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ON THE GROUP OF ISOMETRIES OF THE PLANE WITH GENERALIZED ABSOLUTE VALUE METRIC

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ABSTRACT. In this work, we show that the group of isometries of the plane with generalized absolute value metric is either the semi-direct product of the group D_4 and $T(2)$ or the semi-direct product of the Dihedral group D_8 and $T(2)$ where D_4 and D_8 are the (Euclidean) symmetry groups of the square and the regular octagon, respectively; and $T(2)$ is the group of all translations of the plane.

1. Introduction. It is well known that the group of isometries of Euclidean plane $E(2)$ is the semi-direct product of its two subgroups $O(2)$ (the orthogonal group) and $T(2)$, where $O(2)$ is the symmetry group of the unit circle and $T(2)$ (the translation group) consists of all translations of the plane [**1, 3, 6**]. The group of isometries of the taxicab and CC-plane have been given in [**4, 2**], respectively.

Now, we introduce a family of distances, d_g, which include Chinese checkers' distance, taxicab distance and maximum distance as special cases.

Definition 1. Let $\mathbf{R}^2$ denote the analytical plane, let $X = (x_1, y_1)$, and let $Y = (x_2, y_2)$ be any two points in $\mathbf{R}^2$. Then, for $a, b \in \mathbf{R}$, $a \geq b \geq 0 \neq a$

$$d_g(X,Y) := a \max\{|x_1 - x_2|, |y_1 - y_2|\} + b \min\{|x_1 - x_2|, |y_1 - y_2|\}$$

is called the d_g-*distance* between X and Y.

We write $\mathbf{R}_g^2 = (\mathbf{R}^2, d_g)$ for the plane with generalized absolute value distance defined above. Clearly, there are a lot of distance functions for constants a and b. The following lemma shows that each of these d_g-distances gives a metric.

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Proposition 1. *Every d_g-distance determines a metric.*

Proof. Obviously, d_g is positive definite and symmetric. Thus, we will satisfy triangle inequality for $X = (x_1, y_1)$, $Y = (x_2, y_2)$, $Z = (x_3, y_3)$. Since

$$(1) \qquad |x_1 - x_2| = |x_1 - x_3 + x_3 - x_2| \leq |x_1 - x_3| + |x_3 - x_2|$$

and

$$(2) \qquad |y_1 - y_2| = |y_1 - y_3 + y_3 - y_2| \leq |y_1 - y_3| + |y_3 - y_2|,$$

one obtains

$$\begin{aligned} d_g(X,Y) &= a\max\{|x_1 - x_2|, |y_1 - y_2|\} \\ &\quad + b\min\{|x_1 - x_2|, |y_1 - y_2|\} \\ &= a\max\{|x_1 - x_3 + x_3 - x_2|, |y_1 - y_3 + y_3 - y_2|\} \\ &\quad + b\min\{|x_1 - x_3 + x_3 - x_2|, |y_1 - y_3 + y_3 - y_2|\} \\ &\leq a\max\{|x_1 - x_3| + |x_3 - x_2|, |y_1 - y_3| + |y_3 - y_2|\} \\ &\quad + b\min\{|x_1 - x_3| + |x_3 - x_2|, |y_1 - y_3| + |y_3 - y_2|\} = k. \end{aligned}$$

One can easily see that d_g satisfies the triangle inequality by examining the following cases:

Case I. If $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \geq |y_3 - y_2|$, then

$$\begin{aligned} d_g(X,Y) \leq k &= a(|x_1 - x_3| + |x_3 - x_2|) + b(|y_1 - y_3| + |y_3 - y_2|) \\ &= a|x_1 - x_3| + b|y_1 - y_3| + a|x_3 - x_2| + b|y_3 - y_2| \\ &= d_g(X,Z) + d_g(Z,Y). \end{aligned}$$

Case II. If $|x_1 - x_3| \leq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$, then similarly $d_g(X,Y) \leq d_g(X,Z) + d_g(Z,Y)$.

Case III. If $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$, then there are two possible situations:

i) Let $|x_1 - x_3| + |x_3 - x_2| \geq |y_1 - y_3| + |y_3 - y_2|$. Then

$$d_g(X,Y) \leq k = a(|x_1 - x_3| + |x_3 - x_2|) + b(|y_1 - y_3| + |y_3 - y_2|).$$

Using $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$,

$$\begin{aligned} d_g(X, Z) &= a\,|x_1 - x_3| + b\,|y_1 - y_3| \\ d_g(Z, Y) &= a\,|y_3 - y_2| + b\,|x_3 - x_2|\,. \end{aligned}$$

$d_g(X, Y) \leq d_g(X, Z) + d_g(Y, Z)$ if and only if

$$\begin{aligned} d_g(X, Y) \leq k &= a(|x_1 - x_3| + |x_3 - x_2|) + b(|y_1 - y_3| + |y_3 - y_2|) \\ k &\leq a(|x_1 - x_3| + |y_3 - y_2|) + b(|y_1 - y_3| + |x_3 - x_2|) \\ &\Longleftrightarrow a(|x_3 - x_2| - |y_3 - y_2|) + b(|y_3 - y_2| - |x_3 - x_2|) \leq 0 \\ &\Longleftrightarrow (a - b)(|x_3 - x_2| - |y_3 - y_2|) \leq 0 \\ &\Longleftrightarrow |x_3 - x_2| \leq |y_3 - y_2| \end{aligned}$$

which is our general assumption for this case.

ii) Let $|x_1 - x_3| + |x_3 - x_2| \leq |y_1 - y_3| + |y_3 - y_2|$.

$$d_g(X, Y) \leq k = a(|y_1 - y_3| + |y_3 - y_2|) + b(|x_1 - x_3| + |x_3 - x_2|).$$

Using $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$,

$$\begin{aligned} d_g(X, Z) &= a\,|x_1 - x_3| + b\,|y_1 - y_3| \\ d_g(Z, Y) &= a\,|y_3 - y_2| + b\,|x_3 - x_2|\,. \\ d_g(X, Y) &\leq d_g(X, Z) + d_g(Y, Z) \Leftrightarrow \end{aligned}$$

$$\begin{aligned} d_g(X, Y) \leq k &= a(|y_1 - y_3| + |y_3 - y_2|) + b(|x_1 - x_3| + |x_3 - x_2|) \\ k &\leq a(|x_1 - x_3| + |y_3 - y_2|) + b(|y_1 - y_3| + |x_3 - x_2|) \\ &\Longleftrightarrow a(|y_1 - y_3| - |x_1 - x_3|) + b(|x_1 - x_3| - |y_1 - y_3|) \leq 0 \\ &\Longleftrightarrow (a - b)(|y_1 - y_3| - |x_1 - x_3|) \leq 0 \\ &\Longleftrightarrow |y_1 - y_3| \leq |x_1 - x_3|\,, \end{aligned}$$

which is our general assumption for this case.

Case IV. If $|x_1 - x_3| \leq |y_1 - y_3|$ and $|x_3 - x_2| \geq |y_3 - y_2|$, then there are two possible situations:

i) $|x_1 - x_3| + |x_3 - x_2| \geq |y_1 - y_3| + |y_3 - y_2|$.

ii) $|x_1 - x_3| + |x_3 - x_2| \leq |y_1 - y_3| + |y_3 - y_2|$.

One can easily give a proof for Case IV as in Case III.

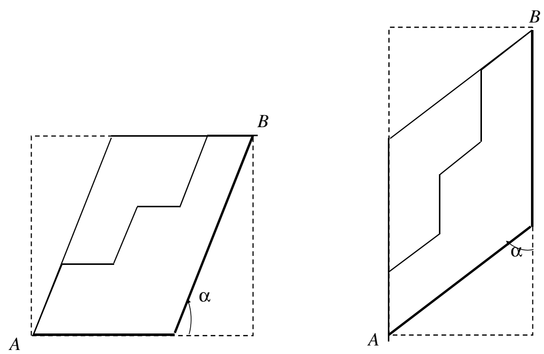

FIGURE 1. Some ways from A to B.

According to the definition of the d_g-metric, the shortest way between points A and B is the union of a vertical or a horizontal line segment and a line segment with the slope $\pm(2ab)/(a^2-b^2)$, as shown in Figure 1. Thus, the shortest distance d_g from A to B is a constant a multiple of the sum of the Euclidean lengths of two such line segments.

Generally, the d_g-distance is equal to a multiple of the Euclidean distance between A and B according to the slope of the line segment of AB. Especially if the slope of AB is 0, ∞, then the d_g-distance is equal to the constant a multiple of the Euclidean distance between A and B. If $b/a=\sqrt{2}-1$ and if the slope of the segment AB is 0, ∓ 1, ∞, then the d_g-distance is equal to the constant a multiple of the Euclidean distance between A and B. (For the sake of shortness we denote the slope of the vertical lines by ∞.)

Consequently, d_g is the constant a multiple of the taxicab metric(d_T), the maximum metric(d_L) and the Chinese checkers metric (d_c) for $a=b$, $b=0$ and $b/a=\sqrt{2}-1$, respectively.

The unit circle in $\mathbf{R}_g^2$ is the set of points (x,y) in the plane which satisfy the equation

$$a\max\{|x|,|y|\}+b\min\{|x|,|y|\}=1,$$

which is a square or an octagon with vertices $A_1(1/a,0)$, $A_2(1/(a+b),1/(a+b))$, $A_3(0,1/a)$, $A_4(-1/(a+b),1/(a+b))$, $A_5(-1/a,0)$, $A_6(-1/(a+b),-1/(a+b))$, $A_7(0,-1/a)$, $A_8(1/(a+b),-1/(a+b))$ as shown in Figure 2.

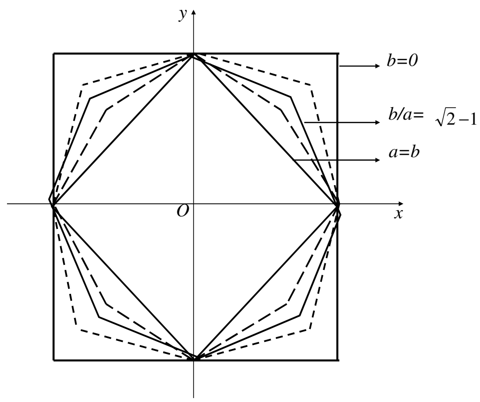

FIGURE 2. The graph of d_g-unit circle.

Points A_1, A_3, A_5 and A_7 of the d_g-unit circle lie on the Euclidean circle $x^2 + y^2 = 1/a^2$. All vertices of the d_g unit circle lie on this Euclidean circle if and only if $b/a = \sqrt{2} - 1$.

In the remaining part of this work, we will study the isometries of $\mathbf{R}_g^2$, determine its group of isometries and give some properties of $\mathbf{R}_g^2$.

2. Isometries of the plane $\mathbf{R}_g^2$. Since an isometry of a plane is defined to be a transformation which preserves the distances in the plane, an isometry of $\mathbf{R}_g^2$ is therefore an isometry of the real plane with respect to a d_g-metric.

Proposition 1. *Every Euclidean translation is an isometry of* $\mathbf{R}_g^2$.

Proof. Let $T_A : \mathbf{R}_g^2 \to \mathbf{R}_g^2$ be such that $T_A(X) = A + X$ is the translation as in the real plane $\mathbf{R}^2$, where $A = (a_1, a_2)$ and $X = (x_1, y_1) \in \mathbf{R}_g^2$. For $X = (x_1, y_1)$ and $Y = (x_2, y_2) \in \mathbf{R}_g^2$, we have

$$\begin{aligned} d_g(T_A(X), T_A(Y)) &\\ &= a \max\{|(a_1+x_1) - (a_1+x_2)|, |(a_2+y_1) - (a_2+y_2)|\} \\ &\quad + b \min\{|(a_1+x_1) - (a_1+x_2)|, |(a_2+y_1) - (a_2+y_2)|\} \\ &= a \max\{|x_1 - x_2|, |y_1 - y_2|\} + b \min\{|x_1 - x_2|, |y_1 - y_2|\} \\ &= d_g(X, Y). \end{aligned}$$

That is, T_A is an isometry. □

Since the plane geometry with a d_g-metric is the study of Euclidean points, lines and angles in $\mathbf{R}_g^2$, we use the following definition and lemma to find the reflections:

Definition. Let P and l be a point and a line in $\mathbf{R}_g^2$, and let Q denote the point on l such that PQ is perpendicular to l. If P' is a point in the opposite side of the line l with respect to P such that $d_g(P,Q) = d_g(P',Q)$, then P' is called the reflection of P.

Notice that it is enough to consider the lines passing through the origin as an axis of reflection because of Proposition 1.

The following lemma will be useful to determine reflections in $\mathbf{R}_g^2$.

Lemma 2. *Let l be the line through the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in the analytical plane, and let d_E denote the Euclidean metric. If l has slope m, then*

$$d_g(A,B) = \rho(m)\, d_E(A,B), \text{ where } \rho(m) = \begin{cases} (a + b\,|m|)/\sqrt{m^2+1} & \text{if } |m| \le 1 \\ (a\,|m| + b)/\sqrt{m^2+1} & \text{if } |m| \ge 1. \end{cases}$$

Proof. If l is parallel to the x- or y-axis, then $d_g(A,B) = a d_E(A,B)$ and $\rho(m) = a$. So $d_g(A,B) = \rho(m)\, d_E(A,B)$. If l is not parallel to the x- and y-axis, then $x_1 \ne x_2$ and $y_1 \ne y_2$, $m = (y_1 - y_2)/(x_1 - x_2)$, where m is the slope of l, and

$$\begin{aligned} d_g(A,B) &= a \max\{|x_1 - x_2|, |y_1 - y_2|\} \\ &\quad + b \min\{|x_1 - x_2|, |y_1 - y_2|\} \\ &= \begin{cases} |x_1 - x_2|\,(a + b\,|m|) & \text{if } |m| \le 1 \\ |x_1 - x_2|\,(a\,|m| + b) & \text{if } |m| \ge 1. \end{cases} \end{aligned}$$

Similarly,

$$d_E(A,B) = |x_1 - x_2|\sqrt{1+m^2}, \text{ for all } m \in \mathbf{R},$$

and consequently the given equality is valid. □

The above proposition says that d_g-distance along any line is some positive constant multiple of Euclidean distance along the same line.

Furthermore, one can immediately obtain the following:

Corollary 3. *If A, B, X are any three collinear points in $\mathbf{R}^2$, then $d_E(X, A) = d_E(X, B)$ if and only if $d_g(X, A) = d_g(X, B)$.*

Corollary 4. *If A, B and X are any three distinct collinear points in the real plane, then*

$$\frac{d_g(X, A)}{d_g(X, B)} = \frac{d_E(X, A)}{d_E(X, B)}.$$

That is, the ratios of the Euclidean and d_g-distances along a line are the same.

Notice that the latter corollary gives us the validity of the theorems of Menelaus and Ceva in $\mathbf{R}_g^2$. The following proposition determines the reflections which preserve the distance in $\mathbf{R}_g^2$.

Proposition 5. *A reflection by the line $y{=}mx$ in $\mathbf{R}_g^2$ is an isometry if and only if $m \in \{0, \pm 1, \infty\}$ when $b/a \neq \sqrt{2} - 1$ or $m \in \{0, \pm 1, \pm(\sqrt{2} - 1), \pm(\sqrt{2} + 1), \infty\}$ when $b/a = \sqrt{2} - 1$.*

Proof. Let $b/a \neq \sqrt{2} - 1$. Consider the Euclidean reflection φ by the line $y = mx$,

$$\begin{aligned} \varphi(P) &= \varphi(x, y) = P' = (x', y') \\ &= \left(\frac{(1 - m^2)x + 2my}{1 + m^2}, \frac{2mx + (-1 + m^2)y}{1 + m^2} \right). \end{aligned}$$

If $Q{=}\overline{PP'} \cap \{(x, y) : y = mx\}$, then $d_E(P, Q) = d_E(P', Q)$ implies $d_g(P, Q) = d_g(P', Q)$ by Corollary 3. That is, P' is the d_g-reflection of P. Using Proposition 1, one can say that φ is an isometry of $\mathbf{R}_g^2$ if and only if $d_g(O, P) = d_g(O, P')$. We claim that

$$d_g(O, P) = d_g(O, P') \Longleftrightarrow m \in \{0,\ \pm 1,\ \infty\}\,.$$

If $d_g(O,P) = d_g(O,P')$, then $m_1 m_2 = 1$ or $m_1 + m_2 = 0$. From the relation among m, m_1 and m_2, $m \in \{-1, 0, 1, \infty\}$ is obtained.

Conversely, if $m \in \{0,\ \pm 1,\ \infty\}$, then $d_g(O,P) = d_g(O,P')$.

i) If $m = 0$, then $P' = (x, -y)$ for $P = (x, y)$.

$$\begin{aligned} d_g(O,P) &= a\max\{|x|,|y|\} + b\min\{|x|,|y|\} \\ &= a\max\{|x|,|-y|\} + b\min\{|x|,|-y|\} \\ &= d_g(O,P'). \end{aligned}$$

ii) If $m = 1$, then $P' = (y, x)$ for $P = (x, y)$.

$$\begin{aligned} d_g(O,P) &= a\max\{|x|,|y|\} + b\min\{|x|,|y|\} \\ &= a\max\{|y|,|x|\} + b\min\{|y|,|x|\} \\ &= d_g(O,P'). \end{aligned}$$

iii) If $m = -1$, then $P' = (-y, -x)$ for $P = (x, y)$.

$$\begin{aligned} d_g(O,P) &= a\max\{|x|,|y|\} + b\min\{|x|,|y|\} \\ &= a\max\{|-y|,|-x|\} + b\min\{|-y|,|-x|\} \\ &= d_g(O,P'). \end{aligned}$$

iv) If $m \to \infty$, then $d_g(O,P) = d_g(O,P')$ from $\rho(m_1) = \rho(m_2)$.

Now, let $b/a = \sqrt{2}-1$. Then, the d_g-distance is a constant a multiple of the d_c-distance. Thus, $m \in \{0,\ \pm 1,\ \pm(\sqrt{2}-1),\ \pm(\sqrt{2}+1),\ \infty\}$ and the result is clear from [**2**].

Proposition 6. *The set of isometric rotations in* $\mathbf{R}_g^2$ *is*

$$R_g = \left\{0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}\right\} \text{ when } \frac{b}{a} \neq \sqrt{2}-1$$

or

$$R_g = \left\{0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4},\ \frac{3\pi}{2},\ \frac{7\pi}{4}\right\} \text{ when } \frac{b}{a} = \sqrt{2}-1.$$

Proof. Let $b/a \neq \sqrt{2}-1$. In order to find the isometric rotations in $\mathbf{R}_g^2$, it is sufficient to determine the rotations which preserve the lengths of the sides of the d_g-unit circle. Consider the points $A_1 = (1/a, 0)$ and $A_2 = (1/(a+b), 1/(a+b))$ on the unit circle of $\mathbf{R}_g^2$. Rotating A_1 and A_2 by an angle θ, we get

$$r_\theta(A_1) = \left(\frac{1}{a}\cos\theta, \frac{1}{a}\sin\theta\right)$$
$$r_\theta(A_2) = \frac{1}{a+b}(\cos\theta - \sin\theta, \sin\theta + \cos\theta\Big).$$

Clearly, $d_g(A_1, A_2) = (a^2+b^2)/(a(a+b))$. If r_θ preserves d_g-distance, we must look for θ which implies $d_g(r_\theta(A_1), r_\theta(A_2)) = (a^2+b^2)/(a(a+b))$. Thus,

$$\begin{aligned} d_g(r_\theta(A_1), r_\theta(A_2)) &= a\max\left\{\left|\left(\frac{1}{a} - \frac{1}{a+b}\right)\cos\theta + \frac{1}{a+b}\sin\theta\right|,\right. \\ &\qquad \left.\left|\Big(\frac{1}{a} - \frac{1}{a+b}\Big)\sin\theta - \frac{1}{a+b}\cos\theta\right|\right\} \\ &\quad + b\min\left\{\left|\left(\frac{1}{a} - \frac{1}{a+b}\right)\cos\theta + \frac{1}{a+b}\sin\theta\right|,\right. \\ &\qquad \left.\left|\left(\frac{1}{a} - \frac{1}{a+b}\right)\sin\theta - \frac{1}{a+b}\cos\theta\right|\right\} \\ &= \frac{a^2+b^2}{a(a+b)} \end{aligned}$$

Let $\alpha = (b/(a(a+b)))\cos\theta + (1/(a+b))\sin\theta$ and $\beta = (b/(a(a+b)))\sin\theta - (1/(a+b))\cos\theta$. Now, consider the following cases:

i) Let $|\alpha| \geq |\beta|$.

If $\alpha \geq 0$ and $\beta \leq 0$, then there is no θ.

If $\alpha \geq 0$ and $\beta \geq 0$, then $\sin\theta = 1$ which implies $\theta = \pi/2$.

If $\alpha \leq 0$ and $\beta \geq 0$, then there is no θ.

If $\alpha \leq 0$ and $\beta \leq 0$, then $\sin\theta = -1$ which implies $\theta = 3\pi/2$.

ii) Let $|\alpha| < |\beta|$. Similar to case (i), one gets $\theta = 0$ and $\theta = \pi$.

It follows from (i) and (ii) that $\theta \in \{0,\ \pi/2,\ \pi,\ 3\pi/2\}$.

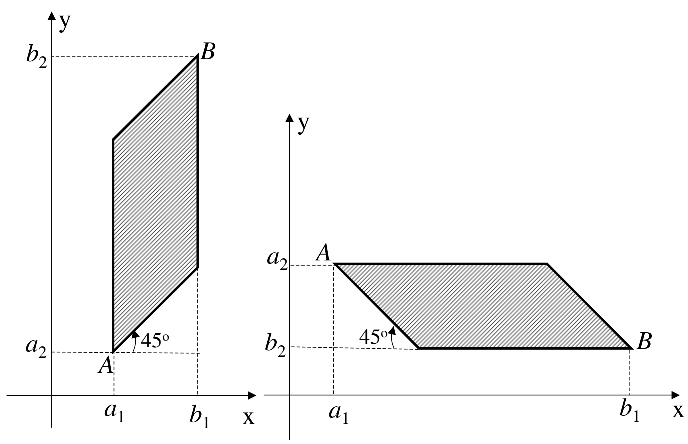

FIGURE 3.

Now, let $b/a = \sqrt{2} - 1$. In this case the d_g-distance is a constant a multiple of the d_c-distance and the required result is clear from [**2**].

Consequently, if $b/a \neq \sqrt{2} - 1$, then we have the orthogonal group $O_g(2)$, consisting of four reflections and four rotations which gives us D_4. D_4 is the Euclidean symmetry group of the square. If $b/a = \sqrt{2}-1$, we have the orthogonal group $O_g(2)$, consisting of eight reflections and eight rotations which gives us D_8. The dihedral group D_8 is the Euclidean symmetry group of the regular octagon.

Now, let us show that all isometries of $\mathbf{R}_g^2$ are in $T(2).O_g(2)$.

Definition. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$ be two points in $\mathbf{R}_g^2$. The minimum distance set of A, B is defined by

$$\{X \mid d_g(A, X) + d_g(B, X) = d_g(A, B)\}$$

and denoted by $\overset{\diamond}{AB}$ (Figure 3).

Let m_{AB} denote the slope of the line through the points A and B. If $m_{AB} = 0, \mp 1$ or the line is vertical, the set $\overset{\diamond}{AB}$ is the line segment joining A and B, that is, $\overset{\diamond}{AB} = AB$. We call $\overset{\diamond}{AB}$ the standard parallelogram with diagonal AB.

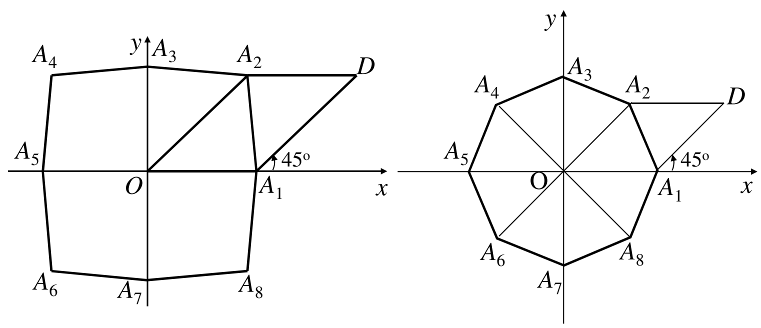

FIGURE 4.

Proposition 7. *Let $\phi : \mathbf{R}_g^2 \to \mathbf{R}_g^2$ be an isometry, and let $\overset{\diamond}{AB}$ be the standard parallelogram. Then*

$$\phi\left(\overset{\diamond}{AB}\right) = \overset{\diamond}{\phi(A)\phi(B)}.$$

Proof. Let $Y \in \phi(\overset{\diamond}{AB})$. Then,

$$\begin{aligned} Y \in \phi\left(\overset{\diamond}{AB}\right) &\iff \text{there exists } X \in \overset{\diamond}{AB} \text{ such that } Y = \phi(X) \\ &\iff d_g(A,X) + d_g(X,B) = d_g(A,B) \\ &\iff d_g(\phi(A),\phi(X)) + d_g(\phi(X),\phi(B)) = d_g(\phi(A),\phi(B)) \\ &\iff Y = \phi(X) \in \overset{\diamond}{\phi(A)\phi(B)}. \qquad \square \end{aligned}$$

Corollary 8. *Let $\phi : \mathbf{R}_g^2 \to \mathbf{R}_g^2$ be an isometry, and let $\overset{\diamond}{AB}$ be a standard parallelogram. Then ϕ maps vertices to vertices and preserves the lengths of sides of $\overset{\diamond}{AB}$.*

Proposition 9. *Let $f : \mathbf{R}_g^2 \to \mathbf{R}_g^2$ be an isometry such that $f(O) = O$. Then $f \in R_g$ or $f \in S_g$.*

Proof. Let $b/a \neq \sqrt{2}-1$. Let $A_1 = ((1/a), 0)$, $A_2 = ((1/a+b), (1/a+b))$, $D = ((2a+b/a(a+b)), (1/a+b))$, and consider the standard parallelogram $\overset{\diamond}{OD}$.

It is clear from Figure 4 that $f(A_1) \in A_iA_{i+1}$. Since f is an isometry by Proposition 7, $f(A_1)$ and $f(A_2)$ must be the vertices of the standard parallelogram $\overset{\diamond}{Of(D)}$. Also, when the slope of the parallel sides of standard parallelogram is 0 or ∞, the slope of the other parallel sides is 1 or -1. Therefore, if $f(A_1) \in A_iA_{i+1}$, then $f(A_1) = A_i$, $i = 1, 3, 5, 7$, and $f(A_2) = A_j$, $j = 2, 4, 6, 8$.

Case 1. If $f(A_1) = A_1$, then $f(A_2) = A_2$ or $f(A_2) = A_8$.

Subcase 1.1. If $f(A_2) = A_2$, then f is a rotation with $\theta = 0$.

Subcase 1.2. If $f(A_2) = A_8$, then f is a reflection by the line $y = 0$.

Case 2. If $f(A_1) = A_3$, then $f(A_2) = A_2$ or $f(A_2) = A_4$.

Subcase 2.1. If $f(A_2) = A_2$, then f is a reflection by the line $y = x$.

Subcase 2.2. If $f(A_2) = A_4$, then f is a rotation with $\theta = \pi/2$.

Case 3. If $f(A_1) = A_5$, then $f(A_2) = A_4$ or $f(A_2) = A_6$.

Subcase 3.1. If $f(A_2) = A_4$, then f is a reflection by the line $x = 0$.

Subcase 3.2. If $f(A_2) = A_6$, then f is a rotation with $\theta = \pi$.

Case 4. If $f(A_1) = A_7$, then $f(A_2) = A_6$ or $f(A_2) = A_8$.

Subcase 4.1. If $f(A_2) = A_6$, then f is a reflection by the line $y = -x$.

Subcase 4.2. If $f(A_2) = A_8$, then f is a rotation with $\theta = 3\pi/2$.

When $b/a = \sqrt{2}-1$, d_g-distance is constant a multiple of d_c-distance. Thus, the required result is clear from [**2**].

Theorem 10. *Let $f : \mathbf{R}_g^2 \to \mathbf{R}_g^2$ be an isometry. Then there exists a unique $T_A \in T(2)$ and $g \in O_g(2)$ such that $f = T_A \circ g$.*

Proof. Let $f(O) = A$ where $A = (a_1, a_2)$. Define $g = T_{-A} \circ f$. We know that g is an isometry and $g(O) = O$. Thus, $g \in O_g(2)$ and $f = T_A \circ g$ by Proposition 9. The proof of uniqueness is trivial.

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