue 0 always contains the eigenspace for eigenvalue -1 of its complement. The main result of this paper is that this statement can be extended to trees and forests.

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2. Preliminaries. Let K be a field. Let $K^{n\times m}$ denote the set of $n \times m$ matrices with entries from K and $K^n = K^{n\times 1}$. Further, let $J_{n,m} \in K^{n\times m}$ and $j_n \in K^n$ have entries only equal to one. Let $I_n \in K^n$ denote the identity matrix. Indices may be omitted where clear from the context.

For the general basics of graph theory, the reader is referred to sources like [**1, 5, 9**]. The foundations of algebraic graph theory are treated in [**2, 8**].

Throughout, we will only consider finite, simple, loopless (undirected) graphs. Observe that the complement of a graph $G = (V, E)$ is denoted by $\overline{G} = (\overline{V}, \overline{E})$. $N_G(x)$ is the set of neighbors of vertex x in G and $\overline{N}_G(x) = N_{\overline{G}}(x)$ is the set of nonneighbors of x in G.

Let G be a graph with vertex set $V = \{x_1, \dots, x_n\}$. Then we define the adjacency matrix $A(G) = (a_{kl})$ by $a_{kl} = 1$ if x_k and x_l are adjacent and 0 otherwise.

The eigenvalues of a graph G are the roots of the characteristic polynomial $\chi(x;G) = \det(A(G) - xI)$. The eigenspace $\ker(A(G) - \lambda I)$ of eigenvalue λ is denoted by $\operatorname{Eig}(\lambda, G)$. If G is fixed, then we will write $E_\lambda = \ker(A(G) - \lambda I)$ and $\overline{E}_\lambda = \ker(A(\overline{G}) - \lambda I)$, respectively, for the eigenspaces of G and $\overline{G}$.

Since $A(G)$ is symmetric, it follows that the eigenvalues of a graph are real and that the multiplicity of a root of $\chi(x;G)$ equals the dimension of the corresponding eigenspace. Interpreting graph eigenvectors as vertex weights, i.e., as functions $V(G) \to \mathbf{R}$, it is possible to derive a notion of graph eigenvectors and eigenspaces that does not depend on the chosen vertex order.

3. Motivational results. In this section, let A be an $n \times n$ matrix over fixed field K and $\mu \in K$. We will consider matrices of the form $A - \mu J$ and study their relationship with matrix A. To be more precise, we study the relationship between the respective kernels.

We will afterwards apply our findings to graphs to motivate why it is interesting to study their relationship to eigenspaces E_0 and $\overline{E}_{-1}$.

Lemma 1. *Let $\mu \neq 0$. Then*

$$\begin{aligned}\ker A \cap \ker(A - \mu J) &= \{x \in \ker A : j^T x = 0\} \\ &= \{x \in \ker(A - \mu J) : j^T x = 0\}.\end{aligned}$$

Proof. This is straightforward. □

Corollary 2. *Let* $\dim\ker A = d_1$ *and* $\dim\ker(A-\mu J) = d_2$*. Then*

$$\dim(\ker A \cap \ker(A-\mu J)) \geq \max\{d_1, d_2\} - 1.$$

Proof. Observe that the additional condition $j^T x = 0$ poses an at most one-dimensional restriction for any subspace. □

Theorem 3. 1. $|\operatorname{rk} A - \operatorname{rk}(A-\mu J)| \leq 1$,
2. $\operatorname{rk} A < \operatorname{rk}(A-\mu J) \Rightarrow \ker(A-\mu J) \subsetneqq \ker A$,
3. $\operatorname{rk} A > \operatorname{rk}(A-\mu J) \Rightarrow \ker(A-\mu J) \supsetneqq \ker A$.

Proof. Let, for example, $\operatorname{rk} A < \operatorname{rk}(A-\mu J)$. It follows that $d_1 = d_2 + 1$. By a suitable linear combination we can choose a basis of $\ker A$ such that at least $d_1 - 1$ basis vectors have vanishing component sum. But then, by Lemma 1 we have found a basis for $\ker(A-\mu J)$. □

At this point we know that if the two kernels have different dimensions the smaller kernel is contained in the other one. But in the case of equal dimensions the kernels need not necessarily be identical. Examples are easy to find.

If, however, we know that the matrix A is symmetric, the two kernels must be identical:

Theorem 4. *Let* $A = A^T$ *and* $\operatorname{rk} A = \operatorname{rk}(A-\mu J)$*. Then,*

$$\ker A = \ker(A-\mu J).$$

Proof. For $\mu = 0$ this is obvious. Therefore, let $\mu \neq 0$, and assume that $\ker A \neq \ker(A-\mu J)$.

Since the kernels have equal dimension it cannot be that one kernel contains the other because then they would be identical. Therefore, there exists a vector $b \in \ker(A-\mu J) \setminus \ker A$.

Let $s = \mu J^T b$. Then $Ab = (A - \mu J)b + \mu Jb = \mu Jb = sj$. We see that necessarily $s \neq 0$ because otherwise $b \in \ker A$ by Lemma 1.

Now let $\tilde{A} = (A \mid j)$. Then $j \in \operatorname{im} A$ if and only if $\operatorname{rk} \tilde{A} = \operatorname{rk} A$. But $A((1/s)b) = j$; therefore, $\operatorname{rk} \tilde{A} = \operatorname{rk} A$.

Let $A' = \tilde{A}^T$. Then $\ker A' \subseteq \ker A$. By $\operatorname{rk} A' = \operatorname{rk} \tilde{A}^T = \operatorname{rk} \tilde{A} = \operatorname{rk} A$, we even see that $\ker A' = \ker A$. Hence, $j^T x = 0$ for all $x \in \ker A$. By Lemma 1 this means that $\ker A \subseteq \ker(A - \mu J)$.

Since we have assumed that the kernels have equal dimension the theorem follows. $\square$

Remark 5. Consider the following congruence relation. Given a fixed matrix M, we will say that the two matrices A and B are congruent modulo M, $A \equiv B$, if there exists $\mu \in K$ such that $A - B = \mu M$. Now let $M = J$. Then $K^{n \times n}$ gets partitioned into congruence classes $C(A) = \{A - \mu J : \mu \in K\}$.

These classes possess a rather interesting property that is straightforward to show: Within each class $C(A)$ there occur at most two different matrix ranks. Moreover, the lower of these two ranks is attained for exactly one member of that class.

Let us transfer our results to eigenspaces of graphs and their complements.

From $A(\overline{G}) = J - A(G) - I$ it follows directly that $\overline{E}_{-1-\lambda} = \ker(J - (A(G) - \lambda I))$. Based on this observation and the fact that $A(G)$ is symmetric, our previous results can be immediately applied to eigenspaces of graphs if we substitute $A - \lambda I$ for the matrix A.

Theorem 6.

1. $|\dim \overline{E}_{-\lambda-1} - \dim E_\lambda\,| \leq 1$,
2. $\dim \overline{E}_{-\lambda-1} < \dim E_\lambda \Rightarrow \overline{E}_{-\lambda-1} \subsetneqq E_\lambda$,
3. $\dim \overline{E}_{-\lambda-1} > \dim E_\lambda \Rightarrow \overline{E}_{-\lambda-1} \supsetneqq E_\lambda$,
4. $\dim \overline{E}_{-\lambda-1} = \dim E_\lambda \Rightarrow \overline{E}_{-\lambda-1} = E_\lambda$.

The first part of this theorem can also be found in [**3**]. However, a different proof technique is employed.

Theorem 6 implies that for an arbitrary graph at least one of the eigenspaces E_0 and $\overline{E}_{-1}$ is contained in the other one.

For an r-regular graph it is well-known that j is an eigenvector for eigenvalue r. Consequently, if x is an eigenvector of an r-regular graph for eigenvalue $\lambda \neq r$, then the sum over its components vanishes (note the symmetry of adjacency matrices). From Lemma 1 we may therefore immediately deduce the following result:

Lemma 7. *Let G be regular and neither a complete nor a null graph. Then,*

$$E_0 = \overline{E}_{-1}, \qquad E_{-1} = \overline{E}_0.$$

For nonregular graphs equality of E_0 and $\overline{E}_{-1}$ cannot generally be expected, but one may ask if there exist graph classes with a uniform direction of inclusion. For paths this is easy to see.

Lemma 8. *For all paths*

$$\overline{E}_{-1} \subseteq E_0$$

holds.

Proof. Let P be a path on n vertices. It is readily checked that $\overline{E}_{-1} \neq \{0\}$ if and only if $n \equiv 3 \mod 4$. In this case $\overline{E}_{-1}$ is spanned by $(1, 0, -1, 0, 1, \dots)$, a vector with a vanishing component sum. $\square$

4. Main result. The main result of this section is that for trees the eigenspace E_0 always contains $\overline{E}_{-1}$, extending Lemma 8.

For the following lemma recall that a matrix is called *totally unimodular* if the determinant of every square submatrix of it is either 0, 1 or -1, cf. [**2**].

Lemma 9. *Let G be a graph with adjacency matrix $A \in \mathbf{R}^{n\times n}$. If A is totally unimodular, then there exists no solution $x \in \mathbf{R}^n$ for*

$$A'x = j_{n+1}$$

where

$$A' = \begin{pmatrix} j_n^T \\ A \end{pmatrix}.$$

Proof. We need to show that

$$\operatorname{rk} \begin{pmatrix} 1 & j_n^T \\ j_n & A \end{pmatrix} = \operatorname{rk} A' + 1.$$

Let $r = \operatorname{rk} A$. Then we may assume without loss of generality that

$$A = \begin{pmatrix} B & B_1 \\ B_1^T & B_2 \end{pmatrix}$$

with invertible matrix $B \in \mathbf{R}^{r \times r}$ and matrices $B_1 \in \mathbf{R}^{r \times (n-r)}$, $B_2 \in \mathbf{R}^{(n-r) \times (n-r)}$.

It follows from basic linear algebra and the symmetry of A that there exists a matrix $R \in \mathbf{R}^{(n-r) \times r}$ such that

$$PAP^T = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$P = \begin{pmatrix} I_r & 0 \\ R & I_{n-r} \end{pmatrix}.$$

Further, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 1 & j_n^T \\ j_n & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P^T \end{pmatrix} = \begin{pmatrix} 1 & j_r^T & y^T \\ j_r & B & 0 \\ y & 0 & 0 \end{pmatrix} \tag{1}$$

and

$$A'P^T = \begin{pmatrix} j_r^T & y^T \\ B & 0 \\ B_1^T & 0 \end{pmatrix}, \tag{2}$$

where $y = Rj_r + j_{n-r} \in \mathbf{R}^{n-r}$.

Case 1. Let $y \neq 0$. It follows from equation (2) that $\operatorname{rk} A' = r + 1$. Moreover,

$$\operatorname{rk} \begin{pmatrix} 1 & j_r^T & y^T \\ j_r & B & 0 \end{pmatrix} = r + 1.$$

Together with equation (1) and $y \neq 0$ it follows from this equation that

$$\operatorname{rk} \begin{pmatrix} 1 & j_n^T \\ j_n & A \end{pmatrix} = r + 2.$$

Case 2. Let $y = 0$. By equation (2) we have $\operatorname{rk} A' = r$. By virtue of equation (1) it now suffices to show that

$$\begin{pmatrix} 1 & j_r^T \\ j_r & B \end{pmatrix}$$

is invertible. It follows from

$$\begin{pmatrix} 1 & -j_r^T B^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 1 & j_r^T \\ j_r & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B^{-1} j_r & I_n \end{pmatrix} = \begin{pmatrix} 1 - j_r^T B^{-1} j_r & 0 \\ 0 & B \end{pmatrix}$$

that it is equivalent to prove $1 - j_r^T B^{-1} j_r \neq 0$.

There exists an induced subgraph H of G with adjacency matrix B. The graph H must be bipartite because otherwise it would contain an induced odd cycle (whose adjacency matrix would correspond to a submatrix of B, and therefore of A, with determinant 2). Since H is bipartite it follows that the spectrum of B is symmetric around zero [**2**]. The same holds for the spectrum of B^{-1} because its eigenvalues are the inverses of the eigenvalues of B (with the same multiplicities). Therefore, $\operatorname{tr} B^{-1} = 0$. Since, like B, the matrix B^{-1} is symmetric and totally unimodular, it now readily follows that the sum $j_r^T B^{-1} j_r$ of all entries of B^{-1} is even, completing the proof. □

Lemma 10. *Let G be a graph with adjacency matrix A. If A is totally unimodular, then* $\ker(J - A) \subseteq \ker J$.

Proof. Suppose that G has n vertices, and let $x = (x_1, \dots, x_n)^T \notin \ker J$. Then $j^T x \neq 0$. Assume without loss of generality that $j^T x = 1$,

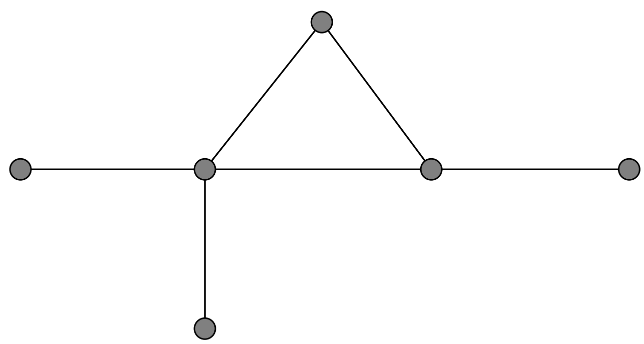

FIGURE 1. Graph with $\overline{E}_{-1} \subseteq E_0$ but without t.u. adjacency matrix.

i.e., $Jx = j$. We will show that $x \notin \ker(J - A)$. By assumption, this is equivalent to $Ax \neq j_n$. But this follows directly from Lemma 9. $\square$

Theorem 11. *For all forests,*

$$\overline{E}_{-1} \subseteq E_0$$

holds.

Proof. It is well known and follows by an easy induction that the adjacency matrix of a forest is totally unimodular. Therefore, the result follows from Lemmas 1 and 10. $\square$

Let us close this section with some remarks.

Remark 12. Total unimodularity of the adjacency matrix is sufficient but not necessary for $\overline{E}_{-1} \subseteq E_0$. Figure 1 shows a graph with $\overline{E}_{-1} \subseteq E_0$ but whose adjacency matrix is not totally unimodular.

Remark 13. Consider a symmetric matrix $A \in \mathbf{R}^{n \times n}$. It follows from basic linear algebra that $Ax = j$ is solvable for $x \in \mathbf{R}^n$ if and only if $\ker A \subseteq \ker J$, which by Lemma 1 is equivalent to $\ker A \subseteq \ker(A - \mu J)$.

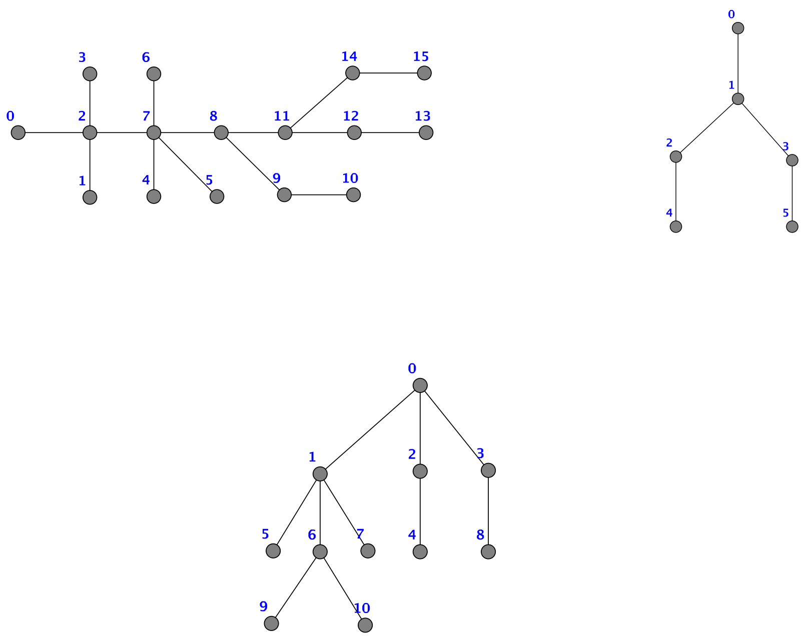

FIGURE 2. Example trees with (a) $E_{-1} \supsetneq \overline{E}_0$, (b) $E_{-1} = \overline{E}_0$ and (c) $E_{-1} \subsetneq \overline{E}_0$.

Therefore, by Theorem 11, we see that $\overline{E}_{-1} = E_0$ holds for a forest with adjacency matrix A if and only if $Ax = j$ is solvable. This is an interesting condition because it requires finding vertex weights such that for every vertex the sum over the weights of its neighbors is one.

Remark 14. The relation between the eigenspaces E_{-1} and $\overline{E}_0$ can be arbitrary even for trees. This means that a result like Theorem 11 does not hold for tree complements. In Figure 2 three representative examples are shown.

5. Conclusion. In the previous sections we have studied the relationship between certain eigenspaces of a graph and its complement. We have shown that one of the eigenspaces E_0 and $\overline{E}_{-1}$ is contained in the other one. For most regular graphs these eigenspaces are even equal. For other graph classes there may not be equality but at least a uniform direction of the inclusion. For instance, we have proven that the eigenspace for eigenvalue 0 of a tree always contains the eigenspace for eigenvalue -1 of its complement.

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