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VARIATION OF THE RADON TRANSFORM

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ABSTRACT. Estimates are obtained for the variation of the Radon transform. The analysis is motivated in part by Kakeya-Besicovitch sets in the plane.

1. Introduction. Given a compact set $E \subset \mathbf{R}^n$, let $E(x)$ denote its characteristic function and $\widehat{E}(\theta, t)$ its Radon transform obtained by integrating $E(x)$ over the hyperplane $\langle \theta, x \rangle = t$ where $\theta \in S^{n-1}$ and $t \in \mathbf{R}^1$. Integrating $E(x)$ over the hyperplane and then with respect to t yields

$$\mu(E) = \int_{\mathbf{R}^n} E(x)\, dx = \int_{-\infty}^{\infty} \widehat{E}(\theta, t)\, dt \tag{1.1}$$

for all $\theta \in S^{n-1}$ where μ denotes Lebesgue measure on $\mathbf{R}^n$. We seek estimates of $\mu(E)$ in terms of the variation of its Radon transform. Given a FIXED direction $\theta \in S^1$ and $\varepsilon > 0$, it is easy to find a set E in the plane satisfying $\widehat{E}(\theta, t) = 1$ for $|t| < \varepsilon$ and 0 otherwise. By (1.1), $\mu(E) = 2\varepsilon$ with ε arbitrary while the variation of $\widehat{E}(\theta, t)$ as a function of t equals 2. As these examples suggest, any estimate of $\mu(E)$ based on the variation of its Radon transform requires integration over almost every direction. Integrals with respect to θ are always over S^{n-1}.

A problem originally stated by Kakeya was to find a set of smallest area in the plane containing a line segment of unit length in every direction. Besicovitch solved the problem by finding a set of measure 0 having this property. More precisely, there is a sequence $\{E_k\}$ of measurable sets in the plane, each containing a line segment of unit length, such that $\mu(E_k) \to 0$ as $k \to \infty$. In our notation, for each $\theta \in S^1$ and $k \geq 1$, we have $\widehat{E_k}(\theta, t) \geq 1$ for some t. The construction along with some of the history of the problem can be found in [**4**].

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By contrast, any compactly supported function $f(x) \in L^2(\mathbf{R}^n)$ with $n \geq 3$ satisfies the inequality

(1.2)
$$\int_{S^{n-1}} \sup_t \left|\hat{f}(\theta,t)\right| \, d\theta \leq A_n\left(\int_{\mathbf{R}^n} |f(x)|\, dx + \left(\int_{\mathbf{R}^n} [f(x)]^2\, dx\right)^{1/2}\right)$$

where the constant A_n depends only on the dimension n. The proof of (1.2), as outlined in [**2**], depends on Fourier transform methods. We establish an inequality that is similar to (1.2). Our methods are different and, at least for the odd dimensional case, independent of the Fourier transform. Where best estimates are possible, numerical values of the constants are computed. The natural setting is absolutely integrable functions with compact support. Estimates for characteristic functions are obtained as a special case.

2. Preliminaries. If $f(x) \in L^1(\mathbf{R}^n)$ has compact support, then so does its Radon transform $\hat{f}(\theta,t)$. Since $f(x)$ is absolutely integrable, $\hat{f}(\theta,t)$ is defined for almost every t for each $\theta \in S^{n-1}$. Let $\operatorname{Var}\hat{f}(\theta) = \sup \Sigma_j |\hat{f}(\theta, y_{j+1}) - \hat{f}(\theta,t_j)|$ for fixed θ. If $\partial\hat{f}(\theta,t)/\partial t$ is absolutely integrable with respect to t, then

(2.1)
$$2\|\hat{f}(\theta)\|_\infty = 2\sup_t \left|\hat{f}(\theta,t)\right| \, dt \leq \int_{-\infty}^{\infty} \left|\partial\hat{f}(\theta,t)/\partial t\right| \, dt = \operatorname{Var} f(\hat{\theta}).$$

In fact, integration of the positive and negative parts of $\partial\hat{f}(\theta,t)/\partial t$ yield monotone functions, each of which has variation greater than or equal to $\|\hat{f}(\theta)\|_\infty$. These observations extend to the derivatives of $\hat{f}(\theta,t)$, for almost every θ where $\operatorname{Var}^k\hat{f}(\theta)$ denotes the variation of the kth partial derivative with respect to t. Thus, $2\|D^k\hat{f}(\theta)\|_\infty \leq \operatorname{Var}^k\hat{f}(\theta)$ where the operator $D^k = \partial^k/\partial t^k$.

Remark 2.1. A simple modification of the Kakeya-Besicovitch construction shows that $\operatorname{Var}\hat{f}(\theta)$ can be made arbitrarily large while $f(x)$ is bounded in the L^1 norm. If we put $f_k(x) = a_k E_k(x)$ where $1/a_k = \mu(E_k)$, then the integral over $\mathbf{R}^2$ of each $f_k(x) \geq 0$ equals 1 for all $k \geq 1$. By (2.1), $\operatorname{Var}\hat{f}_k(\theta)$ grows without bound as $k \to \infty$.

The requirement that $\widehat{E}(\theta,t) \geq \alpha$ with $\alpha = 1$ is merely a matter of scale in the Kakeya-Besicovitch construction. In this paper, we scale and translate so that $f(x)$ and $E(x)$ vanish for $\|x\| > 1$. Of course, this places an upper bound on $\mu(E)$ independent of the variation of the Radon transform. Specifically, if $B = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ then $\mu(B) = \pi^{n/2}/\Gamma(n/2+1)$ where Γ denotes the gamma function. The Radon transform of $B(x)$ will be needed in what follows. A computation yields

$$\widehat{B}(\theta,t) = \widehat{B}(t) = \Omega_{n-1}(1-t^2)^{(n-1)/2} \tag{2.2}$$

for $|t| \leq 1$ and 0 otherwise, where $\Omega_{n-1} = \pi^{(n-1)/2}/\Gamma(n/2+1/2)$.

For n odd, the Plancherel theorem for the Radon transform can be stated as follows. Given compactly supported functions $f(x) \in L^p(\mathbf{R}^n)$ and $g(x) \in L^q(\mathbf{R}^n)$ with $1/p + 1/q = 1$,

$$\int_{\mathbf{R}^n} f(x)g(x)\,dx = c_n \int_{S^{n-1}} \left[\int_{-\infty}^{\infty} D^m \hat{f}(\theta,t) D^m \hat{g}(\theta,t)\,dt\right] d\theta \tag{2.3}$$

where $m = (n-1)/2$ and $c_n = [2(2\pi)^{n-1}]^{-1}$. The integral on the righthand side of (2.3) is to be understood in the formal sense only unless additional information about $f(x)$ or $g(x)$ is available. If we put $f(x) = g(x)$, then the mapping of $f(x) \in L^2(\mathbf{R}^n)$ to $D^m \hat{f}(\theta,t) \in L^2(S^{n-1} \times \mathbf{R}^1)$ is an isometry [**3**, page 29]. Support $f(x) \subset B$ implies that $\hat{f}(\theta,t)$ vanishes for $|t| > 1$. Both $\hat{f}(\theta,t)$ and its derivatives up to order m are absolutely integrable for almost every θ. The derivatives of $\hat{f}(\theta,t)$ are absolutely continuous up to order $m-1$ for these directions. In particular, if $f(x) \in L^2(\mathbf{R}^3)$ then $\hat{f}(\theta,t)$ is absolutely continuous for almost every θ.

The L^2 isometry for n even is from $f(x) \in L^2(\mathbf{R}^n)$ to the derivative of $\hat{f}(\theta,t)$ of half integer order $(n-1)/2$. Since support $f(x) \subset B$, we have $D^m \hat{f}(\theta,t) \in L^2(S^{n-1} \times \mathbf{R}^1)$ for $m = n/2 - 1$. This implies the absolute continuity of the derivatives of order $< m$ as in the odd dimensional case. However, $D^m \hat{f}(\theta,t)$ need not be continuous if n is even. We avoid derivatives of fractional order by replacing $\hat{f}(\theta,t)$ by an integral of fractional order

$$\hat{f}_I(\theta,t) = (2\pi)^{-1/2} \int_{-1}^{1} \frac{\hat{f}(\theta,\tau)}{|t-\tau|^{1/2}}\,d\tau.$$

Remark 2.2. While $\hat{f}(\theta,\tau)$ vanishes for $|\tau| > 1$, $\hat{f}_I(\theta,t)$ can have compact support only if the integral of $f(x)$ over $\mathbf{R}^n$ equals 0. In particular, $\widehat{E}_I$ is not compactly supported for any nonvoid compact set E.

The operator $D^{m+1/2}$ applied to $\hat{f}_I$ is isometric to $D^m\hat{f}$. This can be shown, for example, by looking at the corresponding Fourier multiplier. Granting for the moment that $\hat{f}_I(\theta,t)$ is the Radon transform of some $f_I(x) \in L^2(\mathbf{R}^n)$, we can write the Plancherel theorem as

$$\int_{\mathbf{R}^n} [f_I(x)]^2\,dx = c_n \int_{S^{n-1}} \int_{-1}^{1} [D^m \hat{f}(\theta,t)\,dt]^2\,d\theta$$

where $m = n/2 - 1$ and $c_n = [2(2\pi)^{n-1}]^{-1}$.

Now we show that $\hat{f}_I(\theta,t)$ is the Radon transform of

$$f_I(x) = d_n \int_B \frac{f(y)}{\|x-y\|^{n-1/2}}\,dy$$

where $d_n = (2\pi^n)^{-1/2}\Gamma(n/2 - 1/4)\,\Gamma(1/4)$. Indeed, both $f_I(x)$ and $\hat{f}_I(\theta,t)$ are obtained as convolution integrals. A straightforward integration gives

$$\frac{\pi^{(n-1)/2}\,\Gamma(1/4)}{\Gamma(n/2-1/4)}|t|^{-1/2}$$

as the Radon transform of $\|x\|^{-n+1/2}$. The computed value of d_n follows directly from this. Regarding the integrability of f_I, we appeal to the Hardy-Littlewood-Sobelev theorem on fractional integration [**5**, page 119]. Since support $f(x) \subset B$, $f(x) \in L^2(\mathbf{R}^n)$ implies that $f(x)$ is in $L^p(\mathbf{R}^n)$ for $p = 2n/(n+1)$. It follows that $f_I(x) \in L^2(\mathbf{R}^n)$ and

$$A_n \int_B [f(x)]^2\,dx \ge \int_{S^{n-1}} \int_{-1}^{1} [D^m \hat{f}(\theta,t)]^2 dt\,d\theta \tag{2.4}$$

for some constant A_n. The last inequality requires Jensen's inequality applied to $f(x)$ as well as the Hardy-Littlewood-Sobelev theorem.

As in the odd dimensional case, the Plancherel theorem can be applied to different functions. Given compactly supported functions $f(x) \in L^p(\mathbf{R}^n)$ and $g(x) \in L^q(\mathbf{R}^n)$ with $1/p + 1/q = 1$,

(2.5)
$$\int_{\mathbf{R}^n} f(x)g(x)\,dx = -c_n \int_{S^{n-1}} \left[\int_{-\infty}^{\infty} D^{m+1}\hat{f}(\theta,t)D^m\hat{g}_H(\theta,t)\,dt\right] d\theta$$

where $m = n/2 - 1$ and $\hat{g}_H$ denotes the Hilbert transform of $\hat{g}$ with respect to t. Subject to an integration by parts, (2.5) follows from the Radon inversion formula [**3**, page 27]. The operator D commutes with the Hilbert transform so we can carry out the operations in either order to compute $D^m\hat{g}_H(\theta,t)$. For the application we make, $D^m\hat{g}_H(\theta,t)$ is continuous. The distribution of finite order, formally written as $D^{m+1}\hat{f}(\theta,t)$, need not be identifiable with an ordinary function.

3. Main results. We treat the odd dimensional case first. Inequality (3.1) below applies to the extended reals.

Theorem 3.1. *Suppose that $f(x) \in L^1(\mathbf{R}^n)$ for n odd and support $f(x) \subset B$. Then*

$$\int_B [f(x)]^2\,dx \geq \frac{c_n}{2}\int_{S^{n-1}} [\mathrm{Var}^{m-1}\hat{f}(\theta)]^2\,d\theta \tag{3.1}$$

where $m = (n-1)/2$. If, in addition, $D^{m-1}\hat{f}(\theta,t)$ is continuous in t for almost every $\theta \in S^{n-1}$, then

$$\left|\int_B f(x)\,dx\right| \leq c_n(2\pi)^m(2m)\int_{S^{n-1}} \|D^{m-1}\hat{f}(\theta)\|_\infty\,d\theta. \tag{3.2}$$

Proof. If $f(x) \notin L^2(B)$, then the lefthand side of (3.1) is infinite. Otherwise, the L^2 isometry and Jensen's inequality give

$$\int_B [f(x)]^2\,dx \geq \frac{c_n}{2}\int_{S^{n-1}} \left(\int_{-1}^{1} \left|D^m\hat{f}(\theta,t)\right|\,dt\right)^2 d\theta$$

so inequality (3.1) follows from (2.1).

To establish (3.2), put $g(x) = B(x)$ in (2.3) and integrate by parts to obtain

$$\int_B f(x)g(x)\,dx = -c_n\int_{S^{n-1}} \left[\int_{-1}^{1} D^{m-1}\hat{f}(\theta,t)D^{m+1}\hat{g}(\theta,t)\,dt\right] d\theta.$$

For n odd, $\Omega_{n-1} = \pi^m/m!$ so (2.2), and the formula of Rodrigues, give

$$D^m \widehat{B}(t) = (-1)^m (2\pi)^m p_m(t)$$

where $p_m(t)$ denotes the Legendre polynomial of degree m. We put $D^{m+1}\widehat{B}(t) = d\mu(t)$ where the measure μ has jumps at the endpoints $t = \pm 1$. For almost every fixed θ, $D^{m-1}\hat{f}(\theta, t)$ is continuous in t so

$$\left| \int_{-1}^{1} D^{m-1}\hat{f}(\theta, t)\, d\mu(t) \right| \leq \|D^{m-1}\hat{f}(\theta)\|_\infty \|\mu\| \tag{3.3}$$

where $\|\mu\|$ denotes the variation of the measure μ.

We can disregard the endpoints since $D^{m-1}\hat{f}(\theta, t) = 0$ for $t = \pm 1$. The variation of $p_1(t) = t$ for $-1 \leq t \leq 1$ is 2 so $\|\mu\| = 4\pi$ if $m = 1$. A simple overestimate for $m > 1$ is obtained by multiplying $(2\pi)^m$ by $2m$ where we use the fact that $|p_m(t)| \leq 1$ for $|t| \leq 1$. Integrate (3.3) over S^{n-1} and move the absolute values outside the integral. This yields (3.2) which ends the proof. □

It is easy to show that the inequalities of Theorem 3.1 are best possible on $\mathbf{R}^3$. The function $f(x) = \|x\|^{-1}$ for $0 < \|x\| \leq 1$ and 0 otherwise makes (3.1) an equality. The relevant example for (3.2) is the characteristic function of the set $E_a = \{x \in \mathbf{R}^3 : a \leq \|x\| \leq 1\}$ where $0 < a < 1$. The lefthand side of (ii) equals $4\pi(1-a^3)/3$ and the righthand side is $2\pi(1-a^2)$. In the limit as $a \to 1$, the ratio of these quantities approaches 1.

If $E \subset \mathbf{R}^3$ satisfies $\|\widehat{E}(\theta)\|_\infty \geq \alpha$ for all $\theta \in S^2$, then $\operatorname{Var}\widehat{E}(\theta) \geq 2\alpha$. This implies that $\mu(E) \geq \alpha^2/\pi$ by inequality (3.1). For $n = 5$, the estimate depends on $\operatorname{Var}^1\widehat{E}(\theta)$. If $\|\widehat{E}(\theta)\|_\infty \geq \alpha$, then we also have $\|D^1\widehat{E}(\theta)\|_\infty \geq \alpha$. This follows from the mean value theorem since, for each θ, both $\widehat{E}(\theta, t)$ and its derivative are continuous and vanish for $|t| > 1$. This extends to derivatives of higher order for $n > 5$, providing a lower bound for $\mu(E)$.

Corollary 3.2. *If the Radon transform of $E \subset \mathbf{R}^n$ for n odd satisfies $\|\widehat{E}(\theta)\|_\infty \geq \alpha$ for all $\theta \in S^{n-1}$, then $\mu(E) \geq \alpha^2/[\Gamma(n/2)2^{n-2}\pi^{n/2-1}]$.*

The even dimensional case is divided into an analysis for $n = 2$ and $n \geq 4$. For a set $E \subset \mathbf{R}^2$, (2.4) gives

$$A_2\mu(E) \geq \int_0^{2\pi} \left(\int_{-1}^1 [\widehat{E}(\theta, t)]^2 \, dt \right) d\theta$$

for some constant A_2. But we also have

$$\begin{aligned} [\mu(E)]^2 &= \frac{1}{2\pi} \int_0^{2\pi} \left[\int_{-1}^1 \widehat{E}(\theta, t) \, dt \right]^2 d\theta, \\ &\leq \frac{1}{4\pi} \int_0^{2\pi} \left(\int_{-1}^1 [\widehat{E}(\theta, t)]^2 \, dt \right) d\theta \end{aligned}$$

by (1.1) and Jensen's inequality. Thus, convergence of $\mu(E_k)$ to 0 for some sequence $\{E_k\}$ is equivalent to the convergence of $\{\widehat{E}_k\}$ to 0 in $L^2(S^{n-1} \times \mathbf{R}^1)$.

Theorem 3.3. *Suppose that $f(x) \in L^1(\mathbf{R}^n)$ for $n \geq 4$ and support $f(x) \subset B$. Then*

$$A_n \int_B [f(x)]^2 \, dx \geq \frac{c_n}{2} \int_{S^{n-1}} [\mathrm{Var}^{m-1} \hat{f}(\theta)]^2 \, d\theta$$

for some constant A_n where $m = n/2 - 1$.

Proof. For $f(x) \in L^2(\mathbf{R}^n)$, apply (2.4) and Jensen's inequality as in the odd dimensional case. The inequality is valid in the extended reals if $f(x) \notin L^2(\mathbf{R}^n)$ so this ends the proof. □

Theorem 3.1 and Corollary 3.2 place a lower bound on $\mu(E)$ in terms of the integral of $[\mathrm{Var}\, \widehat{E}(\theta)]^2$. By (2.1) and Jensen's inequality applied to S^{n-1}, this improves on (1.2). For $f(x) \in L^2(\mathbf{R}^n)$, Theorem 3.1 provides a double inequality. Indeed, as noted in Section 2, $f(x) \in L^2(\mathbf{R}^n)$ for n odd implies the continuity in t of $D^{m-1}\hat{f}(\theta, t)$ for almost every $\theta \in S^{n-1}$.

The even dimensional analog of (3.2), obtained from (2.5), is the inequality

$$\left| \int_B f(x) \, dx \right| \leq c_n M_n \int_{S^{n-1}} \mathrm{Var}^m \hat{f}(\theta) \, d\theta \tag{3.4}$$

where

$$M_n = \max_{-1 \le t \le 1} \left| D^m \widehat{B}_H(t) \right|.$$

The boundedness of M_n follows from the continuity of $D^m \widehat{B}_H(t)$ which we now establish. The mth derivative of $\widehat{B}(t)$ can be written as a polynomial in t times $(1-t^2)^{1/2}$ for $|t| \le 1$. Thus, $D^m \widehat{B}(t)$ is Lipschitz continuous of order $1/2$ which implies the continuity of its Hilbert transform.

The estimate is made more precise by computing M_n, or at least finding a numerical upper bound. To accomplish this, we use the fact that

$$\widehat{B}(t) = \frac{1}{\pi} \int_0^\infty \Upsilon(\gamma) \cos(\gamma t) \, d\gamma$$

where the spherically symmetric function $\Upsilon(\gamma)$ is the n-dimensional Fourier transform of $B(x)$. This is easily expressed in terms of Bessel functions. Specifically, for n even,

$$\Upsilon(\gamma) = 2\pi \gamma^{-m} \int_0^1 J_m(r\gamma) r^{m+1} \, dr,$$

where J_m denotes the Bessel function of order $m = n/2 - 1$.

Theorem 3.4. *Suppose that $f(x) \in L^1(\mathbf{R}^n)$ for $n = 2(m+1)$ and support $f(x) \subset B$. Then inequality* (3.4) *holds with*

$$M_n \le 2 \int_0^\infty \gamma^{-1} |J_{m+1}(\gamma)| \, d\gamma. \tag{3.5}$$

Proof. It is sufficient to show that $\Upsilon(\gamma)$ is the one-dimensional inverse Fourier transform of a function with L^1 norm equal to the quantity on the righthand side of (3.5). This integral converges since $J_{m+1}(\gamma)/\gamma^{m+1}$ is bounded and $J_{m+1}(\gamma)$ is asymptotic to $\gamma^{-1/2}$ as $\gamma \to \infty$.

By Sonine's integral formula [**1**, page 98], $\Upsilon(\gamma) = 2\pi \gamma^{-(m+1)} J_{m+1}(\gamma)$. Applying the operator D^m to $\widehat{B}$ and computing a Hilbert transform corresponds to the Fourier multiplier $(i\gamma)^m$ signum (γ). It follows that $D^m \widehat{B}_H(t)$ is continuous with parity opposite that of m and maximum value $\leq$ the righthand side of (3.5) for all t. The inequality is evidently satisfied for $|t| \leq 1$ so this ends the proof. $\square$

We carry out a computation for $n = 2$. The Hilbert transform of $\widehat{B}$ is the odd function

$$\widehat{B}_H(t) = \frac{2}{\pi} \int_{-1}^{1} \frac{(1-\tau^2)^{1/2}}{t-\tau} \, d\tau$$

that attains its maximum at $t = 1$. An integration gives $M_2 = \widehat{B}_H(1) = 2$, which is an improvement over (3.5) for $m = 0$. For $n = 2$, this sharpens inequality (3.4) to

$$\left| \int_B f(x) \, dx \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Var} \hat{f}(\theta) \, d\theta. \tag{3.6}$$

The function $F(x) = (1-\|x\|^2)^{-1/2}$ for $\|x\| < 1$ and 0 otherwise shows that this inequality is best possible. Indeed, the Radon transform of $F(x)$ is $\widehat{F}(t) = \pi$ for $|t| < 1$ and 0 otherwise. Since $\operatorname{Var} \widehat{F}(\theta) = 2\pi$ for all $\theta \in S^1$, the righthand side of (3.6) equals 2π. The integral of $F(x)$ over $\mathbf{R}^2$ gives the same value.

By Remark 2.1, the righthand side of (3.6) can be made arbitrarily large with the lefthand side held constant.

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