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A CONVERSE THEOREM FOR HILBERT-JACOBI FORMS

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1. Introduction and statement of results. Doi and Naganuma, see [**6**], constructed a lifting map from elliptic modular forms to Hilbert modular forms in the case of a real quadratic field with narrow class number one. A converse theorem for Hilbert modular forms was one of their basic tools. This gives rise to the question of constructing a lifting map in the case of Jacobi forms. Here we do the first step in that direction and prove a converse theorem for Hilbert-Jacobi forms.

Studying the connection between functions that satisfy certain transformation laws and the functional equation of their associated L-functions has value on its own and a long history. In a celebrated paper, see [**9**], Hecke showed that the automorphy of a cusp form with respect to $\mathrm{SL}_2(\mathbf{Z})$ is equivalent to the functional equation of its associated L-functions. That only one functional equation is needed is in a way atypical and highly depends on the fact that $\mathrm{SL}_2(\mathbf{Z})$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This situation already changes if one considers cusp forms with respect to a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ which have a character. In this case the functional equation of twists is required, see [**18**].

Hecke's work has inspired an astonishing number of people and a lot of generalizations of his "converse theorem" have been made, e.g., generalizations to Hilbert modular forms as mentioned above, see [**6**], Siegel modular forms, see [**2, 10**], or Jacobi forms, see [**14, 15**]. Maass showed an analogue of Hecke's result for nonholomorphic modular forms, see [**13**]. He proved that these correspond to certain L-functions in quadratic fields. An outstanding generalization of a converse theorem for $\mathrm{GL}\,(n)$ was done by Jacquet and Langlands for $n = 2$, see [**11**],

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Jacquet, Piatetski-Shapiro and Shalika for $n = 3$, see [**12**], and Cogdell and Piatetski-Shapiro for general n, see [**5**].

In this paper, we prove a converse theorem for Hilbert-Jacobi cusp forms over a totally real number field K of degree $g := [K : \mathbf{Q}]$ with discriminant D_K and narrow class number 1. The case $g = 1$, i.e., Jacobi forms over $\mathbf{Q}$ as considered by Eichler and Zagier, see [**7**], is treated in two interesting papers by Martin, see [**14, 15**]. To describe our result, we consider functions $\phi(\tau, z)$ from $\mathbf{H}^g \times \mathbf{C}^g$ into $\mathbf{C}$ that have a Fourier expansion with certain conditions on the Fourier coefficients, see (3.4), (3.5) and (3.6). We show that ϕ is a Hilbert-Jacobi cusp form (for the definition see Section 2) if and only if certain Dirichlet series $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$, see (3.9), satisfy functional equations. More precisely, we show the following.

Theorem 1.1. *Let k be an integer, and let $m \in \mathfrak{d}_K{}^{-1}$ be the inverse different from K. A function ϕ satisfying* (3.3)*,* (3.4)*,* (3.5) *and* (3.6) *is a Hilbert-Jacobi cusp form of weight k and index m if and only if for all ν satisfying* (3.1) *and* (3.2) *and for all $r \in \mathfrak{d}_K{}^{-1}/(2m\mathcal{O}_K)$ the functions $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$, see Definition* 3.2*, have analytic continuations to the whole complex plane, are bounded in every vertical strip and satisfy the functional equations*

$$\mathcal{L}(s, \phi, r, \chi_{m,\nu}) = \frac{1}{\sqrt{D_K}}\, i^{-kg}\, \mathbf{N}(2m)^{-1/2} \times \sum_{\mu \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r)\mathcal{L}(k - s - 1/2, \phi, \mu, \chi_{m,-\nu}),$$

see Section 2 *for the definition of* $\mathbf{N}$ *and* $e_{2m}(\cdot)$*.*

We proceed as follows. In Section 2 we recall basic facts about Hilbert-Jacobi cusp forms. In particular, we show that these have a theta decomposition, see (2.3), where the involved theta series satisfy some transformation law, see Lemma 2.1. Section 3 deals with certain characters of Hecke type and the Dirichlet series needed for the converse theorem. In Section 4, we prove Theorem 1.1.

2. Basic facts about Hilbert-Jacobi cusp forms. We let K be a totally real number field of degree $g := [K : \mathbf{Q}]$ and denote by $\mathcal{O}_K$,

$\mathcal{O}_K^\times$, $\mathfrak{d}_K$, and D_K its ring of integers, units, different, and discriminant, respectively. We denote the jth embedding, $1 \le j \le g$, of an element $l \in K$ by $l^{(j)}$. An element $l \in K$ is said to be totally positive, $l > 0$, if all its embeddings into $\mathbf{R}$ are positive.

Let us now briefly recall some basic facts about Hilbert-Jacobi cusp forms, see also [**16**]. We put $\Gamma_K := \mathrm{SL}_2(\mathcal{O}_K)$. Let the Hilbert-Jacobi group be defined as the set $\Gamma_K^J := \Gamma_K \ltimes (\mathcal{O}_K \times \mathcal{O}_K)$, with the group multiplication

$$\gamma_1 \cdot \gamma_2 := \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, (\lambda_1, \mu_1) \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + (\lambda_2, \mu_2) \right),$$

where we put

$$\gamma_i := \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, (\lambda_i, \mu_i) \right) \in \Gamma_K^J, \quad \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma_K,$$

and $(\lambda_i, \mu_i) \in \mathcal{O}_K \times \mathcal{O}_K$. The Hilbert-Jacobi group is generated by the following three types of elements

$$\left(\begin{pmatrix} \varepsilon & \lambda \\ 0 & \varepsilon^{-1} \end{pmatrix}, (0,0) \right), \tag{2.1}$$
$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (0,0) \right),$$

and

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (\lambda, \mu) \right),$$

where $\lambda, \mu \in \mathcal{O}_K$ and $\varepsilon \in \mathcal{O}_K^\times$, see [**1, 4, 17**].

The Hilbert-Jacobi group acts on $\mathbf{H}^g \times \mathbf{C}^g$ ($\mathbf{H}$ is the usual upper half-plane) by

$$\begin{aligned} &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau, z) \\ &\qquad := \left(\left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{a^{(g)}\tau_g + b^{(g)}}{c^{(g)}\tau_g + d^{(g)}} \right), \right. \\ &\qquad\qquad \left. \left(\frac{z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{z_g + \lambda^{(g)}\tau_g + \mu^{(g)}}{c^{(g)}\tau_g + d^{(g)}} \right) \right), \end{aligned}$$

where

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma_K^J,$$

$\tau = (\tau_1, \dots, \tau_g) \in \mathbf{H}^g$ and $z = (z_1, \dots, z_g) \in \mathbf{C}^g$. Throughout this paper, we write $\tau = u + iv$, $z = x + iy$, $\tau_j = u_j + iv_j$ and $z_j = x_j + iy_j$, $1 \leq j \leq g$.

Let $k \in \mathbf{N}$, $m \in \mathfrak{d}_K{}^{-1}$ totally positive,

$$\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma_K^J,$$

and a function $\phi : \mathbf{H}^g \times \mathbf{C}^g \to \mathbf{C}$. Then we define

$$\begin{aligned} \phi|_{k,m}\gamma(\tau, z) := \mathbf{N}(c\tau + d)^{-k} & \\ & \cdot e\left(-\left(\frac{cm(z + \lambda\tau + \mu)^2}{c\tau + d} + m\tau\lambda^2 + 2m\lambda z\right)\right) \\ & \cdot \phi(\gamma \circ (\tau, z)), \end{aligned}$$

where for $\alpha \in K$ and for $z \in \mathbf{C}^g$, we define $\mathbf{N}(\alpha z) := \prod_{j=1}^g (\alpha^{(j)} z_j)$, $\mathrm{tr}\,(az) := \sum_{j=1}^g a^{(j)} z_j$ and $e(\alpha z) := e^{2\pi i \mathrm{tr}\,(\alpha z)}$.

A holomorphic function $\phi : \mathbf{H}^g \times \mathbf{C}^g \to \mathbf{C}$ is called a *Hilbert-Jacobi cusp form* of weight k and index m if $\phi|_{k,m}\gamma(\tau, z) = \phi(\tau, z)$ for all $\gamma \in \Gamma_K^J$ and if it has a Fourier expansion of the form

$$\sum_{\substack{n, r \in \mathfrak{d}_K{}^{-1} \\ 4nm - r^2 > 0}} c(n, r)\, e\,(n\tau + rz)\,.$$

In [**16**], m is chosen to be in $\mathcal{O}_K$, but our choice $m \in \mathfrak{d}_K{}^{-1}$ seems more natural since in this way the coefficients of Hilbert-Siegel modular forms are examples for Jacobi forms as in the classical case.

If ϕ is a Hilbert-Jacobi cusp form, then the transformation $(\tau, z) \to (\tau, z + \lambda\tau + \mu)$ leads to

$$c(n, r) = c(n + \lambda r + \lambda^2 m, r + 2\lambda m), \quad \text{for all} \quad \lambda \in \mathcal{O}_K. \tag{2.2}$$

From this, we can deduce that

$$\phi(\tau, z) = \sum_{r \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} f_r(\tau)\, \vartheta_{m,r}(\tau, z), \tag{2.3}$$

where, for $r \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)$, we define

$$f_r(\tau) := \sum_{\substack{n \in \mathfrak{d}_K{}^{-1} \\ 4nm - r^2 > 0}} c(n,r)\, e_{4m}\left(\left(4nm - r^2\right)\tau\right), \tag{2.4}$$

$$\vartheta_{m,r}(\tau, z) := \sum_{\lambda \in \mathcal{O}_K} e_{4m}\left((r + 2\lambda m)^2\,\tau + 4m\,(r + 2\lambda m)\,z\right), \tag{2.5}$$

and where, for $\alpha, \beta \in K$, $\beta \neq 0$ and $z \in \mathcal{C}^g$, we define $e_\beta(\alpha z) := e(\beta^{-1}\alpha z)$.

The theta series $\vartheta_{m,r}$ satisfy the following transformation law.

Lemma 2.1. *If $m \in \mathfrak{d}_K{}^{-1}$ is totally positive and $\mu \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)$, then we have*

$$\vartheta_{m,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{1}{\sqrt{D_K}}\mathbf{N}\left((\tau/i)^{1/2}\right) \cdot \mathbf{N}\,(2m)^{-1/2} \cdot e\left(\frac{m \cdot z^2}{\tau}\right) \sum_{r \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r)\vartheta_{m,r}(\tau, z),$$

where we put $(\tau/i)^{1/2} := ((\tau_1/i)^{1/2}, \ldots, (\tau_g/i)^{1/2})$, and we take the principal value of the square root, namely, $-\pi/2 < \arg(w) \leq \pi/2$ for $w \in \mathbf{C}$.

From Lemma 2.1, we obtain

Corollary 2.2. *A function $\phi : \mathbf{H}^g \times \mathbf{C}^g$ having a decomposition of the form (2.3) satisfies*

$$\phi\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \mathbf{N}(\tau)^k e\left(\frac{mz^2}{\tau}\right)\phi(\tau, z)$$

if and only if

$$f_r(\tau) = \frac{1}{\sqrt{D_K}} i^{-kg}\,\mathbf{N}\left((\tau/i)^{1/2-k}\right)\mathbf{N}(2m)^{-1/2} \times \sum_{\mu \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) f_\mu\left(-\frac{1}{\tau}\right), \tag{2.6}$$

for all $r \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)$. *In particular, if* ϕ *is a Hilbert-Jacobi cusp form, then* ϕ *satisfies* (2.6).

Exactly as in the case of elliptic modular forms, one can show:

Lemma 2.3. *Assume that* ϕ *is a Hilbert-Jacobi cusp form, with* f_r *defined as in* (2.4). *Let* c_1 *be a positive real number, and let* S *be the subset of* $\mathbf{H}^g$ *such that for all* $\tau \in S$ *the components* v_j, $1 \le j \le g$, *are larger than* c_1. *Then we have*

$$|f_r(\tau)| \ll_{\phi,c_1} e^{-c_2\left(\sum_{j=1}^{g} v_j\right)}, \tag{2.7}$$

where c_2 *is a positive constant, and where the constant implied in* $\ll_{\phi,c_1}$ *depends on* ϕ *and on* c_1.

Lemma 2.4. *If* ϕ *is a Hilbert-Jacobi cusp form of weight* k *and index* m, *then the function*

$$g(\tau, z) := \mathbf{N}(v)^{k/2} \exp\left(-2\pi \operatorname{tr}\left(\frac{my^2}{v}\right)\right) \phi(\tau, z)$$

is bounded on $\mathbf{H}^g \times \mathbf{C}^g$.

By using Lemma 2.4, we have the following.

Lemma 2.5. *If* ϕ *is a Hilbert-Jacobi cusp form of weight* k *and index* m *with Fourier coefficients* $c(n,r)$, *then* $|c(n,r)| \ll_\phi \mathbf{N}(4mn - r^2)^{k/2}$.

3. Hecke-type characters and Dirichlet series. For the remainder we assume that k is an integer. For $m \in \mathfrak{d}_K{}^{-1}$, we let T_m be the subgroup of $\mathcal{O}_K^\times$ defined by

$$T_m := \left\{\varepsilon \in \mathcal{O}_K^\times \mid \varepsilon - 1 \in 2m\mathfrak{d}_K\right\}.$$

We have that $\varepsilon \in \mathcal{O}_K^\times$ is in T_m if and only if $\varepsilon r - r \in 2m\mathcal{O}_K$ for every $r \in \mathfrak{d}_K{}^{-1}/(2m\mathcal{O}_K)$.

We let $u_1, \dots, u_{g-1}$ be a basis of T_m^2, where $T_m^2 := \{\varepsilon^2 \mid \varepsilon \in T_m\}$. We take $\varepsilon_1, \dots, \varepsilon_{g-1} \in T_m$ which satisfy $\varepsilon_l^2 = u_l$ for $l = 1, \dots, g-1$. If m is not a generator of the different inverse, then T_m does not contain -1; hence, the ε_l are uniquely determined. If m is a generator of the different inverse, then T_m contains -1, and we choose $\varepsilon_l > 0$ as a solution of the above equation.

For integers N_l, $1 \leq l \leq g-1$, we choose pure imaginary solutions $\nu_1, \dots, \nu_g$ which satisfy the following equations

$$\sum_{j=1}^{g} \nu_j = 0, \tag{3.1}$$

$$\sum_{j=1}^{g} \nu_j \log\left(u_l^{(j)}\right) = 2\pi i\left(N_l + \frac{1}{2}\delta_l\right), \tag{3.2}$$

where we put $\delta_l = 0$ or 1 if $\mathbf{N}(\varepsilon_l)^k = 1$ or -1, respectively. For any integers N_l, $l = 1, \dots, g-1$, we have a solution to (3.1) and (3.2), because

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ \log(u_1^{(1)}) & \cdots & \log(u_1^{(g)}) \\ \vdots & \cdots & \vdots \\ \log(u_{g-1}^{(1)}) & \cdots & \log(u_{g-1}^{(g)}) \end{pmatrix} = (-1)^{g+1} g \cdot \det((\log(u_l^{(j)}))_{l,j=1,\dots,g-1}) \neq 0,$$

where the last inequality can be obtained from the fact that basis elements u_l are multiplicatively independent.

For $x \in K$ and $\nu := (\nu_1, \dots, \nu_g)$ satisfying (3.1) and (3.2), we set

$$\chi_{m,\nu}(x) := \prod_{j=1}^{g} \left|x^{(j)}\right|^{\nu_j}.$$

To define the Dirichlet series needed, we consider functions $\phi(\tau, z)$ from $\mathbf{H}^g \times \mathbf{C}^g$ into $\mathbf{C}$ that have a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathfrak{d}_K^{-1} \\ 4nm - r^2 > 0}} c(n, r) e(n\tau + rz) \tag{3.3}$$

that is absolutely and locally uniformly convergent. We regard $c(n,r) = 0$ unless $4nm - r^2 > 0$ or unless $n, r \in \mathfrak{d}_K{}^{-1}$. Moreover, we demand that its Fourier coefficients satisfy

$$c(n,r) = c(n + \lambda r + \lambda^2 m, r + 2\lambda m) \quad \text{for all } \lambda \in \mathcal{O}_K, \tag{3.4}$$

$$c(\varepsilon^2 n, \varepsilon r) = \mathbf{N}(\varepsilon)^k c(n,r), \quad \text{for all } \varepsilon \in \mathcal{O}_K^\times), \tag{3.5}$$

$$c(n,r) \ll_\phi \mathbf{N}(4nm - r^2)^M \tag{3.6}$$

for an integer M.

Lemma 3.1. (1) *Condition* (3.4) *implies that we can decompose* $\phi(\tau, z)$ *as in* (2.3).

(2) *Conditions* (3.4) *and* (3.5) *imply by the definition of* T_m *that*

$$c_r(N) := c\left(\frac{N + r^2}{4m}, r\right), \quad N \in \mathfrak{d}_K{}^{-2},$$

is well defined on $r \in \mathfrak{d}_K{}^{-1}/(2m\mathcal{O}_K)$, *where we put* $\mathfrak{d}_K{}^{-2} := \mathfrak{d}_K{}^{-1} \cdot \mathfrak{d}_K{}^{-1}$.

(3) ϕ *is a Hilbert-Jacobi cusp form if and only if* (3.3), (3.4), (3.5) *and* (3.6) *hold, and if* ϕ *satisfies the transformation law*

$$\phi\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \mathbf{N}(\tau)^k e\left(\frac{mz^2}{\tau}\right)\phi(\tau, z). \tag{3.7}$$

(4) *From Corollary* 2.2, *we see that a function* ϕ *satisfying* (3.3), (3.4), (3.5) *and* (3.6) *is a Hilbert-Jacobi cusp form if and only if* (2.6) *is satisfied for all* $r \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)$.

Proof. These are straightforward. We omit the proof of this Lemma. □

Let us now define the Dirichlet series needed for Theorem 1.1.

Definition 3.2. For a function ϕ satisfying (3.3), (3.4), (3.5) and (3.6), $r \in \mathfrak{d}_K{}^{-1}/(2m\mathcal{O}_K)$, and ν satisfying (3.1) and (3.2), we define

$$L(s,\phi,r,\chi_{m,\nu}) := \sum_{\alpha \in \mathfrak{d}_K{}^{-2}/T_m^2} \chi_{m,\nu}(\alpha) \cdot c_r(\alpha) \cdot \mathbf{N}(\alpha)^{-s}, \tag{3.8}$$

$$\begin{aligned}\mathcal{L}(s,\phi,r,\chi_{m,\nu}) := 2^{gs}\pi^{-gs} &\prod_{j=1}^{g} \Gamma(s-\nu_j)\mathbf{N}(m)^s \\ &\times \prod_{j=1}^{g} \left(m^{(j)}\right)^{-\nu_j} L(s,\phi,r,\chi_{m,\nu}).\end{aligned} \tag{3.9}$$

Due to (3.6), the series $L(s,\phi,r,\chi_{m,\nu})$ is absolutely convergent for $\sigma = \mathrm{Re}\,(s) > M+1$.

We have the following lemma.

Lemma 3.3. *For $\sigma > M+1$, we have the identity*

$$\mathcal{L}(s,\phi,r,\chi_{m,\nu}) = \int_{T_m^2 \backslash \mathbf{R}_+^g} f_r(iy)\mathbf{N}(y^{s-\nu})\frac{dy}{\mathbf{N}(y)}, \tag{3.10}$$

where $f_r(\tau)$ is the form defined in (2.4) *with Fourier coefficients $c(n,r)$.*

Proof. This can be directly calculated by using the Fourier expansion of $f_r(iy)$ and by using the relation $\mathbf{N}(u_l^\nu) = \mathbf{N}(\varepsilon_l)^k$ for $l = 1,\dots,g-1$, where u_l and ε_l are defined in the beginning of this section. We leave the details to the reader, see also [**3**, page 87]. □

4. Proof of Theorem 1.1. Theorem 1.1 follows directly from the two lemmas proven in this section.

Lemma 4.1. *If ϕ is a Hilbert-Jacobi cusp form, $r \in \mathfrak{d}_K{}^{-1}/(2m\mathcal{O}_K)$ and ν satisfies* (3.1) *and* (3.2), *then the functions $\mathcal{L}(s,\phi,r,\chi_{m,\nu})$ have* *analytic continuations to the whole complex plane. They are of rapid decay and satisfy the functional equations*

(4.1)

$$\mathcal{L}(s,\phi,r,\chi_{m,\nu}) = \frac{1}{\sqrt{D_K}}\, i^{-kg}\,\mathbf{N}(2m)^{-1/2} \times \sum_{\mu\in(\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r)\mathcal{L}(k-s-1/2,\phi,\mu,\chi_{m,-\nu}).$$

Proof. To prove the analytic continuation of $\mathcal{L}(s,\phi,r,\chi_{m,\nu})$, we show that the righthand side of (3.10) is analytic for all s. For this, we separate the integral into a part with $\mathbf{N}(y) \geq 1$ and a part with $\mathbf{N}(y) \leq 1$. Using the transformation law of f_r, one can see that it is enough to consider the part with $\mathbf{N}(y) \geq 1$. To estimate this, we use the variables $y_0 \in \mathbf{R}_+$ and $t = (t_1, \dots, t_{g-1}) \in \mathbf{R}^{g-1}$, where

$$y_j := y_0 \cdot e^{\sum_{l=1}^{g-1} t_l \log\left(u_l^{(j)}\right)}.$$

Then a fundamental domain of $T_m^2\backslash\mathbf{R}_+^g$ is given by the inequalities $y_0 > 0$ and $0 \leq t_l < 1$, $l = 1, \dots, g-1$, and the part with $\mathbf{N}(y) \geq 1$ is given by $y_0 \geq 1$. The analyticity now follows if we use Lemma 2.3, since for $c > 0$ and $\sigma \in \mathbf{R}$ arbitrary the integral $\int_1^\infty e^{-cy}y^\sigma\,dy$ is convergent. The boundedness of $\mathcal{L}(s,\phi,r,\chi_{m,\nu})$ in every vertical strip also follows from this convergence.

Moreover, by using the transformation law of f_r and Lemma 3.3, equation (4.1) follows since $1/y$ runs through $T_m^2 \setminus \mathbf{R}_+^g$ if y does. □

Lemma 4.2. *Assume that ϕ is a function satisfying* (3.3), (3.4), (3.5) *and* (3.6), *and that for all $r \in \mathfrak{d}_K{}^{-1}/(2m\mathcal{O}_K)$ and for all ν satisfying* (3.1) *and* (3.2) *the series $\mathcal{L}(s,\phi,r,\chi_m)$ have analytic continuations, satisfy* (4.1) *and are of rapid decay. Then ϕ is a Hilbert-Jacobi cusp form of weight k and of index m.*

Proof. By analytic continuation it is enough to show (2.2) for $\tau = iy$. We parametrize the integrals as before and use the Mellin inversion formula to get, for σ sufficiently large

(4.2)

$$\int_{[0,1]^{g-1}} f_r(iy_0{\cdot}e^{tR})e^{-\nu tR}\,dt = \frac{1}{2gR_m\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}(s/g,\phi,r,\chi_{m,\nu})y_0^{-s}\,ds,$$

where

$$R_m := \det((\log(u_l^{(j)}))_{l,j=1,\dots,g-1})$$

$$f_r\left(iy_0 \cdot e^{tR}\right) := f_r\left(iy_0 e^{\sum_{l=1}^{g-1} t_l \log\left(u_l^{(1)}\right)}, \cdots, iy_0 e^{\sum_{l=1}^{g-1} t_l \log\left(u_l^{(g)}\right)}\right),$$

$$e^{-\nu tR} := \prod_{j=1}^{g} \prod_{l=1}^{g-1} e^{-\nu_j t_l \log\left(u_l^{(j)}\right)} = \prod_{l=1}^{g-1} e^{-2\pi i (N_l + (1/2)\delta_l) t_l}, \tag{4.3}$$

where N_l and δ_l appeared in (3.2). Applying (4.1) and making the substitution $s \to g(k-1/2-s)$ gives that the righthand side of (4.2) equals

$$\frac{1}{\sqrt{D_K}} i^{-kg}\, \mathbf{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \times \frac{1}{2gR_m\pi i} \int_{g(k-1/2)-\sigma-i\infty}^{g(k-1/2)-\sigma+i\infty} \mathcal{L}(s/g, \phi, \mu, \chi_{m,\nu}) y_0^s \, ds.$$

If $\mathrm{Re}\,(s) > M+1$, the series $L(s, \phi, r, \chi_{m,\nu})$ is absolutely convergent, and the series $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$ is of rapid decay for $|\mathrm{Im}\,(s)| \to \infty$. Also, $\mathcal{L}(s, \phi, r, \chi_{m,-\nu})$ is bounded in every vertical strip and has a functional equation. By using the Phragmén-Lindelöf principle, we can conclude that $\mathcal{L}(s, \phi, r, \chi_{m,-\nu})$ is of uniformly rapid decay for $|\mathrm{Im}\,(s)| \to \infty$ in every vertical strip. Hence, we use Cauchy's theorem and shift the path of integration to the line $\mathrm{Re}\,(s) = \sigma$. Thus, the lefthand side of (4.2) equals

$$\frac{1}{\sqrt{D_K}} i^{-kg}\, \mathbf{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \times \frac{1}{2gR_m\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}(s/g, \phi, \mu, \chi_{m,-\nu}) y_0^s \, ds.$$

But the latter integral equals

$$2gR_m\pi i \int_{[0,1]^{g-1}} f_\mu(iy_0^{-1} \cdot e^{-tR}) e^{-\nu tR} \, dt.$$

Thus,

$$
\begin{aligned}
(4.4)\quad & \int_{[0,1]^{g-1}} f_r(iy_0 \cdot e^{tR}) e^{-\nu tR}\, dt \\
&= \frac{1}{\sqrt{D_K}} i^{-kg}\, \mathbf{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \\
&\qquad \times \int_{[0,1]^{g-1}} f_\mu(iy_0^{-1} \cdot e^{-tR}) e^{-\nu tR}\, dt.
\end{aligned}
$$

We now let

$$
\begin{aligned}
g_r(t) := f_r(iy_0 \cdot e^{tR}) &- \frac{1}{\sqrt{D_K}} i^{-kg}\, \mathbf{N}(2m)^{-1/2} \\
&\times \sum_{\mu \in (\mathfrak{d}_K{}^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} f_\mu(iy_0^{-1} \cdot e^{-tR}).
\end{aligned}
$$

To prove the lemma, it suffices to show that $g_r(t)$ is identically zero. But this follows since the function

$$\hat{g}_r(t) := g_r(t) \prod_{l=1}^{g-1} e^{-\pi i\, \delta_l\, t_l}$$

has period 1 in every component of t and all $(N_1, \ldots, N_{g-1})$th Fourier coefficients of $\hat{g}_r(t)$ are 0 due to (4.3) and (4.4). □

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