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SOME APPLICATIONS OF CERTAIN CHARACTER SUMS

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ABSTRACT. In this paper, by using certain character sums, the mean value of the type

$$\sum_{a=1}^{p-1} a^n f^2(a)$$

is studied, and a few asymptotic formulae are obtained.

1. Introduction. The following sum

$$S_\chi(n) = \sum_{a=1}^{q} a^n \chi(a)$$

appears frequently in number theory, where χ is a nonprincipal primitive character modulo q, and has been studied by several experts. For example, for $q \equiv 3 \bmod 4$ being a prime p and χ being the Legendre symbol, Ayoub, Chowla and Walum [**1**] have proved that $S_\chi(n) < 0$ for $n = 1, 2$ and for $n \geq p-2$. Fine [**2**] has shown that, for $n > 2$, there exist infinitely many primes $p \equiv 3 \bmod 4$ with $S_\chi(n) > 0$; and infinitely many with $S_\chi(n) < 0$.

Williams [**18**] proved that

$$S_\chi(n) = O(p^{n+1/2} \log p)$$

for χ being the Legendre symbol modulo p. Toyoizumi [**4**] used the generalized Bernoulli numbers to express $S_\chi(n)$ in terms of Gaussian

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sums and Dirichlet L-functions as follows:

$$(1.1)\quad \sum_{a=1}^{q} a^n \chi(a)$$
$$= \begin{cases} 2q^n\tau(\chi)\sum_{1\le m\le n/2}(\binom{n}{2m-1}(2m-1)!L(2m,\overline{\chi})/ \\ \qquad\qquad (-1)^{m+1}(2\pi)^{2m}) & \text{if } \chi(-1)=1; \\ 2q^n\tau(\chi)\sum_{0\le m\le (n-1/2)}(\binom{n}{2m}(2m)!L(2m+1,\overline{\chi})/ \\ \qquad\qquad (-1)^{m+1}(2\pi)^{2m+1}i) & \text{if } \chi(-1)=-1, \end{cases}$$

where $\tau(\chi) = \sum_{a=1}^{q}\chi(a)\mathrm{e}(a/q)$ is the Gaussian sum, $\mathrm{e}(y) = \mathrm{e}^{2\pi i y}$, $L(s,\chi)$ is the Dirichlet L-function corresponding to χ, and $(n/m) = (n!/m!(n-m)!)$.

In this paper, by using (1.1), the mean value of the type

$$\sum_{a=1}^{p-1} a^n f^2(a)$$

is studied, and a few asymptotic formulae are obtained. First in Section 2, we give some mean value theorems for Dirichlet L-functions, which will be used later. We will study the mean value of Cochrane sums in Section 3. Next, the mean value of the difference between an integer and its inverse modulo a prime p is considered. Finally, in Section 5, an asymptotic formula for the mean value of a problem of Lehmer is obtained.

2. Some lemmas. We need the following lemmas:

Lemma 2.1. *Let $q \ge 3$ be an integer. Then we have the asymptotic formulae*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^4 = \frac{5\pi^4}{144}\phi(q)\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\exp\left(\frac{3\ln q}{\ln\ln q}\right)\right);$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2)\,|L(1,\chi)|^4 = \frac{\pi^4}{36}\phi(q)\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\exp\left(\frac{3\ln q}{\ln\ln q}\right)\right);$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(4)\,|L(1,\chi)|^4 = \frac{11\pi^4}{576}\phi(q)\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\exp\left(\frac{3\ln q}{\ln\ln q}\right)\right),$$

where $\phi(q)$ *is the Euler function,* $\prod_{p|q}$ *denotes the product over all prime divisors of* q *and* $\exp(y) = e^y$.

Proof. This is Lemma 2 of [**5**].

Lemma 2.2. *Let* $p \geq 3$ *be a prime. Then, for any integer* t *with* $(t,p)=1$*, we have*

$$\sum_{\substack{a=1 \\ p\nmid a+t}}^{p-1} e\left(\frac{r\overline{a} + s\overline{a+t}}{p}\right) \ll \sqrt{p},$$

where $\overline{a}$ *is defined by the equation* $a\overline{a} \equiv 1 \bmod p$.

Proof. See Lemma 3 of [**6**].

Lemma 2.3. *Let* $p \geq 3$ *be a prime,* $j \geq i \geq 0$ *and* $m \geq 1$ *integers. Then we have the estimate*:

$$\Psi := \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \chi_1(2^i)\tau^2(\chi_1)L^2(1,\overline{\chi}_1) \times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2\neq\chi_0}} \chi_2(2^j)\tau^2(\chi_2)L^2(1,\overline{\chi}_2)\tau(\overline{\chi}_1\overline{\chi}_2)L(2m,\chi_1\chi_2) \ll p^{7/2}\log^4 p.$$

Proof. Let $d(n) = \sum_{d|n} 1$ be the divisor function. Then, for any nonprincipal character χ modulo p and parameter $N \geq p$, applying

Abel's identity we can have

$$L^2(1,\overline{\chi}) = \sum_{n=1}^{+\infty} \frac{\overline{\chi}(n)d(n)}{n} = \sum_{1\le n\le N} \frac{\overline{\chi}(n)d(n)}{n} + \int_N^{+\infty} \frac{\sum_{N<n\le y}\overline{\chi}(n)d(n)}{y^2}\,\mathrm{d}y.$$

Since

$$\begin{aligned}\sum_{N<n\le y} \overline{\chi}(n)\,d(n) &= 2\sum_{n\le\sqrt{y}}\overline{\chi}(n)\sum_{m\le y/n}\overline{\chi}(m) \\ &\quad - 2\sum_{n\le\sqrt{N}}\overline{\chi}(n)\sum_{m\le N/n}\overline{\chi}(m) \\ &\quad - \Big(\sum_{n\le\sqrt{y}}\overline{\chi}(n)\Big)^2 + \Big(\sum_{n\le\sqrt{N}}\overline{\chi}(n)\Big)^2 \\ &\ll p\log^2 p,\end{aligned}$$

then

$$L^2(1,\overline{\chi}) = \sum_{1\le n\le N}\frac{\overline{\chi}(n)d(n)}{n} + O\Big(\frac{p\log^2 p}{N}\Big).$$

Therefore,

(2.1)

$$\begin{aligned}\Psi &= \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \chi_1(2^i)\tau^2(\chi_1)\Big(\sum_{1\le n_1\le N}\frac{\overline{\chi}_1(n_1)d(n_1)}{n_1} + O\Big(\frac{p\log^2 p}{N}\Big)\Big) \\ &\quad\times \sum_{\substack{\chi_2 \bmod p\\ \chi_2(-1)=-1\\ \chi_1\chi_2\ne\chi_0}} \chi_2(2^j)\tau^2(\chi_2)\Big(\sum_{1\le n_2\le N}\frac{\overline{\chi}_2(n_2)d(n_2)}{n_2} + O\Big(\frac{p\log^2 p}{N}\Big)\Big) \\ &\quad\times \tau(\overline{\chi}_1\overline{\chi}_2)\sum_{n_3=1}^{+\infty}\frac{\chi_1\chi_2(n_3)}{n_3^{2m}} \\ &= \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \chi_1(2^i)\tau^2(\chi_1)\sum_{1\le n_1\le N}\frac{\overline{\chi}_1(n_1)d(n_1)}{n_1}\end{aligned}$$

$$\times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2\neq\chi_0}} \chi_2(2^j)\tau^2(\chi_2) \sum_{1\le n_2\le N} \frac{\overline{\chi}_2(n_2)d(n_2)}{n_2}$$
$$\times \tau(\overline{\chi}_1\overline{\chi}_2) \sum_{n_3=1}^{+\infty} \frac{\chi_1\chi_2(n_3)}{n_3^{2m}}$$
$$+ O\left(\frac{p^{(11)/2}\log^2 p \log^2 N}{N}\right) + O\left(\frac{p^{(13)/2}\log^4 p}{N^2}\right)$$
$$:= \Omega + O\left(\frac{p^{(11)/2}\log^2 p \log^2 N}{N}\right) + O\left(\frac{p^{(13/2)}\log^4 p}{N^2}\right).$$

From the properties of Gaussian sums, we have

$$\tau^2(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right) \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b}{p}\right) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(b) e\left(\frac{a+b\overline{a}}{p}\right).$$

Then

$$\begin{aligned}\tau^2(\chi_1)\tau^2(\chi_2)\tau(\overline{\chi}_1\overline{\chi}_2) &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi_1(b) e\left(\frac{a+b\overline{a}}{p}\right) \\ &\quad \times \sum_{c=1}^{p-1}\sum_{d=1}^{p-1} \chi_2(d) e\left(\frac{c+d\overline{c}}{p}\right) \sum_{f=1}^{p-1} \overline{\chi}_1\overline{\chi}_2(f) e\left(\frac{f}{p}\right) \\ &= \sum_{f=1}^{p-1} e\left(\frac{f}{p}\right) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi_1(b) e\left(\frac{a+bf\overline{a}}{p}\right) \\ &\quad \times \sum_{c=1}^{p-1}\sum_{d=1}^{p-1} \chi_2(d) e\left(\frac{c+df\overline{c}}{p}\right).\end{aligned}$$

Noting that for $(ab,p)=1$, by the orthogonality relations for character sums modulo p we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a)\overline{\chi}(b) = \begin{cases} (p-1)/2 & \text{if } a \equiv b \mod p; \\ -(p-1)/2 & \text{if } a \equiv -b \mod p; \\ 0 & \text{otherwise.} \end{cases}$$

This gives

(2.2)

$$
\begin{aligned}
\Omega = & \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \chi_1(2^i)\tau^2(\chi_1) \sum_{1\le n_1\le N} \frac{\overline{\chi}_1(n_1)d(n_1)}{n_1} \\
& \times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2\ne\chi_0}} \chi_2(2^j)\tau^2(\chi_2) \sum_{1\le n_2\le N} \frac{\overline{\chi}_2(n_2)d(n_2)}{n_2}\tau(\overline{\chi}_1\overline{\chi}_2) \\
& \times \sum_{n_3=1}^{+\infty} \frac{\chi_1\chi_2(n_3)}{n_3^{2m}} = \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \chi_1(2^i)\tau^2(\chi_1) \\
& \times \sum_{1\le n_1\le N} \frac{\overline{\chi}_1(n_1)d(n_1)}{n_1} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1}} \chi_2(2^j)\tau^2(\chi_2) \\
& \times \sum_{1\le n_2\le N} \frac{\overline{\chi}_2(n_2)d(n_2)}{n_2}\tau(\overline{\chi}_1\overline{\chi}_2) \\
& \times \sum_{n_3=1}^{+\infty} \frac{\chi_1\chi_2(n_3)}{n_3^{2m}} + O\left(p^3\log^4 N\right) \\
= & \sum_{1\le n_1\le N} \frac{d(n_1)}{n_1} \sum_{1\le n_2\le N} \frac{d(n_2)}{n_2} \sum_{n_3=1}^{+\infty} \frac{1}{n_3^{2m}} \\
& \times \sum_{a=1}^{p-1} \mathrm{e}\left(\frac{a}{p}\right) \sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \mathrm{e}\left(\frac{c}{p}\right) \sum_{d=1}^{p-1}\sum_{f=1}^{p-1} \mathrm{e}\left(\frac{f(1+b\overline{a}+d\overline{c})}{p}\right) \\
& \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \chi_1(2^i b n_3)\overline{\chi}_1(n_1) \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1}} \chi_2(2^j d n_3)\overline{\chi}_2(n_2) \\
& + O(p^3\log^4 N) \\
= & \frac{(p-1)^2}{4} \sum_{\substack{1\le n_1\le N \\ (n_1,p)=1}} \frac{d(n_1)}{n_1} \sum_{\substack{1\le n_2\le N \\ (n_2,p)=1}} \frac{d(n_2)}{n_2}
\end{aligned}
$$

$$\times \sum_{\substack{n_3=1\\(n_3,p)=1}}^{+\infty} \frac{1}{n_3^{2m}} \sum_{a=1}^{p-1} \mathrm{e}\left(\frac{a}{p}\right) \sum_{\substack{b=1\\2^i bn_3\equiv n_1 \bmod p}}^{p-1} \sum_{c=1}^{p-1} \mathrm{e}\left(\frac{c}{p}\right)$$

$$\times \sum_{\substack{d=1\\2^j dn_3\equiv n_2 \bmod p}}^{p-1} \sum_{f=1}^{p-1} \mathrm{e}\left(\frac{f(1+b\overline{a}+d\overline{c})}{p}\right)$$

$$- \frac{(p-1)^2}{4} \sum_{\substack{1\le n_1\le N\\(n_1,p)=1}} \frac{d(n_1)}{n_1} \sum_{\substack{1\le n_2\le N\\(n_2,p)=1}} \frac{d(n_2)}{n_2} \sum_{\substack{n_3=1\\(n_3,p)=1}}^{+\infty} \frac{1}{n_3^{2m}}$$

$$\times \sum_{a=1}^{p-1} \mathrm{e}\left(\frac{a}{p}\right) \sum_{\substack{b=1\\2^i bn_3\equiv n_1 \bmod p}}^{p-1} \sum_{c=1}^{p-1} \mathrm{e}\left(\frac{c}{p}\right)$$

$$\times \sum_{\substack{d=1\\2^j dn_3\equiv -n_2 \bmod p}}^{p-1} \sum_{f=1}^{p-1} \mathrm{e}\left(\frac{f(1+b\overline{a}+d\overline{c})}{p}\right) - \frac{(p-1)^2}{4}$$

$$\times \sum_{\substack{1\le n_1\le N\\(n_1,p)=1}} \frac{d(n_1)}{n_1} \sum_{\substack{1\le n_2\le N\\(n_2,p)=1}} \frac{d(n_2)}{n_2} \sum_{\substack{n_3=1\\(n_3,p)=1}}^{+\infty} \frac{1}{n_3^{2m}}$$

$$\times \sum_{a=1}^{p-1} \mathrm{e}\left(\frac{a}{p}\right) \sum_{\substack{b=1\\2^i bn_3\equiv -n_1 \bmod p}}^{p-1} \sum_{c=1}^{p-1} \mathrm{e}\left(\frac{c}{p}\right)$$

$$\times \sum_{\substack{d=1\\2^j dn_3\equiv n_2 \bmod p}}^{p-1} \sum_{f=1}^{p-1} \mathrm{e}\left(\frac{f(1+b\overline{a}+d\overline{c})}{p}\right) + \frac{(p-1)^2}{4}$$

$$\times \sum_{\substack{1\le n_1\le N\\(n_1,p)=1}} \frac{d(n_1)}{n_1} \sum_{\substack{1\le n_2\le N\\(n_2,p)=1}} \frac{d(n_2)}{n_2} \sum_{\substack{n_3=1\\(n_3,p)=1}}^{+\infty} \frac{1}{n_3^{2m}}$$

$$\times \sum_{a=1}^{p-1} \mathrm{e}\left(\frac{a}{p}\right) \sum_{\substack{b=1\\2^i bn_3\equiv -n_1 \bmod p}}^{p-1} \sum_{c=1}^{p-1} \mathrm{e}\left(\frac{c}{p}\right)$$

$$\begin{aligned}
&\times \sum_{\substack{d=1\\ 2^j dn_3\equiv -n_2 \bmod p}}^{p-1} \sum_{f=1}^{p-1} \mathrm{e}\left(\frac{f(1+b\overline{a}+d\overline{c})}{p}\right)\\
&+O\left(p^3\log^4 N\right)\\
:=\ &\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4+O\left(p^3\log^4 N\right).
\end{aligned}$$

By the properties of reduced residue systems, we get

$$\begin{aligned}
\Upsilon_1 =\ & \frac{(p-1)^2}{4}\sum_{\substack{1\le n_1\le N\\ (n_1,p)=1}}\frac{d(n_1)}{n_1}\sum_{\substack{1\le n_2\le N\\ (n_2,p)=1}}\frac{d(n_2)}{n_2}\sum_{\substack{n_3=1\\ (n_3,p)=1}}^{+\infty}\frac{1}{n_3^{2m}}\\
&\times\sum_{a=1}^{p-1}\mathrm{e}\left(\frac{a}{p}\right)\sum_{\substack{b=1\\ 2^i bn_3\equiv n_1 \bmod p}}^{p-1}\sum_{c=1}^{p-1}\mathrm{e}\left(\frac{c}{p}\right)\\
&\times \sum_{\substack{d=1\\ 2^j dn_3\equiv n_2 \bmod p}}^{p-1} \sum_{f=1}^{p-1} \mathrm{e}\left(\frac{f(1+b\overline{a}+d\overline{c})}{p}\right)\\
=\ & \frac{(p-1)^2}{4}\sum_{\substack{1\le n_1\le N\\ (n_1,p)=1}}\frac{d(n_1)}{n_1}\sum_{\substack{1\le n_2\le N\\ (n_2,p)=1}}\frac{d(n_2)}{n_2}\sum_{\substack{n_3=1\\ (n_3,p)=1}}^{+\infty}\frac{1}{n_3^{2m}}\\
&\times\sum_{a=1}^{p-1}\mathrm{e}\left(\frac{a}{p}\right)\sum_{c=1}^{p-1}\mathrm{e}\left(\frac{c}{p}\right)\sum_{f=1}^{p-1}\mathrm{e}\left(\frac{f(1+\overline{2^i}n_1\overline{n_3a}+\overline{2^j}n_2\overline{n_3c})}{p}\right)\\
=\ & \frac{(p-1)^2p}{4}\sum_{\substack{1\le n_1\le N\\ (n_1,p)=1}}\frac{d(n_1)}{n_1}\sum_{\substack{1\le n_2\le N\\ (n_2,p)=1}}\frac{d(n_2)}{n_2}\sum_{\substack{n_3=1\\ (n_3,p)=1}}^{+\infty}\frac{1}{n_3^{2m}}\\
&\times\sum_{\substack{a=1\\ 1+\overline{2^i}n_1\overline{n_3a}+\overline{2^j}n_2\overline{n_3c}\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1}\mathrm{e}\left(\frac{a+c}{p}\right)+O\left(p^2\log^4 N\right).
\end{aligned}$$

From the congruence equation,

$$1+\overline{2^i}n_1\overline{n_3a}+\overline{2^j}n_2\overline{n_3c}\equiv 0 \bmod p$$

we have $c \equiv -n_2\overline{2^{j-i}n_1\overline{a}+2^jn_3} \bmod p$. Then, by Lemma 2.2, we get

$$\begin{aligned}\sum_{\substack{a=1\\ 1+\overline{2^i}n_1\overline{n_3}\overline{a}+\overline{2^j}n_2\overline{n_3}\overline{c}\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \mathrm{e}\left(\frac{a+c}{p}\right) &= \sum_{\substack{a=1\\ p\nmid 2^{j-i}n_1\overline{a}+2^jn_3}}^{p-1} \mathrm{e}\left(\frac{a-n_2\overline{2^{j-i}n_1\overline{a}+2^jn_3}}{p}\right)\\ &= \sum_{\substack{a=1\\ p\nmid \overline{a}+2^jn_3}}^{p-1} \mathrm{e}\left(\frac{2^{j-i}n_1a-n_2\overline{\overline{a}+2^jn_3}}{p}\right)\\ &= \sum_{\substack{a=1\\ p\nmid a+2^jn_3}}^{p-1} \mathrm{e}\left(\frac{2^{j-i}n_1\overline{a}-n_2\overline{a+2^jn_3}}{p}\right)\\ &\ll \sqrt{p}.\end{aligned}$$

Therefore,

$$\Upsilon_1 \ll p^{7/2}\log^4 N. \tag{2.3}$$

Similarly, we can deduce

$$\Upsilon_2, \Upsilon_3, \Upsilon_4 \ll p^{7/2}\log^4 N. \tag{2.4}$$

Then from (2.2)–(2.4) we can have

$$\Omega \ll p^{7/2}\log^4 N. \tag{2.5}$$

Now, taking $N = p^2$ in (2.1) and (2.5), we immediately get

$$\Psi \ll p^{7/2}\log^4 p.$$

This completes the proof of Lemma 2.3. □

3. Mean value of Cochrane sums. For a positive integer q and an arbitrary integer h, the Cochrane sums are defined by

$$c(h,q) = {\sum_{a=1}^{q}}' \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{\overline{a}h}{q}\right)\right),$$

where $\sum'_a$ denotes the summation over all a such that $(a,q)=1$, and

$$((x)) = \begin{cases} x-[x]-1/2 & \text{if } x \text{ is not an integer;} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

The second author and Yi Yuan [**16**] gave the following upper bound estimate

$$|c(h,q)| \ll \sqrt{q}d(q)\ln^2 q$$

and

$$\sum_{h=1}^{p-1} c^2(h,p) = \frac{5}{144}p^2 + O\left(p\exp\left(\frac{4\ln p}{\ln\ln p}\right)\right).$$

In [**7**], the second author found that there are some relationships between $c(h,q)$ and Kloosterman sums

$$K(m,n;q) = {\sum_{a=1}^{q}}' \mathrm{e}\left(\frac{ma+n\overline{a}}{q}\right).$$

For example, if q is a square-full number, then we have

$${\sum_{h=1}^{q}}' c(h,q)K(h,1;q) = -\frac{1}{2\pi^2}q\phi(q) + O\left(q\exp\left(\frac{3\ln q}{\ln\ln q}\right)\right).$$

For general integer $q \geq 3$, he [**8**] obtained the asymptotic formula

$$\text{(3.1)}\quad {\sum_{h=1}^{q}}' c(h,q)K(h,1;q) = -\frac{1}{2\pi^2}q\phi(q)\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right) + O\left(q^{(3/2)+\varepsilon}\right),$$

where $\prod_{p\|q}$ denotes the product over all prime divisors of q with $p|q$ and $p^2 \nmid q$, and ε is any fixed positive number. The authors [**15**] proved that the error term in (3.1) is $O(q^{1+\varepsilon})$.

In this section, we study the mean value

$$\sum_{h=1}^{p-1} h^n c^2(h,p),$$

and give an asymptotic formula.

Theorem 3.1. *Let $p \geq 3$ be a prime, n a positive integer. Then we have*

$$\sum_{h=1}^{p-1} h^n c^2(h,p) = \frac{5}{144(n+1)} p^{n+2} + O\left(p^{n+(3/2)} \log^4 p\right).$$

Proof. From [**16**, Lemma 1] we know that

$$c(h,p) = -\frac{1}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(h) \tau^2(\chi) L^2(1, \overline{\chi}).$$

Then from (1.1), Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}
\sum_{h=1}^{p-1} h^n c^2(h,p) &= \frac{1}{\pi^4(p-1)^2} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \tau^2(\chi_1) L^2(1, \overline{\chi}_1) \\
&\quad \times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1}} \tau^2(\chi_2) L^2(1, \overline{\chi}_2) \sum_{h=1}^{p-1} h^n \overline{\chi}_1 \overline{\chi}_2(h) \\
&= \frac{p^2}{\pi^4(p-1)^2} \sum_{h=1}^{p-1} h^n \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^4 \\
&\quad + \frac{2p^n}{\pi^4(p-1)^2} \sum_{1 \leq m \leq (n/2)} \frac{\binom{n}{2m-1}(2m-1)!}{(-1)^{m+1}(2\pi)^{2m}} \\
&\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \tau^2(\chi_1) L^2(1, \overline{\chi}_1) \\
&\quad \times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2 \neq \chi_0}} \tau^2(\chi_2) L^2(1, \overline{\chi}_2) \tau(\overline{\chi}_1 \overline{\chi}_2) L(2m, \chi_1\chi_2) \\
&= \frac{5}{144(n+1)} p^{n+2} + O\left(p^{n+(3/2)} \log^4 p\right).
\end{aligned}$$

This proves Theorem 3.1. □

4. An integer and its inverse modulo p. Let $q > 2$ and c be two integers with $(c, q) = 1$, and

$$M(q,k,c) = \sideset{}{'}\sum_{\substack{a=1 \\ ab\equiv c \bmod q}}^{q} \sideset{}{'}\sum_{b=1}^{q} (a-b)^{2k}.$$

In [**9**], the second author used the estimates for Kloosterman sums and trigonometric sums to obtain a sharp asymptotic formula for $M(q,k,c)$ as following:

(4.1)
$$M(q,k,c) = \frac{1}{(2k+1)(k+1)}\phi(q)q^{2k} + O\left(4^k q^{(4k+1)/2} d^2(q) \ln^2 q\right).$$

The error term in (4.1) is the best possible. In fact, for $k = 1$, let

$$M(q,1,c) = \frac{1}{6}\phi(q)q^2 + \frac{1}{3}q\prod_{p|q}(1-p) + F(q,1,c).$$

The second author [**10**] used the properties of Dedekind sums and Cochrane sums to give a sharp mean value formula for $F(q,1,c)$:

$$\sideset{}{'}\sum_{c=1}^{q} F^2(q,1,c) = \frac{5}{36}q^3\phi^3(q)\prod_{p^\alpha\|q}\frac{((p+1)^3/p(p^2+1)) - 1/(p^{3\alpha-1})}{1+(1/p)+(1/p^2)} + O\left(q^5 \exp\left(\frac{4\ln q}{\ln\ln q}\right)\right),$$

where $\prod_{p^\alpha\|q}$ denotes the product over all prime divisors of q with $p^\alpha|q$ and $p^{\alpha+1}\nmid q$.

In this section, we study the mean value

$$\sum_{c=1}^{p-1} c^n F^2(p,1,c),$$

and give an asymptotic formula. We shall prove the following:

Theorem 4.1. *Let $p \geq 3$ be a prime, n be a positive integer. Then we have*

$$\sum_{c=1}^{p-1} c^n F^2(p,1,c) = \frac{5}{36(n+1)} p^{n+6} + O\left(p^{n+(11/2)} \log^4 p\right).$$

Proof. From [**10**, Lemma 1] we know that

$$\begin{aligned} F(p,1,c) &= -\frac{2}{(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(c) \left(\sum_{a=1}^{p} a\chi(a) \right)^2 \\ &= \frac{2p^2}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(c) \tau^2(\chi) L^2(1,\overline{\chi}). \end{aligned}$$

Then from (1.1), Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} \sum_{c=1}^{p-1} c^n F^2(p,1,c) &= \frac{4p^4}{\pi^4(p-1)^2} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \tau^2(\chi_1) L^2(1,\overline{\chi}_1) \\ &\quad \times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1}} \tau^2(\chi_2) L^2(1,\overline{\chi}_2) \sum_{c=1}^{p-1} c^n \overline{\chi}_1 \overline{\chi}_2(c) \\ &= \frac{4p^6}{\pi^4(p-1)^2} \sum_{c=1}^{p-1} c^n \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^4 \\ &\quad + \frac{8p^{n+4}}{\pi^4(p-1)^2} \sum_{1 \leq m \leq (n/2)} \frac{\binom{n}{2m-1}(2m-1)!}{(-1)^{m+1}(2\pi)^{2m}} \\ &\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \tau^2(\chi_1) L^2(1,\overline{\chi}_1) \\ &\quad \times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2 \neq \chi_0}} \tau^2(\chi_2) L^2(1,\overline{\chi}_2) \tau(\overline{\chi}_1 \overline{\chi}_2) L(2m, \chi_1\chi_2) \end{aligned}$$

$$= \frac{5}{36(n+1)} p^{n+6} + O\left(p^{n+(11/2)} \log^4 p\right).$$

This proves Theorem 4.1. $\square$

5. On a problem of D.H. Lehmer. Let $q > 2$ be an odd number and c any integer with $(c,q) = 1$. For each integer a with $1 \le a \le q$ and $(a,q) = 1$, we know that there exists one and only one b with $1 \le b \le q$ such that $ab \equiv c \bmod q$. Let $N(q,c)$ be the number of cases in which a and b are of opposite parity, that is,

$$N(q,c) = \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{\substack{b=1 \\ ab\equiv c \bmod q \\ 2\nmid a+b}}^{q} 1.$$

For $c = 1$ and $q = p$ an odd prime, Lehmer [**3**] asked us to find $N(p,1)$ or at least to say something nontrivial about it. In references [**11, 12**], the second author proved that

$$N(q,1) = \frac{1}{2}\phi(q) + O\left(q^{1/2} d^2(q) \ln^2 q\right). \tag{5.1}$$

For any nonnegative integer n, let

$$N(q,1,n) = \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{\substack{b=1 \\ ab\equiv 1 \bmod q \\ 2\nmid a-b}}^{q} (a-b)^{2n}.$$

The second author [**13**] proved the following asymptotic formula:

$$N(q,1,n) = \frac{1}{(2n+1)(2n+2)}\phi(q) q^{2n} + O\left(4^n q^{2n+(1/2)} d^2(q) \ln^2 q\right).$$

For any fixed positive integer c with $(c,q) = 1$, define

$$E(q,c) = N(q,c) - \frac{1}{2}\phi(q).$$

The second author [**17**] showed that, for any odd integer $q > 2$,

$$\sum_{c=1}^{q}{}' E^2(q,c) = \frac{3}{4}\phi^2(q) \prod_{p^\alpha \| q} \frac{((p+1)^3/p(p^2+1)) - (1/p^{3\alpha-1})}{1+(1/p)+(1/p^2)} + O\left(q \exp\left(\frac{4\ln q}{\ln\ln q}\right)\right).$$

This proves the error term in (5.1) is the best possible. In [**14**], he found that there exists some close relation between the error term $E(q,c)$ and Kloosterman sums, and obtained the following hybrid mean value formula:

$$\sum_{c=1}^{q}{}' E(q,c)K(\overline{4}c,1;q) = \frac{4}{\pi^2} q\phi(q) \prod_{p\|q}\left(1 - \frac{1}{p(p-1)}\right) + O\left(q^{(3/2)+\varepsilon}\right).$$

In this section, we study the mean value

$$\sum_{c=1}^{p-1} c^n E^2(p,c),$$

and give an asymptotic formula.

Theorem 5.1. *Let $p \geq 3$ be a prime, n be a positive integer. Then we have*

$$\sum_{c=1}^{p-1} c^n E^2(p,c) = \frac{3}{4(n+1)} p^{n+2} + O\left(p^{n+(3/2)} \log^4 p\right).$$

Proof. From [**5**, Lemma 1] we know that

$$E(p,c) = \frac{2}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(c)\left(1-2\chi(2)\right)^2 \tau^2(\chi) L^2(1,\overline{\chi}).$$

Then from (1.1), Lemma 2.1 and Lemma 2.3, we have

$$\sum_{c=1}^{p-1} c^n E^2(p,c) = \frac{4}{\pi^4(p-1)^2} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left(1-2\chi_1(2)\right)^2 \tau^2(\chi_1) L^2(1,\overline{\chi}_1)$$

$$\times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1}} \left(1-2\chi_2(2)\right)^2 \tau^2(\chi_2) L^2(1,\overline{\chi}_2) \sum_{c=1}^{p-1} c^n \overline{\chi}_1 \overline{\chi}_2(c)$$

$$= \frac{4p^2}{\pi^4(p-1)^2} \sum_{c=1}^{p-1} c^n \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |1-2\chi(2)|^2 |L(1,\chi)|^4$$

$$+ \frac{8p^n}{\pi^4(p-1)^2} \sum_{1\le m\le (n/2)} \frac{\binom{n}{2m-1}(2m-1)!}{(-1)^{m+1}(2\pi)^{2m}}$$

$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \left(1-2\chi_1(2)\right)^2 \tau^2(\chi_1) L^2(1,\overline{\chi}_1)$$

$$\times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2\neq\chi_0}} \left(1-2\chi_2(2)\right)^2 \tau^2(\chi_2)$$

$$\times L^2(1,\overline{\chi}_2)\tau(\overline{\chi}_1\overline{\chi}_2) L(2m,\chi_1\chi_2)$$

$$= \frac{3}{4(n+1)} p^{n+2} + O\left(p^{n+(3/2)} \log^4 p\right).$$

This completes the proof of Theorem 5.1. □

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