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CRITICAL CURVE OF THE NON-NEWTONIAN POLYTROPIC FILTRATION EQUATIONS COUPLED VIA NONLINEAR BOUNDARY FLUX

ZHAOYIN XIANG, CHUNLAI MU AND YULAN WANG

ABSTRACT. This paper is concerned with the critical curve of non-Newtonian polytropic filtration equations coupled via the nonlinear boundary flux. We obtain the critical global existence curve by constructing various self-similar supersolutions and subsolutions. The critical Fujita curve is conjectured with the aid of some new results.

1. Introduction. In this paper, we consider the following doubly degenerate parabolic equations

$$(1.1)\qquad \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\bigg(\bigg|\frac{\partial u^{m_1}}{\partial x}\bigg|^{p_1-2}\frac{\partial u^{m_1}}{\partial x}\bigg), \qquad x>0, \quad 0<t<T,\\ \frac{\partial v}{\partial t} &= \frac{\partial}{\partial x}\bigg(\bigg|\frac{\partial v^{m_2}}{\partial x}\bigg|^{p_2-2}\frac{\partial v^{m_2}}{\partial x}\bigg), \qquad x>0, \quad 0<t<T,\end{aligned}$$

coupled via the nonlinear boundary flux

$$(1.2)\qquad \begin{aligned} -\bigg|\frac{\partial u^{m_1}}{\partial x}\bigg|^{p_1-2}\frac{\partial u^{m_1}}{\partial x}\bigg|_{x=0} &= v^{q_1}(0,t), \quad 0<t<T,\\ -\bigg|\frac{\partial v^{m_2}}{\partial x}\bigg|^{p_2-2}\frac{\partial v^{m_2}}{\partial x}\bigg|_{x=0} &= u^{q_2}(0,t), \quad 0<t<T,\end{aligned}$$

where $m_i > 1$, $p_i > 2$ and $q_i > 0$, $i = 1, 2$, are parameters. We consider initial data

$$(1.3)\qquad u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x > 0,$$

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which are assumed to be continuous, nonnegative and compactly supported in $\mathbf{R}_+$.

Parabolic systems like (1.1)–(1.3) appear in population dynamics, chemical reactions, heat transfer, and so on. In particular, equations (1.1) may be used to describe the nonstationary flows in a porous medium of fluids with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions. In this case, equations (1.1) are called non-Newtonian polytropic filtration equations. See [**5, 10, 12**] and references therein.

For systems (1.1)–(1.3), the local in time existence and the comparison principle of nonnegative weak solutions, defined in the usual integral way, can be easily established as for instance in [**2, 5, 10, 12**]. In this work, we are interested in the large time behavior of solutions of the nonlinear boundary problem (1.1)–(1.3) and investigate the critical global existence curve and the critical Fujita curve, a subject that has deserved a great deal of attention in recent years, see for example the books [**9, 10**], the surveys [**1, 6**] and the references therein.

As a precedent we have the work of Wang et al. [**11**], where they study the single-equation case

$$(1.4)\qquad \begin{gathered} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial u^{m_1}}{\partial x}\right), \quad x>0, \quad 0<t<T, \\ -\left|\frac{\partial u^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial u^{m_1}}{\partial x}\bigg|_{x=0} = u^{q_1}(0,t), \quad 0<t<T, \\ u(x,0)=u_0(x), \quad x>0, \end{gathered}$$

where $m_1 > 1$, $p_1 > 2$. They show that if $0 < q_1 \le q_0 := ((m_1+1)(p_1-1))/p_1$, then all nonnegative solutions of (1.4) are global in time, while for $q_1 > ((m_1+1)(p_1-1))/p_1$ there are solutions with finite time blow-up. Thus, q_0 is the critical global existence exponent. Moreover, it was also shown that $q_c := (m_1+1)(p_1-1)$ is a critical exponent of Fujita type. Precisely, q_c has the following properties: if $q_0 < q_1 < q_c$, then all nontrivial nonnegative solutions blow up in a finite time, while global nontrivial nonnegative solutions exist if $q_1 > q_c$. These results of [**11**] are the extensions of those of Galaktionov and Levine [**3**], in which the authors dealt with three classical cases of (1.4): the heat equation ($p_1 = 2, m_1 = 1$); the porous medium equation ($p_1 = 2$) and the p-Laplace equation ($m_1 = 1$).

Another extension of [**3**] is the recent work of Quirós and Rossi [**8**], where they considered systems (1.1)–(1.3) with $p_1 = p_2 = 2$,

$$
\begin{aligned}
&u_t = (u^{m_1})_{xx}\,, \qquad v_t = (v^{m_2})_{xx}\,,\\
&x > 0, \quad 0 < t < T,\\
&-(u^{m_1})_x(0,t) = v^{q_1}(0,t), \qquad -(v^{m_2})_x(0,t) = u^{q_2}(0,t),\\
&0 < t < T,\\
&u(x,0) = u_0(x), \qquad v(x,0) = v_0(x),\\
&x > 0.
\end{aligned}
\tag{1.5}
$$

Denoting by

$$
\begin{aligned}
\gamma_1 &= \frac{2q_1 + m_2 + 1}{(m_1+1)(m_2+1) - 4q_1q_2},\\
\tau_1 &= \frac{q_1(m_1 - 1 - 2q_2) + m_1(m_2+1)}{(m_1+1)(m_2+1) - 4q_1q_2},\\
\gamma_2 &= \frac{2q_2 + m_1 + 1}{(m_1+1)(m_2+1) - 4q_1q_2},\\
\tau_2 &= \frac{q_2(m_2 - 1 - 2q_1) + m_2(m_1+1)}{(m_1+1)(m_2+1) - 4q_1q_2},
\end{aligned}
$$

they proved that the solutions of (1.5) are global if $4q_1q_2 \le (m_1 + 1)(m_2+1)$ and may blow up in finite time if $4q_1q_2 > (m_1+1)(m_2+1)$. In the case of $4q_1q_2 > (m_1+1)(m_2+1)$, if $\gamma_1 + \tau_1 \le 0$ or $\gamma_2 + \tau_2 \le 0$, then every nontrivial nonnegative solution of (1.5) blows up in a finite time, while global nonnegative solutions exist if $\gamma_1 + \tau_1 > 0$ and $\gamma_2 + \tau_2 > 0$. Therefore, the critical global existence curve is $4q_1q_2 = (m_1+1)(m_2+1)$ and the critical Fujita type curve is described by $\min\{\gamma_1 + \tau_1, \gamma_2 + \tau_2\} = 0$.

Motivated by the above cited works, in this paper we will construct various kinds of self-similar supersolutions and subsolutions to obtain the critical global existence curve of system (1.1)–(1.3). The critical curve of Fujita type is conjectured with the aid of some new results. These results seem to be natural extensions of [**3, 8, 11, 13, 14**]. We remark that, by using our methods, which are essentially due to (1.4), it is easy to deal with the system coupled through nonlinear boundary conditions as [**14, 15**].

To state our results, we need to introduce the following numbers. Let

$$\begin{aligned}\alpha_1 &= \frac{(p_1-1)(p_2-1)(m_2+1)+(p_2-1)p_1q_1}{(p_1-1)(p_2-1)(m_1+1)(m_2+1)-p_1p_2q_1q_2},\\ \beta_1 &= m_1\alpha_1 - \frac{q_1}{p_1-1}\alpha_2,\\ \alpha_2 &= \frac{(p_1-1)(p_2-1)(m_1+1)+(p_1-1)p_2q_2}{(p_1-1)(p_2-1)(m_1+1)(m_2+1)-p_1p_2q_1q_2},\\ \beta_2 &= m_2\alpha_2 - \frac{q_2}{p_2-1}\alpha_1,\end{aligned}$$

if $q_1q_2 \neq ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1p_2$. The values $\alpha_1, \alpha_2, \beta_1, \beta_2$ are the exponents of self-similar solutions to problem (1.1)–(1.3).

Our results read as follows.

Theorem 1.1. (i) *If* $q_1q_2 \leq ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1p_2$, *then every nonnegative solution of system* (1.1)–(1.3) *is global in time*;

(ii) *If* $q_1q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1p_2$, *then the system* (1.1)–(1.3) *exists with solutions blowing up in a finite time.*

Remark 1.1. The results of Theorem 1.1 for system (1.1)–(1.3) do coincide with those of [**8, 11, 13, 14**]. Theorem 1.1 shows that the critical global existence curve of system (1.1)–(1.3) is $q_1q_2 = ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1p_2$.

Theorem 1.2. *Assume* $q_1q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1p_2$.

(i) *If* $\min\{\alpha_1+\beta_1, \alpha_2+\beta_2\} > 0$, *then there exists a global solution to system* (1.1)–(1.3);

(ii) *If* $\max\{\alpha_1+\beta_1, \alpha_2+\beta_2\} < 0$, *then every nonnegative nontrivial solution of* (1.1)–(1.3) *blows up in finite time.*

Remark 1.2. The results of Theorem 1.2 for system (1.1)–(1.3) do coincide with those of [**8, 11, 13, 14**]. The restriction $\max\{\alpha_1+\beta_1, \alpha_2+\beta_2\} < 0$ in (ii) is rather technical. It comes from the construction of self-similar solutions, which are similar to the so-called Zel'dovich-Kompaneetz-Barenblatt profile. We believe that the critical Fujita curve should be given by $\min\{\alpha_1+\beta_1,\alpha_2+\beta_2\}=0$.

The rest of this paper is organized as follows. In Section 2, we consider the global existence curve (Theorem 1.1). The proof of Theorem 1.2 is the subject of Section 3.

2. Critical global existence curve. In this section, we characterize when the solutions to problem (1.1)–(1.3) are global in time for any initial data or they may blow up for some initial values. By constructing self-similar solutions and using comparison arguments, the critical global existence curve is obtained.

Proof of Theorem 1.1 (i). It is enough to construct global supersolutions with initial data as large as needed. To this purpose, we look for a globally defined in time strict supersolution of self-similar form

$$\begin{gathered}\overline{u}(x,t)=e^{\kappa_1 t}\left(M+e^{-L_1 x e^{-\kappa_2 t}}\right)^{1/m_1},\\ \overline{v}(x,t)=e^{\kappa_3 t}\left(M+e^{-L_2 x e^{-\kappa_4 t}}\right)^{1/m_2},\\ x\geq 0,\quad t\geq 0,\end{gathered}$$

where $M=\max\{\|u_0\|_\infty^{m_1}+1,\|v_0\|_\infty^{m_2}+1\}$, and the constants $\kappa_i>0$, $i=1,2,3,4$, and $L_i>0$, $i=1,2$, are to be determined. Clearly, we have

$$\overline{u}(x,0)\geq u_0(x),\qquad \overline{v}(x,0)\geq v_0(x),\quad x\geq 0.$$

After a series of computations, we have

$$\begin{aligned}&\frac{\partial\overline{u}}{\partial t}\geq\kappa_1 e^{\kappa_1 t}\left(M+e^{-L_1 x e^{-\kappa_2 t}}\right)^{1/m_1}\geq\kappa_1 M^{\frac{1}{m_1}}e^{\kappa_1 t},\\ &\left|\frac{\partial\overline{u}^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial\overline{u}^{m_1}}{\partial x}=-L_1^{p_1-1}e^{(p_1-1)(m_1\kappa_1-\kappa_2)t}e^{-(p_1-1)L_1 x e^{-\kappa_2 t}},\\ &\frac{\partial}{\partial x}\left(\left|\frac{\partial\overline{u}^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial\overline{u}^{m_1}}{\partial x}\right)\leq L_1^{p_1}(p_1-1)e^{((p_1-1)m_1\kappa_1-p_1\kappa_2)t},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \overline{v}}{\partial t} &\geq \kappa_3 e^{\kappa_3 t}\left(M + e^{-L_3 x e^{-\kappa_4 t}}\right)^{1/m_2} \geq \kappa_3 M^{1/m_2} e^{\kappa_3 t},\\ \left|\frac{\partial \overline{v}^{m_2}}{\partial x}\right|^{p_2-2}\frac{\partial \overline{v}^{m_2}}{\partial x} &= -L_2^{p_2-1} e^{(p_2-1)(m_2\kappa_3-\kappa_4)t} e^{-(p_2-1)L_2 x e^{-\kappa_4 t}},\\ \frac{\partial}{\partial x}\left(\left|\frac{\partial \overline{v}^{m_2}}{\partial x}\right|^{p_2-2}\frac{\partial \overline{v}^{m_2}}{\partial x}\right) &\leq L_2^{p_2}(p_2-1)e^{((p_2-1)m_2\kappa_3-p_2\kappa_4)t}.\end{aligned}$$

Therefore, we see that $(\overline{u}, \overline{v})$ is a supersolution of system (1.1)–(1.3) provided that

$$\begin{aligned}\kappa_1 M^{1/m_1} e^{\kappa_1 t} &\geq L_1^{p_1}(p_1-1)e^{((p_1-1)m_1\kappa_1-p_1\kappa_2)t},\\ \kappa_3 M^{1/m_2} e^{\kappa_3 t} &\geq L_2^{p_2}(p_2-1)e^{((p_2-1)m_2\kappa_3-p_2\kappa_4)t}\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}L_1^{p_1-1}e^{(p_1-1)(m_1\kappa_1-\kappa_2)t} &\geq (M+1)^{q_1/m_2}e^{\kappa_3 q_1 t},\\ L_2^{p_2-1}e^{(p_2-1)(m_2\kappa_3-\kappa_4)t} &\geq (M+1)^{q_2/m_1}e^{\kappa_1 q_2 t}.\end{aligned} \tag{2.2}$$

One can see that (2.2) holds if

$$(p_1-1)(m_1\kappa_1-\kappa_2) \geq \kappa_3 q_1, \qquad (p_2-1)(m_2\kappa_3-\kappa_4) \geq \kappa_1 q_2 \tag{2.3}$$

and

$$L_1^{p_1-1} \geq (M+1)^{q_1/m_2}, \quad L_2^{p_2-1} \geq (M+1)^{q_2/m_1}. \tag{2.4}$$

Firstly, we get (2.4) by taking $L_1 = (M+1)^{q_1/(m_2(p_1-1))}$ and $L_2 = (M+1)^{q_2/(m_1(p_2-1))}$.

Next, we see that (2.1) is valid if

$$\kappa_1 \geq (p_1-1)m_1\kappa_1 - p_1\kappa_2, \quad \kappa_3 \geq (p_2-1)m_2\kappa_3 - p_2\kappa_4$$

and

$$\kappa_1 M^{1/m_1} \geq L_1^{p_1}(p_1-1), \qquad \kappa_3 M^{1/m_2} \geq L_2^{p_2}(p_2-1).$$

To this purpose, we only need to take κ_1 and κ_4 large enough with

$$\kappa_2 = \frac{(p_1-1)m_1-1}{p_1}\kappa_1, \qquad \kappa_3 = \frac{p_2}{(p_2-1)m_2-1}\kappa_4. \tag{2.5}$$

Finally, to obtain (2.3), we substitute (2.5) into (2.3) and then only need to confirm

$$(2.6)\qquad \begin{aligned} (p_1-1)\left(m_1-\frac{m_1(p_1-1)-1}{p_1}\right)\kappa_1 &\geq \frac{p_2q_1}{m_2(p_2-1)-1}\kappa_4,\\ (p_2-1)\left(\frac{m_2p_2}{m_2(p_2-1)-1}-1\right)\kappa_4 &\geq q_2\kappa_1. \end{aligned}$$

It follows from the assumption

$$q_1q_2 \leq \frac{(p_1-1)(p_2-1)(m_1+1)(m_2+1)}{p_1p_2}$$

that (2.6) is true for suitable κ_1, κ_4.

Therefore, we have proved $(\overline{u}, \overline{v})$ is a global supersolution of system (1.1)–(1.3). The global existence of solutions to problem (1.1)–(1.3) follows from the comparison principle.

To prove the nonexistence of global solutions, we construct a blow up self-similar subsolution of system (1.1)–(1.3). Firstly, we have the following lemma.

Lemma 2.1. *If* $q_1q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/(p_1p_2)$, *then there exist compactly supported functions* f_i, $i=1,2$, *such that* $(\underline{u}, \underline{v})$ *is a subsolution of system* (1.1), (1.2), *where*

$$(2.7)\qquad \underline{u}(x,t) = (T-t)^{\alpha_1} f_1(\xi), \qquad \xi = \frac{x}{(T-t)^{\beta_1}},$$

$$(2.8)\qquad \underline{v}(x,t) = (T-t)^{\alpha_2} f_2(\zeta), \qquad \zeta = \frac{x}{(T-t)^{\beta_2}}.$$

Proof. For $\underline{u}$ defined by (2.7), a direct calculation yields

$$\frac{\partial \underline{u}}{\partial t} = (T-t)^{\alpha_1-1}\left(-\alpha_1 f_1(\xi) + \beta_1 \xi f_1'(\xi)\right),$$

$$\left|\frac{\partial \underline{u}^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial \underline{u}^{m_1}}{\partial x} = (T-t)^{(p_1-1)(m_1\alpha_1-\beta_1)}\left|\left(f_1^{m_1}\right)'\right|^{p_1-2}\left(f_1^{m_1}\right)',$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \underline{u}^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial \underline{u}^{m_1}}{\partial x}\right) = (T-t)^{(p_1-1)m_1\alpha_1-p_1\beta_1}\left(|(f_1^{m_1})'|^{p_1-2}(f_1^{m_1})'\right)'.$$

Similarly, for $\underline{v}$ defined by (2.8), we have

$$\frac{\partial \underline{v}}{\partial t} = (T-t)^{\alpha_2-1}\left(-\alpha_2 f_2(\zeta)+\beta_2\zeta f_2'(\zeta)\right),$$

$$\left|\frac{\partial \underline{v}^{m_2}}{\partial x}\right|^{p_2-2}\frac{\partial \underline{v}^{m_2}}{\partial x} = (T-t)^{(p_2-1)(m_2\alpha_2-\beta_2)}|(f_2^{m_2})'|^{p_2-2}(f_2^{m_2})',$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \underline{v}^{m_2}}{\partial x}\right|^{p_2-2}\frac{\partial \underline{v}^{m_2}}{\partial x}\right) = (T-t)^{(p_2-1)m_2\alpha_2-p_2\beta_2}\left(|(f_2^{m_2})'|^{p_2-2}(f_2^{m_2})'\right)'.$$

Noticing that

$$\begin{aligned}\alpha_1-1&=(p_1-1)m_1\alpha_1-p_1\beta_1, \quad (p_1-1)(m_1\alpha_1-\beta_1)=q_1\alpha_2,\\ \alpha_2-1&=(p_2-1)m_2\alpha_2-p_2\beta_2, \quad (p_2-1)(m_2\alpha_2-\beta_2)=q_2\alpha_1,\end{aligned} \tag{2.9}$$

we see $(\underline{u},\underline{v})$ is a subsolution of (1.1), (1.2) if f_1 and f_2 satisfy

$$\begin{aligned}-\alpha_1 f_1(\xi)+\beta_1\xi f_1'(\xi) &\le \left(|(f_1^{m_1})'|^{p_1-2}(f_1^{m_1})'\right)'(\xi),\\ -\alpha_2 f_2(\zeta)+\beta_2\zeta f_2'(\zeta) &\le \left(|(f_2^{m_2})'|^{p_2-2}(f_2^{m_2})'\right)'(\zeta)\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}-|(f_1^{m_1})'|^{p_1-2}(f_1^{m_1})'(0) &\le f_2^{q_1}(0),\\ -|(f_2^{m_2})'|^{p_2-2}(f_2^{m_2})'(0) &\le f_1^{q_2}(0).\end{aligned} \tag{2.11}$$

We choose

$$\begin{aligned}f_1(\xi)&=A_1(a_1-\xi)_+^{(p_1-1)/(m_1(p_1-1)-1)},\\ f_2(\zeta)&=A_2(a_2-\zeta)_+^{(p_2-1)/(m_2(p_2-1)-1)},\end{aligned} \tag{2.12}$$

where A_i, a_i, $i=1,2$, are constants to be determined. It is easy to see that

$$f_1'(\xi) = -\frac{(p_1-1)A_1}{m_1(p_1-1)-1}(a_1-\xi)_+^{(p_1-1)/(m_1(p_1-1)-1)-1},$$

$$\begin{aligned}
|\,(f_1^{m_1})'\,|^{p_1-2}\,(f_1^{m_1})' &= -\left(\frac{m_1(p_1-1)A_1^{m_1}}{m_1(p_1-1)-1}\right)^{p_1-1} \\
&\qquad \times (a_1-\xi)_+^{(p_1-1)/(m_1(p_1-1)-1)}, \\
\left(|\,(f_1^{m_1})'\,|^{p_1-2}\,(f_1^{m_1})'\right)' &= (m_1A_1^{m_1})^{p_1-1}\left(\frac{p_1-1}{m_1(p_1-1)-1}\right)^{p_1} \\
&\qquad \times (a_1-\xi)_+^{(p_1-1)/(m_1(p_1-1)-1-1)},
\end{aligned}$$

and

$$f_2'(\zeta) = -\frac{(p_2-1)A_2}{m_2(p_2-1)-1}(a_2-\zeta)_+^{(p_2-1)/(m_2(p_2-1)-1)-1},$$

$$\begin{aligned}
|\,(f_2^{m_2})'\,|^{p_2-2}\,(f_2^{m_2})' &= -\left(\frac{m_2(p_2-1)A_2^{m_2}}{m_2(p_2-1)-1}\right)^{p_2-1} \\
&\qquad \times (a_2-\zeta)_+^{(p_2-1)/(m_2(p_2-1)-1)}, \\
\left(|\,(f_2^{m_2})'\,|^{p_2-2}\,(f_2^{m_2})'\right)' &= (m_2A_2^{m_2})^{p_2-1}\left(\frac{p_2-1}{m_2(p_2-1)-1}\right)^{p_2} \\
&\qquad \times (a_2-\zeta)_+^{(p_2-1)/(m_2(p_2-1)-1)-1}.
\end{aligned}$$

Therefore, inequalities (2.10) are valid provided that

$$\begin{gathered}
\frac{1-\beta_1}{m_1(p_1-1)-1}\xi + m_1^{p_1-1}A_1^{m_1(p_1-1)-1}\left(\frac{p_1-1}{m_1(p_1-1)-1}\right)^{p_1} + \alpha_1 a_1 \ge 0, \\
0 \le \xi \le a_1, \\
\frac{1-\beta_2}{m_2(p_2-1)-1}\zeta + m_2^{p_2-1}A_2^{m_2(p_2-1)-1}\left(\frac{p_2-1}{m_2(p_2-1)-1}\right)^{p_2} + \alpha_2 a_2 \ge 0, \\
0 \le \zeta \le a_2.
\end{gathered}$$

Here, we use equalities (2.9). To show the above inequalities, we choose a_1 and a_2 with

$$a_1 = c_1 A_1^{m_1(p_1-1)-1}, \quad a_2 = c_2 A_2^{m_2(p_2-1)-1}, \tag{2.13}$$

where

$$\begin{aligned}
c_1 &= \frac{p_1-1}{(m_1(p_1-1)-1)(-\alpha_1)+|\beta_1-1|}\left(\frac{m_1(p_1-1)}{m_1(p_1-1)-1}\right)^{p_1-1} > 0, \\
c_2 &= \frac{p_2-1}{(m_2(p_2-1)-1)(-\alpha_2)+|\beta_2-1|}\left(\frac{m_2(p_2-1)}{m_2(p_2-1)-1}\right)^{p_2-1} > 0.
\end{aligned}$$

Here, we remark that $\alpha_1, \alpha_2 < 0$ under the assumption $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/(p_1 p_2)$.

On the other hand, the boundary conditions in (2.11) are satisfied if we have

$$\tag{2.14} \begin{aligned} c_3 A_1^{m_1(p_1-1)} a_1^{(p_1-1)/(m_1(p_1-1)-1)} &\le A_2^{q_1} a_2^{((p_2-1)q_1)/(m_2(p_2-1)-1)}, \\ c_4 A_2^{m_2(p_2-1)} a_2^{(p_2-1)/(m_2(p_2-1)-1)} &\le A_1^{q_2} a_1^{((p_1-1)q_2)/(m_1(p_1-1)-1)}, \end{aligned}$$

where

$$c_3 = \left(\frac{m_1(p_1-1)}{m_1(p_1-1)-1}\right)^{p_1-1} > 0, \quad c_4 = \left(\frac{m_2(p_2-1)}{m_2(p_2-1)-1}\right)^{p_2-1} > 0.$$

According to (2.13), we see that (2.14) holds provided that we choose A_1, A_2 satisfying

$$\tag{2.15} \begin{aligned} c_3 c_1^{(p_1-1)/(m_1(p_1-1)-1)} A_1^{(m_1+1)(p_1-1)} &\le c_2^{((p_2-1)q_1)/(m_2(p_2-1)-1)} A_2^{p_2 q_1}, \\ c_4 c_2^{(p_2-1)/(m_2(p_2-1)-1)} A_2^{(m_2+1)(p_2-1)} &\le c_1^{((p_1-1)q_1)/(m_1(p_1-1)-1)} A_1^{p_1 q_2}. \end{aligned}$$

The condition $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/(p_1 p_2)$ ensures that we can take A_1 and A_2 large enough such that inequalities (2.15) hold. Therefore, we have shown that $(\underline{u}, \underline{v})$ is a weak subsolution of system (1.1), (1.2).

Proof of Theorem 1.1 (ii). We take the initial data (u_0, v_0) large enough such that $u_0(x) \ge \underline{u}(x,0) = T^{\alpha_1} f_1(x/T^{\beta_1})$ and $v_0(x) \ge \underline{v}(x,0) = T^{\alpha_2} f_2(x/T^{\beta_2})$, where $\underline{u}, \underline{v}$ and f_1, f_2 are defined in Lemma 2.1. According to the construction of $f_1(\xi)$ and $f_2(\zeta)$ in the proof of Lemma 2.1, we see that $\lim_{t\to T^-} \underline{u}(0,t) = +\infty$ and $\lim_{t\to T^-} \underline{v}(0,t) = +\infty$ if $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/(p_1 p_2)$. Then it follows from the comparison principle that a solution blowing up in a finite time exists in system (1.1)–(1.3).

3. Critical curve of Fujita type. In this section, we turn our attention to the results of Fujita type and give the proof of Theorem 1.2.

Proof of Theorem 1.2 (i). We construct the following auxiliary functions

$$\tag{3.1} \overline{u}(x,t) = (\tau+t)^{\alpha_1} f_1(\xi), \quad \overline{v}(x,t) = (\tau+t)^{\alpha_2} f_2(\zeta),$$

where $\tau > 0$, $\xi = x(\tau+t)^{-\beta_1}$ and $\zeta = x(\tau+t)^{-\beta_2}$. Then

$$\frac{\partial \overline{u}}{\partial t} = (\tau+t)^{\alpha_1-1}\left(\alpha_1 f_1(\xi) - \beta_1 \xi f_1'(\xi)\right),$$

$$\left|\frac{\partial \overline{u}^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial \overline{u}^{m_1}}{\partial x} = (\tau+t)^{(p_1-1)(m_1\alpha_1-\beta_1)}|\,(f_1^{m_1})'\,|^{p_1-2}\,(f_1^{m_1})',$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \overline{u}^{m_1}}{\partial x}\right|^{p_1-2}\frac{\partial \overline{u}^{m_1}}{\partial x}\right) = (\tau+t)^{(p_1-1)m_1\alpha_1-p_1\beta_1}\left(|\,(f_1^{m_1})'\,|^{p_1-2}\,(f_1^{m_1})'\right)'.$$

Similarly, we have

$$\frac{\partial \overline{v}}{\partial t} = (\tau+t)^{\alpha_2-1}\left(\alpha_2 f_2(\zeta) - \beta_2 \zeta f_2'(\zeta)\right),$$

$$\left|\frac{\partial \overline{v}^{m_2}}{\partial x}\right|^{p_2-2}\frac{\partial \overline{v}^{m_2}}{\partial x} = (\tau+t)^{(p_2-1)(m_2\alpha_2-\beta_2)}|\,(f_2^{m_2})'\,|^{p_2-2}\,(f_2^{m_2})',$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \overline{v}^{m_2}}{\partial x}\right|^{p_2-2}\frac{\partial \overline{v}^{m_2}}{\partial x}\right) = (\tau+t)^{(p_2-1)m_2\alpha_2-p_2\beta_2}\left(|\,(f_2^{m_2})'\,|^{p_2-2}\,(f_2^{m_2})'\right)'.$$

It follows from (2.9) that $(\overline{u}, \overline{v})$ is a supersolution of (1.1), (1.2) provided that $f_1(\xi)$ and $f_2(\zeta)$ satisfy

$$\begin{aligned}&\left(|\,(f_1^{m_1})'\,|^{p_1-2}\,(f_1^{m_1})'\right)'(\xi) + \beta_1\xi f_1'(\xi) - \alpha_1 f_1(\xi) \le 0,\\ &\left(|\,(f_2^{m_2})'\,|^{p_2-2}\,(f_2^{m_2})'\right)'(\zeta) + \beta_2\zeta f_2'(\zeta) - \alpha_2 f_2(\zeta) \le 0,\end{aligned} \tag{3.2}$$

and

(3.3)

$$-|\,(f_1^{m_1})'\,|^{p_1-2}\,(f_1^{m_1})'(0) \ge f_2^{q_1}(0), \quad -|\,(f_2^{m_2})'\,|^{p_2-2}\,(f_2^{m_2})'(0) \ge f_1^{q_2}(0).$$

Set

(3.4)

$$f_1(\xi) = A_1\left((d_1a_1)^{p_1/(p_1-1)} - (\xi+a_1)^{p_1/(p_1-1)}\right)_+^{(p_1-1)/(m_1(p_1-1)-1)}, \quad d_1 > 1,$$

$$f_2(\zeta) = A_2\left((d_2a_2)^{p_2/(p_2-1)} - (\zeta+a_2)^{p_2/p_2-1}\right)_+^{(p_2-1)/(m_2(p_2-1)-1)}, \quad d_2 > 1.$$

We claim that there exist A_i, a_i such that inequalities (3.2) are valid for f_i, $i = 1, 2$, defined by (3.4). In fact, it is easy to verify that

$$
\begin{aligned}
\left(|(f_1^{m_1})'|^{p_1-2}(f_1^{m_1})'\right)' &= -\left(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\right)^{p_1-1} \\
&\quad \times A_1^{-1}\left(f_1(\xi) + (\xi + a_1) f_1'(\xi)\right), \\
a_1 \le \xi + a_1 &\le d_1 a_1; \\
\left(|(f_2^{m_2})'|^{p_2-2}(f_2^{m_2})'\right)' &= -\left(\frac{m_2 p_2 A_2^{m_2}}{m_2(p_2-1)-1}\right)^{p_2-1} \\
&\quad \times A_2^{-1}\left(f_2(\zeta) + (\zeta + a_2) f_2'(\zeta)\right), \\
a_2 \le \zeta + a_2 &\le d_2 a_2.
\end{aligned}
\tag{3.5}
$$

Substituting (3.5) into (3.2), we have

$$
\begin{aligned}
&\left(\left(\beta_1 - \left(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\right)^{p_1-1} A_1^{-1}\right)(\xi + a_1) - \beta_1 a_1\right) f_1'(\xi) \\
&\qquad - \left(\alpha_1 + \left(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\right)^{p_1-1} A_1^{-1}\right) f_1(\xi) \le 0, \\
&\left(\left(\beta_2 - \left(\frac{m_2 p_2 A_2^{m_2}}{m_2(p_2-1)-1}\right)^{p_2-1} A_2^{-1}\right)(\zeta + a_2) - \beta_2 a_2\right) f_2'(\zeta) \\
&\qquad - \left(\alpha_2 + \left(\frac{m_2 p_2 A_2^{m_2}}{m_2(p_2-1)-1}\right)^{p_2-1} A_2^{-1}\right) f_2(\zeta) \le 0.
\end{aligned}
\tag{3.6}
$$

Notice that $f_1(\xi)$ and $f_2(\zeta)$ are defined by (3.4), and then

$$
\begin{aligned}
f_1'(\xi) &= -\frac{A_1 p_1}{m_1(p_1-1)-1}(\xi + a_1)^{1/(p_1-1)} \\
&\quad \times \left((d_1 a_1)^{p_1/(p_1-1)} - (\xi + a_1)^{p_1/(p_1-1)}\right)^{(p_1-1)/(m_1(p_1-1)-1)-1}, \\
f_2'(\zeta) &= -\frac{A_2 p_2}{m_2(p_2-1)-1}(\zeta + a_2)^{1/(p_2-1)} \\
&\quad \times \left((d_2 a_2)^{p_2/(p_2-1)} - (\zeta + a_2)^{p_2/(p_2-1)}\right)^{(p_2-1)/(m_2(p_2-1)-1)-1}.
\end{aligned}
$$

Therefore, denoting by $z = \xi + a_1$, $\eta = \zeta + a_2$ and noticing (2.7), we transform (3.6) into the following inequalities with respect to z and η:

$$
\begin{aligned}
g_1(z) = {} & \bigg((m_1+1)(p_1-1)\bigg(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\bigg)^{p_1-1} A_1^{-1} - 1\bigg) z^{p_1/(p_1-1)} \\
& + p_1 a_1 \beta_1 z^{1/(p_1-1)} - (m_1(p_1-1)-1) \\
& \times \bigg(\alpha_1 + \bigg(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\bigg)^{p_1-1} A_1^{-1}\bigg)(d_1 a_1)^{p_1/(p_1-1)} \le 0,
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
g_2(\eta) = {} & \bigg((m_2+1)(p_2-1)\bigg(\frac{m_2 p_2 A_2^{m_2}}{m_2(p_2-1)-1}\bigg)^{p_2-1} A_2^{-1} - 1\bigg) \eta^{p_2/(p_2-1)} \\
& + p_2 a_2 \beta_2 \eta^{1/(p_2-1)} - (m_2(p_2-1)-1) \\
& \times \bigg(\alpha_2 + \bigg(\frac{m_2 p_2 A_2^{m_2}}{m_2(p_2-1)-1}\bigg)^{p_2-1} A_2^{-1}\bigg)(d_2 a_2)^{p_2/(p_2-1)} \\
& \le 0.
\end{aligned} \tag{3.8}
$$

To obtain $g_1(z) \le 0$ (inequality (3.7)), we divide its proof into two cases.

Case I. $1/((m_1+1)(p_1-1)) > -\alpha_1$. Then we may take A_1 such that

$$
-\alpha_1 < \bigg(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\bigg)^{p_1-1} A_1^{-1} := l_1 < \frac{1}{(m_1+1)(p_1-1)}.
$$

Thus, for any $z \ge 0$, $g_1(z)$ attains its maximum at $z^* = (a_1\beta_1)/(1-(m_1+1)(p_1-1)l_1)$. Here, we use the assumption $\beta_1 > -\alpha_1 > 0$ whenever $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/(p_1 p_2)$. We impose $g_1(z^*) \le 0$. To do this, we substitute z^* into the definition of $g_1(z)$ and then only need to choose d_1 sufficiently large such that

$$
\begin{aligned}
& d_1^{p_1/(p_1-1)} \\
& \quad > \max\bigg\{\frac{(p_1-1)(1-(m_1+1)(p_1-1)l_1)^{-1/(p_1-1)}\beta_1^{p_1/(p_1-1)}}{(m_1(p_1-1)-1)(\alpha_1+l_1)}, 1\bigg\}.
\end{aligned}
$$

Case II. $1/((m_1+1)(p_1-1)) \le -\alpha_1$. It follows from $\alpha_1+\beta_1>0$ that $\beta_1 > (1/(m_1+1)(p_1-1))$. We take A_1 with

$$\frac{1}{(m_1+1)(p_1-1)} < \left(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\right)^{p_1-1} A_1^{-1} := l_1 < \beta_1.$$

Then $g_1(z)$ is nondecreasing with respect to $z \ge 0$. So for $z \in [a_1, d_1 a_1]$, $g_1(z)$ attains its maximum at $z^* = d_1 a_1$. To obtain $g_1(z) \le 0$ for all $z \in [a_1, d_1 a_1]$, we impose $g(d_1 a_1) \le 0$. To this purpose, we only need to choose d_1 with

$$(l_1 - \beta_1)\ d_1 + \beta_1 < 0,$$

or equivalently,

$$d_1 > \max\left\{\beta_1\left(\beta_1 - l_1\right)^{-1}, 1\right\}.$$

Therefore, we have shown that there exist A_1 and d_1 such that $g_1(z) \le 0$ holds.

Analogously, we may choose A_2 and d_2 such that $g_2(\eta) \le 0$, that is, inequality (3.7) holds. In a word, we have proved our claim. $\square$

Now we consider the boundary condition (3.3). We only need to show (3.9)

$$\begin{aligned}
&\left(\frac{m_1 p_1 A_1^{m_1}}{m_1(p_1-1)-1}\right)^{p_1-1} \left(d_1^{p_1/(p_1-1)} - 1\right)^{(p_1-1)/(m_1(p_1-1)-1)} \\
&\quad \times a_1^{((m_1+1)(p_1-1))/(m_1(p_1-1)-1)} \\
&\quad \ge A_2^{q_1} \left(d_2^{p_2/(p_2-1)} - 1\right)^{((p_2-1)q_1)/(m_2(p_2-1)-1)} a_2^{(p_2 q_1)/(m_2(p_2-1)-1)}, \\
&\left(\frac{m_2 p_2 A_2^{m_2}}{m_2(p_2-1)-1}\right)^{p_2-1} \left(d_2^{p_2/(p_2-1)} - 1\right)^{(p_2-1)/(m_2(p_2-1)-1)} \\
&\quad \times a_2^{((m_2+1)(p_2-1))/(m_2(p_2-1)-1)} \\
&\quad \ge, A_1^{q_2} \left(d_1^{p_1/(p_1-1)} - 1\right)^{((p_1-1)q_2)/(m_1(p_1-1)-1)} a_1^{(p_1 q_2)/(m_1(p_1-1)-1)}.
\end{aligned}$$

For fixed A_1, d_1, A_2 and d_2, the assumption

$$q_1 q_2 > \frac{(p_1-1)(p_2-1)(m_1+1)(m_2+1)}{p_1 p_2}$$

ensures that there exist a_1 and a_2 small enough such that the above inequalities hold.

(ii) We construct the following well known self-similar solutions (which are similar to the so called Zel'dovich-Kompaneetz-Barenblatt profile, see [**3, 5, 9**]) to (1.1)–(1.3) in the form

$$\begin{aligned} u_B(x,t) &= (\tau+t)^{-1/((p_1-1)(m_1+1))} h_1(\xi), \\ \xi &= x(\tau+t)^{-1/((p_1-1)(m_1+1))}, \\ v_B(x,t) &= (\tau+t)^{-1/((p_2-1)(m_2+1))} h_2(\zeta), \\ \zeta &= x(\tau+t)^{-1/((p_2-1)(m_2+1))}, \end{aligned} \tag{3.10}$$

where $\tau > 0$ and

$$\begin{aligned} h_1(\xi) &= C(m_1,p_1)(c_1^{p_1/(p_1-1)} - \xi^{p_1/(p_1-1)})_+^{(p_1-1)/(m_1(p_1-1)-1)}, \\ h_2(\zeta) &= C(m_2,p_2)(c_2^{p_2/(p_2-1)} - \zeta^{p_2/(p_2-1)})_+^{(p_2-1)/(m_2(p_2-1)-1)} \end{aligned} \tag{3.11}$$

with $c_1, c_2 > 0$ and
(3.12)

$$C(m_i,p_i) = \left(\frac{1}{(p_i-1)(m_i+1)} \left(\frac{m_i(p_i-1)-1}{m_i p_i} \right)^{p_i-1} \right)^{1/(m_i(p_i-1)-1)}, \quad i = 1,2.$$

It is easy to check that h_i, $i = 1,2$, satisfy

$$\begin{gathered} \left(|\,(h_1^{m_1})'\,|^{p_1-2}\,(h_1^{m_1})' \right)(\xi) + \frac{1}{(p_1-1)(m_1+1)} (\xi h_1'(\xi) + h_1(\xi)) = 0, \\ h_1'(0) = 0, \\ \left(|\,(h_2^{m_2})'\,|^{p_2-2}\,(h_2^{m_2})' \right)(\zeta) + \frac{1}{(p_2-1)(m_2+1)} (\zeta h_2'(\zeta) + h_2(\zeta)) = 0, \\ h_2'(0) = 0. \end{gathered}$$

Since $u(x,t)$ and $v(x,t)$ are nontrivial and nonnegative, we see $u(0,t_0) > 0$ and $v(0,t_0) > 0$ for some $t_0 \ge 0$ (compare with a Barenblatt solution of the corresponding equations). Noticing that $u(x,t_0), v(x,t_0) > 0$ are continuous, see [**4, 12**], there exist $\tau > 0$ large enough and $c_i > 0$, $i = 1,2$, small enough such that

$$u(x,t_0) \ge u_B(x,t_0), \quad v(x,t_0) \ge v_B(x,t_0), \quad \text{for } x > 0.$$

A direct calculation shows that (u_B, v_B) is a weak subsolution of (1.1)–(1.3) in $(0,\infty)\times(t_0,+\infty)$. By the comparison principle, we obtain that

$$u(x,t) \geq u_B(x,t), \quad v(x,t) \geq v_B(x,t), \quad \text{for } x>0, t\geq t_0.$$

Since $\max\{\alpha_1+\beta_1, \alpha_2+\beta_2\} < 0$, we see

$$T^{\beta_1} \ll T^{-\alpha_1}, \quad T^{\beta_2} \ll T^{-\alpha_2}, \quad \text{for large } T.$$

So there exists a $t^* \geq t_0$ such that

$$\begin{aligned} T^{\beta_1} &\ll (\tau+t^*)^{1/((m_1+1)(p_1-1))} \ll T^{-\alpha_1}, \\ T^{\beta_2} &\ll (\tau+t^*)^{1/((m_2+1)(p_2-1))} \ll T^{-\alpha_2}. \end{aligned} \tag{3.13}$$

Let $(\underline{u}, \underline{v})$ be defined by (2.10) in the proof of Lemma 2.1. The inequalities (3.13) imply that

$$\begin{gathered} \underline{u}(x,0) \leq u_B(x,t^*) \leq u(x,t^*), \\ \underline{v}(x,0) \leq v_B(x,t^*) \leq v(x,t^*), \\ \text{for } x>0. \end{gathered}$$

It follows from Lemma 2.1 and the comparison principle that (u,v) blows up in a finite time. Observing that (3.13) holds for general nontrivial (u_0, v_0), and we know that every nonnegative, nontrivial solution of (1.1)–(1.3) blows up in finite time.

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School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu 610054, P.R. China
Email address: zhaoyin-xiang@sohu.com

Department of Mathematics, Sichuan University, Chengdu 610064, P.R. China
Email address: chunlaimu@yahoo.com.cn

Department of Mathematics, Xihua University, Chengdu 610039, P.R. China
Email address: wangyulan-math@163.com