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ITERATIVE APPROACHES TO COMMON FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we design an iterative algorithm that converges strongly to common fixed points of a sequence of asymptotically nonexpansive mappings in a reflexive Banach space with weakly continuous duality mapping. Our results show that the uniform smoothness requirement imposed on the space in the results of [**9, 11, 29**] is not required.

1. Introduction. Let D be a nonempty closed convex subset of a Banach space X, and let $T : D \to D$ be a mapping. Given an $x_0 \in D$ and a $t \in (0, 1)$, then for a nonexpansive mapping T we can define the contraction $G_t : D \to D$ by $G_t x = tTx + (1-t)x_0$, $x \in D$. By the Banach contraction principle, G_t has a unique fixed point x_t in D, i.e., we have

$$x_t = tTx_t + (1-t)x_0. \tag{1.1}$$

The strong convergence of the path $\{x_t\}$ as $t \to 1$ for nonexpansive mapping T on a bounded D was proved in a Hilbert space independently by Browder [**1**] and Halpern [**8**] in 1967 and in a uniformly smooth Banach space by Reich [**20**]. More recently, it has been studied in various papers, see e.g., [**12, 26, 27, 31**]. Halpern [**8**] and Reich [**20**] studied strong convergence of approximants defined by (1.1) to prove strong convergence of the iteration

$$y_{n+1} = \alpha_n x + (1-\alpha_n)Ty_n, \quad n \in \mathbf{N}, \tag{1.2}$$

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where x and y_1 are elements of D and $\{\alpha_n\}$ is a real sequence such that $0 \leq \alpha_n \leq 1$ and $\alpha_n \to 0$. They obtained partial results and posed problems for the convergence of the sequence defined by (1.2). In 1991, Wittmann [**29**] proved the following result in a Hilbert space:

Theorem 1.1. *Let D be a nonempty closed convex subset of a Hilbert space H, let T be a nonexpansive mapping from D into itself such that $F(T)$, the set of fixed points of T, is nonempty, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ with*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{y_n\}$ defined by (1.2) converges strongly to Px, where P is the metric projection from D onto $F(T)$.

Very recently, Hara, Pillay and Xu [**9**] extended Theorem 1.1 to a finite family of nonexpansive mappings in the same space setting. They proved the following theorem:

Theorem 1.2. *Let D be a nonempty closed convex subset of a Hilbert space H, and let $\{T_n\}$ be a sequence of nonexpansive self-mappings on D such that $\mathcal{F} =: \cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Assume that $V_1, V_2, \dots, V_N : D \to D$ are nonexpansive mappings with the property: for any bounded set $\widetilde{D}$ of D, the following holds*

$$\lim_{n\to\infty} \Big(\sup_{x\in\widetilde{D}} \|T_n x - V_i(T_n x)\| \Big) = 0 \quad \text{for all } i = 1, 2, \dots, N. \tag{1.3}$$

Given $u, x_1 \in D$, define the sequence $\{x_n\}$ in D by

$$x_{n+1} := \alpha_n u + (1-\alpha_n) T_n x_n, \quad n \in \mathbf{N}. \tag{1.4}$$

Then $\{x_n\}$ converges strongly to $P_{F(V)} u \in \cap_{i=1}^{N} F(V_i)$, where $P_{F(V)}$ is the metric projection from H onto $F(V) := \cap_{i=1}^{N} F(V_i)$.

The asymptotically nonexpansive mappings were introduced by Goebel and Kirk [**4**] as a natural generalization of nonexpansive mappings. The strong convergence of the Mann and Ishikawa iteration processes to fixed points of asymptotically nonexpansive mappings has been studied by Huang [**10**], Liu and Kang [**16**], Rhoades [**21**] and Schu [**22, 23**] under the assumption of complete continuity. In [**18**], Osilike and Aniagbosor proved that the theorems of Schu [**23**] and Rhoades [**21**] remain true without the boundedness condition imposed on the domain, provided that the set of fixed points of the operator is nonempty. In fact, Osilike and Aniagbosor proved the following:

Theorem 1.3. *Let D be a nonempty convex subset of a uniformly convex Banach space X, and let $T : D \to D$ be an asymptotically nonexpansive mapping with $F(T) \neq \varnothing$ and sequence of Lipschitz-constants for the iterates, $\{T^n\}$, of T, $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbf{N}$. Let $\{x_n\}$ be the sequence in D generated from arbitrary $x_1 \in D$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \in \mathbf{N}. \tag{1.5}$$

Then $x_n - Tx_n \to 0$ as $n \to \infty$.

In [**24**], Schu considered the strong convergence of almost fixed points $x_n = \mu_n T^n x_n$ of an asymptotically nonexpansive mapping T in a smooth and reflexive Banach space having a weakly sequentially continuous duality mapping. He also studied the convergence of Halpern's iteration process for asymptotically nonexpansive mappings under the admissibility property, cf. [**24**]. To state Schu's result, we need the following definition:

Definition 1.1. Let $\{\varepsilon_n\}$ be a sequence in $(0,\infty)$, and let $\{\mu_n\}$ be another sequence in $(0,1)$. Then $(\{\varepsilon_n\}, \{\mu_n\})$ is called *admissible* [**24**] if

(1) $\{\varepsilon_n\}$ is decreasing;

(2) $\{\mu_n\}$ is strictly increasing with $\lim_{n\to\infty} \mu_n = 1$;

(3) there exists a sequence $\{\beta_n\}$ of natural numbers such that

 (i) $\{\beta_n\}$ is increasing,

 (ii) $\lim_{n\to\infty} \beta_n(1-\mu_n) = \infty$,

(iii) $\lim_{n\to\infty}((1-\mu_{n+\beta_n})/(1-\mu_n)) = 1$, and

(iv) $\lim_{n\to\infty}((\varepsilon_n\beta_n\mu_{n+\beta_n})/(1-\mu_n)) = 0$.

Schu proved the following:

Theorem 1.4 [**24**]. *Let X be a smooth reflexive Banach space possessing a duality mapping $J : X \to X^*$ that is weakly sequentially continuous at* 0; *C a nonempty closed, bounded, and star-shaped with respect to zero*; *$T : C \to C$ an asymptotically nonexpansive with sequence of Lipschitz-constants for the iterates, $\{T^n\}$, of T, $\{k_n\} \subset [1,\infty)$, and $(I-T)$ demi-closed*; *$\lambda_n \in ((1/2), 1)$; $z_0 \in C$; $z_{n+1} := \lambda_{n+1}/k_{n+1}T^n z_n$* for all $n \in \mathbf{N}$;

(a) $\lim_{n\to\infty}\lambda_n = 1$; *$\mu_n := \lambda_n/k_n$ for all $n \in \mathbf{N}$*; *$k_n \le \lambda_n^2/(2\lambda_n - 1)$ for all $n \in \mathbf{N}$*; *$\{(1-\mu_n)/(1-\lambda_n)\}$ bounded*; *$\varepsilon_n \in (0,\infty)$ for each $n \in \mathbf{N}$ such that $(\{\varepsilon_n\}, \{\mu_n\})$ is admissible*;

(b) $||T^n x - T^{n+1}x|| \le \varepsilon_n$ *for all $n \in \mathbf{N}$ and all $x \in C$.*

Then $\{z_n\}$ converges strongly to some fixed point of T.

In Theorem 1.4, the strong convergence of the almost fixed points $x_n = \mu_n T^n x_n$ is applied to the convergence of the iteration process:

$$z_{n+1} = (1-\mu_{n+1})u + \mu_{n+1}T^n z_n, \tag{1.6}$$

where u is an arbitrary element in C, i.e., the iteration parameter μ_n is also the parameter in the almost fixed point equation $x_n = \mu_n T^n x_n$. Moreover, choices of μ_n and λ_n are not simple.

The purpose of this paper is to develop an iterative algorithm to find a solution of the problem:

$$\text{find } x \in D \text{ such that } x \in \bigcap_{n=1}^{\infty} F(T_n), \tag{1.7}$$

for a sequence of asymptotically nonexpansive self-mappings $\{T_n\}$ on D in a Banach space without complete continuity and property (A). More precisely, if $\mathcal{F}$ is the nonempty common fixed point set of a sequence of asymptotically nonexpansive mappings on a closed convex subset of a reflexive Banach space with a weakly continuous duality mapping J_φ, then we are able to design an algorithm that converges strongly to a solution of problem (1.7). Our results improve the main results of Hara, Pillay and Xu [**9**], Jung [**11**] and Wittmann [**29**] by removing the need for uniform smoothness of the space in their results.

2. Preliminaries. A Banach space X is said to satisfy *Opial's condition* [**17**] if, for sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

By a *gauge* we mean a continuous strictly increasing function φ defined on $\mathbf{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. We associate with a gauge φ a (generally multi-valued) *duality mapping* $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) := \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|) \quad \text{and} \quad \|x^*\| = \varphi(\|x\|)\}.$$

Clearly the (normalized) duality mapping J corresponds to the gauge $\varphi(t) = t$. Note that

$$J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x), \quad x \neq 0.$$

Browder [**2**] initiated the study of certain classes of nonlinear operators by means of duality mappings of the form of J_φ. For $t \geq 0$, let

$$\Phi(t) := \int_0^t \varphi(r)\, dr.$$

It is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|\cdot\|)$ at x.

Now let us recall that X is said to have a *weakly continuous duality mapping* if a gauge φ exists such that the duality mapping J_φ is single-valued and continuous from X with the weak topology to X^* with the weak* topology. We know that if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition, see [**7**]. Every l^p, $1 < p < \infty$, space has a weakly continuous duality mapping with the gauge $\varphi(t) = t^{p-1}$.

One sees that Φ is a convex function and

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. We need the subdifferential inequality:

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, j(x+y)\rangle \quad \text{for all } x, y \in X$$

and

$$j(x+y) \in J_\varphi(x+y).$$

For a smooth X we have

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle \quad \text{for all } x, y \in X; \tag{2.1}$$

or, considering the normalized duality mapping J, we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle \quad \text{for all } x, y \in X.$$

Recall that a self-mapping T defined on a subset C of a Banach space X is said to be *demi-closed* if for any sequence $\{u_n\}$ in C the following implication holds:

$$u_n \rightharpoonup u \in C \quad \text{and} \quad \lim_{n\to\infty} \|Tu_n - w\| = 0 \quad \text{implies} \quad Tu = w.$$

The following result can be found in [**5**, page 108].

Lemma 2.1 (Demi-closedness principle). *Let X be a reflexive Banach space which satisfies Opial's condition, C a nonempty closed convex subset of X and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demi-closed on C, where I is the identity mapping.*

The mapping T is said to be *asymptotically regular on C* if

$$\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0 \quad \text{for all } x \in C.$$

The concept of asymptotic regularity is due to Browder and Petryshyn [**3**]. Two very interesting examples of asymptotically regular mappings without fixed points can be found in [**14, 28**].

The mapping T is said to be *uniformly asymptotically regular on* C if $\lim_{n\to\infty}(\sup_{x\in C}\|T^n x - T^{n+1}x\|) = 0$. T is said to be *uniformly asymptotically regular with sequence* $\{\epsilon_n\}$ if $\|T^n x - T^{n+1}x\| \le \varepsilon_n$ for all $x \in C$ and $n \in \mathbf{N}$, where $\varepsilon_n \to 0$ as $n \to \infty$.

In the sequel, we need the following result:

Lemma 2.2. *Let D be a nonempty subset of a Banach space X and $T : D \to D$ a Lipschitzian mapping. If $\{z_n\}$ is a sequence in D with $\lim_{n\to\infty}\|z_n - T^n z_n\| = 0$ and $\lim_{n\to\infty}\|T^n z_n - T^{n+1}z_n\| = 0$, then $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$.*

Proof. Let L be the Lipschitz constant of T. Then

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - T^n z_n\| + \|T^n z_n - T^{n+1}z_n\| + \|T^{n+1}z_n - Tz_n\| \\ &\le (1+L)\|z_n - T^n z_n\| + \|T^n z_n - T^{n+1}z_n\| \\ &\longrightarrow 0 \quad \text{as} \quad n \to \infty. \qquad \square\end{aligned}$$

We should note that the condition $\lim_{n\to\infty}\|T^n z_n - T^{n+1}z_n\| = 0$ is not necessary if T is uniformly asymptotically regular with sequence $\{\varepsilon_n\}$.

Let C be a closed convex subset of X, D a nonempty subset of C and P_D a retraction from C into D, that is, $P_D x = x$ for all $x \in D$. A retraction P_D is said to be *sunny* if $P_D(P_D x + t(x - P_D x)) = P_D x$ for all $x \in C$ and $t \ge 0$. The set D is said to be a *sunny nonexpansive retract of* C if there exists a sunny nonexpansive retraction of C onto D.

In the sequel, we shall need the following results.

Lemma 2.3 [**6**]. *Let C be a nonempty convex subset of a smooth Banach space, D a nonempty subset of C and P_D a retraction from C onto D. Then P_D is sunny and nonexpansive if and only if*

$$\langle x - P_D x, J(z - P_D x)\rangle \le 0 \quad \textit{for all } x \in C \quad \textit{and} \quad z \in D.$$

Lemma 2.4. *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences satisfying the following*:

(a) $\{\alpha_n\} \subset [0,1], \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\{\beta_n\} \subset [1,\infty), \sum_{n=1}^{\infty}(\beta_n - 1) < \infty$;

(c) $\limsup_{n\to\infty} \gamma_n \le 0$.

Let $\{\delta_n\}$ *be a sequence of nonnegative numbers which satisfies the inequality*

$$\delta_{n+1} \le (1-\alpha_n)\beta_n\delta_n + \alpha_n\gamma_n, \quad n \in \mathbf{N}. \tag{2.2}$$

Then $\lim_{n\to\infty} \delta_n = 0$.

Proof. Since $\limsup_{n\to\infty} \gamma_n \le 0$, for a given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\gamma_n \le \varepsilon \quad \text{for all } n \ge n_0.$$

Hence, from (2.2), we infer that

$$\delta_{n+1} \le [(1-\alpha_n)\delta_n + \alpha_n\varepsilon]\beta_n \quad \text{for all } n \ge n_0.$$

Thus, inductively we get that, for all $n \in \mathbf{N}$,

$$\delta_{n+n_0} \le \left(\prod_{i=n_0}^{n+n_0-1} \beta_i\right)\left[\prod_{i=n_0}^{n+n_0-1}(1-\alpha_i)\delta_{n_0} + \left(1 - \prod_{i=n_0}^{n+n_0-1}(1-\alpha_i)\right)\varepsilon\right]. \tag{2.3}$$

Using conditions (i) and (ii), we obtain from (2.3) that

$$\limsup_{n\to\infty} \delta_n = \limsup_{n\to\infty} \delta_{n+n_0} \le \varepsilon.$$

Since ε is an arbitrary positive real number, we conclude that $\delta_n \to 0$ as $n \to \infty$. $\square$

3. Existence of sunny nonexpansive retraction. First, we show that, under certain conditions, the set of fixed points of an asymptotically nonexpansive mapping in a reflexive Banach space is a sunny nonexpansive retract of its domain. Note that for the case of nonexpansive mappings such a condition is not necessary.

Theorem 3.1. *Let X be a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex bounded subset of X and $T : D \to D$ a uniformly asymptotically regular and asymptotically nonexpansive mapping. Then $F(T)$ is a nonempty and sunny nonexpansive retract of D.*

Proof. Let $\{k_n\}$ be Lipschitz constants of $\{T^n\}$. For $u \in D$, define the contraction mapping $T_n : D \to D$ by

$$T_n x = (1 - \lambda_n)u + \lambda_n T^n x \quad \text{for all } n \in \mathbf{N},$$

where $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions: $\lambda_n k_n < 1$ for all $n \in \mathbf{N}$, $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty}(k_n - 1)/(1 - \lambda_n) = 0$. It follows from the contraction mapping principle that, for each $n \in \mathbf{N}$, there exists exactly one point x_n in D such that

$$x_n = (1 - \lambda_n)u + \lambda_n T^n x_n. \tag{3.1}$$

Then

$$\|x_n - T^n x_n\| \le \frac{1 - \lambda_n}{\lambda_n}\|x_n - u\| \le \frac{1 - \lambda_n}{\lambda_n} \operatorname{diam}(D),$$

where $\operatorname{diam}(D)$ is the diameter of D. Hence, $x_n - T^n x_n \to 0$ as $n \to \infty$. We remark from Lemma 2.2 that $x_n - Tx_n \to 0$ as $n \to \infty$.

Since X is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $w \in D$. By demi-closedness of $I - T$, we conclude that $w \in F(T)$.

From (3.1), we arrive at

$$x_n - T^n x_n = (1 - \lambda_n)(u - T^n x_n), \tag{3.2}$$

so that for $y \in F(T)$

$$\begin{aligned}\langle x_n - T^n x_n, J_\varphi(x_n - y)\rangle &= \langle x_n - y + T^n y - T^n x_n, J_\varphi(x_n - y)\rangle \\ &\ge -(k_n - 1)\|x_n - y\|\varphi(\|x_n - y\|) \\ &\ge -(k_n - 1)M'\end{aligned}$$

for some constant $M' \ge 0$. It follows from (3.1) and (3.2) that

(3.3)
$$\langle x_n - u, J_\varphi(x_n - y)\rangle \le \frac{k_n - 1}{1-\lambda_n} M' \quad \text{for all } n \in \mathbf{N} \quad \text{and} \quad y \in F(T).$$

Therefore, for all $i \ge 1$

$$\begin{aligned}
&\|x_{n_i} - w\|\varphi(\|x_{n_i} - w\|) \\
&\qquad = \langle x_{n_i} - w, J_\varphi(x_{n_i} - w)\rangle \\
&\qquad = \langle x_{n_i} - u, J_\varphi(x_{n_i} - w)\rangle + \langle u - w, J_\varphi(x_{n_i} - w)\rangle \\
&\qquad \le \frac{k_{n_i} - 1}{1-\lambda_{n_i}} M' + \langle u - w, J_\varphi(x_{n_i} - w)\rangle.
\end{aligned}$$

From $J_\varphi(x_{n_i} - w) \overset{*}{\to} 0$ weakly and $\lim_{i\to\infty}(k_{n_i} - 1)/(1-\lambda_{n_i}) = 0$, we get that $x_{n_i} \to w$ as $i \to \infty$.

To complete the proof, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to z \ne w$, where $z = Tz$. It follows from (3.3) that

$$\langle z - u, J_\varphi(z - w)\rangle \le 0 \quad \text{and} \quad \langle w - u, J_\varphi(w - z)\rangle \le 0.$$

Adding these two inequalities gives

$$\langle z - w, J_\varphi(z - w)\rangle = \|z - w\|\varphi(\|z - w\|) \le 0,$$

and thus $z = w$. That is, $\{x_n\}$ strongly converges to w. Hence we can define a mapping $P_{F(T)}$ from D onto $F(T)$ by $\lim_{n\to\infty} x_n = P_{F(T)}u$, since u is an arbitrary point of D. Again, from (3.3), we know that

$$\langle u - P_{F(T)}u, J_\varphi(w - P_{F(T)}u)\rangle \le 0 \quad \text{for all } u \in D \quad \text{and} \quad w \in F(T).$$

This proves that $P_{F(T)}$ is a sunny nonexpansive retraction on D by Lemma 2.3. □

Corollary 3.1. *Let X be a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex bounded subset of X and $T : D \to D$ a nonexpansive mapping. Then $F(T)$ is a nonempty and sunny nonexpansive retract of D.*

Proof. In the case of a nonexpansive mapping, (3.1) reduces to

$$x_n = (1 - \lambda_n)u + \lambda_n T x_n \quad \text{for all } n \in \mathbf{N}.$$

By boundedness of D,

$$x_n - T x_n \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

By Lemma 2.1, since $I - T$ is demi-closed at zero, it follows that $F(T) \neq \varnothing$. Moreover, from Theorem 3.1, we conclude that $F(T)$ is a sunny nonexpansive retract of D. □

Remark 3.1. In Theorem 3.1 and Corollary 3.1, boundedness of D can be replaced by the assumption "$F(T)$ is nonempty."

4. Convergence analysis. Let D be a nonempty closed convex subset of a Banach space X, and let $\{T_n\}_{n\in\mathbf{N}}$ be a sequence of asymptotically nonexpansive self-mappings ([**19**]) on D with Lipschitz-constant sequence $\{k_n\}$, i.e., for each $n \in \mathbf{N}$, there exists a positive constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T_n x - T_n y\| \leq k_n \|x - y\|$ for all $x, y \in D$.

We now introduce our iteration scheme:

Given $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

(I) $x_{n+1} := \alpha_n u + (1 - \alpha_n) T_n x_n$, $n \in \mathbf{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Recall that a sequence $\{x_n\}$ in a subset D of a Banach space X is an *approximate fixed point sequence* for a mapping $T : D \to D$ if

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0. \tag{4.1}$$

The approximate fixed point sequence plays an important role in the fixed point theory of nonlinear operators. The importance of approximate fixed point sequences is that once a sequence has been constructed and proved to be an appropriate fixed point sequence for a continuous mapping T, the convergence of that sequence to a fixed point of T is then generally achieved under some compactness-type assumptions either on T or on its domain.

In Theorem 1.2, condition (1.3) gives that the sequence $\{T_n x_n\}$ has an approximate fixed point subsequence in D for the family $\{V_i\}$. Motivated and inspired by condition (1.3), we now consider a more general situation:

AF point property. Let D be a nonempty subset of a Banach space X, and let $\{T_n\}$ be a sequence of self-mappings on D. A sequence $\{z_n\}$ in D is said to have the *approximate fixed point property* (in short, *AF* point property) for $\{T_n\}$ if $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$.

The following extended notion of demi-closedness will prove pertinent to establishing the strong convergence of (I) to a solution of problem (1.7).

Condition $\mathcal{D}$. Let D be a nonempty closed convex subset of a Banach space X and $\{T_n\}$ a sequence of self-mappings on D. A family $\{I - T_n\}$ is said to be *demi-closed at zero* if for every bounded sequence $\{z_n\}$ in D, the following condition holds

($\mathcal{D}$) $z_n - T_n z_n \to 0 \Rightarrow w_w(z_n) \subset \mathcal{F}$, where $w_w(z_n)$ is the set of weak cluster points of the sequence $\{z_n\}$.

We now show that the iterative sequence $\{x_n\}$ defined by (I) enjoys the *AF* point property for $\{T_n\}$, a sequence of asymptotically nonexpansive self-mappings in a Banach space which is not necessarily uniform convex.

Theorem 4.1. *Let D be a nonempty convex subset of a Banach space X and $\{T_n\}$ a sequence of asymptotically nonexpansive self-mappings on D with Lipschitz-constant sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$.*

Given $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) T_n x_n, \quad n \in \mathbf{N}, \tag{4.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \tag{4.3}$$

$$\lim_{n\to\infty}\left|1-\frac{\alpha_n}{\alpha_{n+1}}\right| = 0 \quad \textit{and} \quad \lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_{n+1}} = 0, \tag{4.4}$$
$$\textit{where} \quad \varepsilon_n = \|T_n x_n - T_{n+1}x_n\|.$$

If $\{x_n\}$ *is bounded, then* $x_n - T_n x_n \to 0$ *as* $n \to \infty$.

Proof. Since $\{x_n\}$ and $\{T_n x_n\}$ are bounded, there exists a constant $K \geq 0$ such that

$$\|x_{n+1} - x_n\| \leq K \quad \text{and} \quad \|T_n x_n - u\| \leq K \quad \text{for all } n \in \mathbf{N}.$$

It follows that for all $n \geq 2$

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\qquad = \|(\alpha_n - \alpha_{n-1})(u - T_{n-1}x_{n-1}) \\
&\qquad\qquad + (1-\alpha_n)(T_n x_n - T_{n-1}x_{n-1})\| \\
&\qquad \leq |\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)\|T_n x_n - T_{n-1}x_{n-1}\| \\
&\qquad \leq |\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)(\|T_n x_n - T_n x_{n-1}\| \\
&\qquad\qquad\qquad + \|T_n x_{n-1} - T_{n-1}x_{n-1}\|) \\
&\qquad \leq |\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)k_n\|x_n - x_{n-1}\| \\
&\qquad\quad + \|T_n x_{n-1} - T_{n-1}x_{n-1}\| \\
&\qquad = (1-\alpha_n)k_n\|x_n - x_{n-1}\| + \alpha_n\left(\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right|K + \frac{\varepsilon_{n-1}}{\alpha_n}\right).
\end{aligned}$$

Applying Lemma 2.4, we obtain that $x_n - x_{n+1} \to 0$ as $n \to \infty$. Since

$$\begin{aligned}
\|x_{n+1} - T_n x_n\| &= \alpha_n\|T_n x_n - u\| \\
&\leq \alpha_n \sup_{n\in\mathbf{N}}\|T_n x_n - u\| \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

it follows that

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

Theorem 4.2. *Let D be a nonempty convex subset of a Banach space X and $T : D \to D$ an asymptotically nonexpansive mapping with*

Lipschitz-constant sequence $\{k_n\}$ *for the iterates,* $\{T^n\}$*, of* T*, such that* $\sum_{n=1}^{\infty}(k_n-1)<\infty$*. Given* $u, x_1 \in D$*, define a sequence* $\{x_n\}$ *in* D *by*

$$x_{n+1} := \alpha_n u + (1-\alpha_n)T^n x_n, \quad n \in \mathbf{N},$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying the conditions* (4.3)–(4.4) *with* $\varepsilon_n = \|T^n x_n - T^{n+1}x_n\|$*. If* $\{x_n\}$ *is bounded, then* $x_n - Tx_n \to 0$ *as* $n \to \infty$*.*

As a direct consequence of Theorem 4.1, we have the following:

Corollary 4.1. *Let* D *be a nonempty closed convex subset of a Banach space* X *and* $T: D \to D$ *a nonexpansive mapping. Given* $u, x_1 \in D$*, define a sequence* $\{x_n\}$ *in* D *by*

$$x_{n+1} := \alpha_n u + (1-\alpha_n)Tx_n, \quad n \in \mathbf{N}, \tag{4.5}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$*. If* $\{x_n\}$ *is bounded, then* $x_n - Tx_n \to 0$ *as* $n \to \infty$*.*

We begin with a strong convergence of the iteration scheme (I) for approximation of common fixed points of a finite family of nonexpansive mappings in a Banach space.

Theorem 4.3. *Let* X *be a reflexive Banach space with a weakly continuous duality mapping* J_φ *with gauge* φ*. Let* D *be a nonempty closed convex subset of* X *and* $\{T_n\}$ *a sequence of asymptotically nonexpansive self-mappings on* D *with Lipschitz-constant sequence* $\{k_n\}$ *such that* $\sum_{n=1}^{\infty}(k_n-1)<\infty$ *and* $\mathcal{F} \neq \varnothing$*. Let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *such that*

$$\lim_{n\to\infty}\alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Let us assume that $V_1, V_2, \ldots, V_N : D \to D$ *are nonexpansive mappings with the property:*

for any bounded set $\widetilde{D}$ *of* D*, the following holds*

$$\lim_{n\to\infty}\left(\sup_{x\in\widetilde{D}}\|T_n x - V_i(T_n x)\|\right) = 0 \quad \text{for all} \quad i = 1, 2, \ldots, N. \tag{4.6}$$

Suppose also that $P_{F(V)}$ is the sunny nonexpansive retraction from D onto $F(V) := \cap_{i=1}^{N} F(V_i)$. Given $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

(I) $x_{n+1} := \alpha_n u + (1 - \alpha_n) T_n x_n$, $n \in \mathbf{N}$.

Then $\{x_n\}$ converges strongly to $P_{F(V)} u$.

Proof. We note that condition (4.6) implies that $\varnothing \neq \mathcal{F} \subset F(V)$. Hence, the sunny nonexpansive retraction $P_{F(V)}$ from C onto $F(V)$ is well defined. We proceed with the following steps:

1. $\{x_n\}$ *is bounded.* Let v be an element of $\mathcal{F}$. Set $M := \max\{\|u - v\|, \|x_1 - v\|\}$. Since

$$\|x_2 - v\| \leq k_1 M,$$

and

$$\|x_3 - v\| \leq \alpha_2 \|u - v\| + (1 - \alpha_2) k_2 \|x_2 - v\| \leq k_1 k_2 M.$$

So, in general,

$$\|x_n - v\| \leq \left(\prod_{i=1}^{n-1} k_i \right) M \quad \text{for all } n \geq 2.$$

The condition $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ implies that $\prod_{n=1}^{\infty} k_n = 1$, i.e., the sequence $\{\prod_{i=1}^{n-1} k_i\}$ is bounded. Hence, $\{x_n\}$ is bounded.

2. $\{T_n x_n\}$ *is bounded.* For all $n \in \mathbf{N}$ and for any $v \in \mathcal{F}$,

$$\|T_n x_n\| \leq \|T_n x_n - v\| + \|v\| \leq k_n \|x_n - v\| + \|v\| \leq M_1$$

for some constant $M_1 \geq 0$.

3. $x_{n+1} - T_n x_n \to 0$. From (I) we deduce that

$$\|x_{n+1} - T_n x_n\| = \alpha_n \|T_n x_n - u\| \leq \alpha_n M_2$$

for some constant $M_2 \geq 0$.

4. $\limsup_{n \to \infty} \langle u - P_{F(V)} u, J_\varphi(x_{n+1} - P_{F(V)} u) \rangle \leq 0$. Since X is reflexive and $\{T_n x_n\}$ is bounded, there exists a subsequence $\{T_{n_k} x_{n_k}\}$

of $\{T_n x_n\}$ such that $T_{n_k} x_{n_k} \rightharpoonup z \in D$. By our assumption we infer that for $\widetilde{D} = \{x_n\}$

$$0 = \lim_{n\to\infty} \left(\sup_{x\in\widetilde{D}} \|T_n x - V_i(T_n x)\| \right) \geq \limsup_{n\to\infty} \|T_n x_n - V_i(T_n x_n)\|$$
$$\geq \lim_{k\to\infty} \|T_{n_k} x_{n_k} - V_i(T_{n_k} x_{n_k})\|,$$

and it follows that

$$\lim_{k\to\infty} \|T_{n_k} x_{n_k} - V_i(T_{n_k} x_{n_k})\| = 0 \quad \text{for all} \quad i = 1, 2, \dots, N.$$

By the demi-closedness principle, see Lemma 2.1, we have that $z \in \cap_{i=1}^N F(V_i)$. Since $P_{F(V)}$ is the sunny nonexpansive retraction from D onto $\cap_{i=1}^N F(V_i)$ and J_φ is weakly continuous, it follows from Lemma 2.3 that

$$\begin{aligned}\limsup_{n\to\infty} &\langle u - P_{F(V)}u, J_\varphi(T_n x_n - P_{F(V)}u)\rangle \\ &= \lim_{k\to\infty} \langle u - P_{F(V)}u, J_\varphi(T_{n_k} x_{n_k} - P_{F(V)}u)\rangle \\ &= \langle u - P_{F(V)}u, J_\varphi(z - P_{F(V)}u)\rangle \leq 0.\end{aligned}$$

Hence, by Step 3 we have that

$$\limsup_{n\to\infty} \langle u - P_{F(V)}u, J_\varphi(x_{n+1} - P_{F(V)}u)\rangle \leq 0.$$

Set $\Theta_n := \Phi(\|x_n - P_{F(V)}u\|)$. Observe that

$$\begin{aligned}\Theta_{n+1} &= \Phi(\|\alpha_n(u - P_{F(V)}u) + (1-\alpha_n)(T_n x_n - P_{F(V)}u)\|) \\ &\leq \Phi((1-\alpha_n)\|T_n x_n - P_{F(V)}u\|) \\ &\quad + \alpha_n \langle u - P_{F(V)}u, J_\varphi(x_{n+1} - P_{F(V)}u)\rangle \\ &\leq \Phi((1-\alpha_n)k_n\|x_n - P_{F(V)}u\|) \\ &\qquad + \alpha_n \langle u - P_{F(V)}u, J_\varphi(x_{n+1} - P_{F(V)}u)\rangle \\ &\leq (1-\alpha_n)k_n\Theta_n + \alpha_n\langle u - P_{F(V)}u, J_\varphi(x_{n+1} - P_{F(V)}u)\rangle.\end{aligned}$$

Applying Lemma 2.4 we conclude that $\Theta_n \to 0$, completing the proof. $\square$

The following result is a direct consequence of Theorem 4.3 which generalizes [**9**, Theorem 3.3] from a Hilbert space to a Banach space.

Theorem 4.4. *Let X be a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex subset of X and $\{T_n\}$ a sequence of nonexpansive self-mappings on D such that $\mathcal{F} \neq \varnothing$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Assume that $V_1, V_2, \dots, V_N : D \to D$ are nonexpansive mappings with the property:*

for any bounded subset $\widetilde{D}$ of D, the following holds

$$\lim_{n\to\infty}\left(\sup_{x\in\widetilde{D}} \|T_n x - V_i(T_n x)\|\right) = 0 \quad \textit{for all } i = 1, 2, \dots, N.$$

Suppose also that $P_{F(V)}$ is the sunny nonexpansive retraction from D onto $F(V) =: \cap_{i=1}^{N} F(V_i)$. Given $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

$$x_{n+1} := \alpha_n u + (1-\alpha_n) T_n x_n, \quad n \in \mathbf{N}. \tag{4.7}$$

Then $\{x_n\}$ converges strongly to $P_{F(V)} u$.

As a direct consequence of Theorem 4.4, we have the following corollary which is Theorem 5 of Jung [**11**].

Corollary 4.2. *Let X be a uniformly smooth Banach space with a weakly continuous duality mapping $J : X \to X^*$, D a nonempty closed convex subset of X and $T_n : D \to D$, $n = 1, 2, \dots$, nonexpansive mappings such that $\mathcal{F} := \cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Assume that $V_1, V_2, \dots, V_N : D \to D$ are nonexpansive mappings with the property:*

for any bounded subset $\widetilde{D}$ of D, the following holds

$$\lim_{n\to\infty}\left(\sup_{x\in\widetilde{D}} \|T_n x - V_i(T_n x)\|\right) = 0 \quad \textit{for all } i = 1, 2, \dots, N.$$

Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Given $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

$$x_{n+1} := \alpha_n u + (1-\alpha_n) T_n x_n, \quad n \in \mathbf{N}. \tag{4.8}$$

Then $\{x_n\}$ *converges strongly to* $P_{F(V)}u$, *where* $P_{F(V)}$ *is the sunny nonexpansive retraction of* D *onto* $F(V) := \cap_{i=1}^{N} F(V_i)$.

Using Theorem 4.3, we obtain the strong convergence of our iteration process to the common fixed points of a finite family of asymptotically nonexpansive mappings.

Theorem 4.5. *Let* X *be a reflexive Banach space with a weakly continuous duality mapping* J_φ *with gauge* φ. *Let* D *be a nonempty closed convex subset of* X *and* $\{T_n\}$ *a sequence of asymptotically nonexpansive self-mappings on* D *with Lipschitz-constant sequence* $\{k_n\}$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\mathcal{F} \neq \varnothing$. *Let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *such that*

$$(1-\alpha_n)k_n \leq 1 \quad \text{for all} \quad n \in \mathbf{N}, \qquad \lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let us assume that $V_1, V_2, \dots, V_N : D \to D$ *are asymptotically nonexpansive mappings with the property*:

for any bounded subset $\widetilde{D}$ *of* D, *the following holds*

$$\lim_{n\to\infty} \left(\sup_{x\in\tilde{D}} \|T_n x - V_i(T_n x)\| \right) = 0 \quad \text{for all } i = 1, 2, \dots, N. \tag{4.9}$$

Suppose also that $P_{F(V)}$ *is the sunny nonexpansive retraction from* D *onto* $\cap_{i=1}^{N} F(V_i)$. *Given* $u, x_1 \in D$, *let us define a sequence* $\{x_n\}$ *in* D *by*

(I) $x_{n+1} := \alpha_n u + (1-\alpha_n)T_n x_n$, $n \in \mathbf{N}$.

Then $\{x_n\}$ *strongly converges to* $P_{F(V)}u$.

Proof. From Step 2 of Theorem 4.3, $\{T_n x_n\}$ is bounded. Thus, there exists a subsequence $\{T_{n_k} x_{n_k}\}$ of $\{T_n x_n\}$ such that $T_{n_k} x_{n_k} \rightharpoonup z \in D$. From condition (4.9), we infer that

$$\lim_{k\to\infty} \|T_{n_k} x_{n_k} - V_i(T_{n_k} x_{n_k})\| = 0 \quad \text{for all } i = 1, 2, \dots, N.$$

Since, for each i, $1 \leq i \leq N$, $I - T_i$ is demi-closed at zero, we obtain that $z \in F(V)$. Therefore, Theorem 4.5 follows from Theorem 4.3. □

It is well known that, for every asymptotically nonexpansive mapping T defined on a closed convex bounded subset of a uniformly convex Banach space, $I - T$ is demi-closed at zero, see e.g., [**13, 30**].

Corollary 4.3. *Let X be a uniformly convex Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex bounded subset of X, and let $\{T_n\}$ be a sequence of asymptotically nonexpansive self-mappings on D with Lipschitz-constant sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\mathcal{F} \neq \varnothing$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Let us assume that $V_1, V_2, \dots, V_N : D \to D$ are asymptotically nonexpansive mappings with the property:

for any subset $\widetilde{D}$ of D, the following holds

$$\lim_{n\to\infty}\left(\sup_{x\in\widetilde{D}} \|T_n x - V_i(T_n x)\|\right) = 0 \quad \textit{for all } i = 1, 2, \dots, N. \tag{4.10}$$

Suppose also that $P_{F(V)}$ is the sunny nonexpansive retraction from D onto $\cap_{i=1}^{N} F(V_i)$. Given $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

(I) $x_{n+1} := \alpha_n u + (1 - \alpha_n) T_n x_n$, $n \in \mathbf{N}$.

Then $\{x_n\}$ converges strongly to $P_{F(V)} u$.

Next, we turn our attention to the strong convergence of scheme (I) to common fixed points of an infinite family of asymptotically nonexpansive mappings in a Banach space.

Theorem 4.6. *Let X be a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex subset of X and $\{T_n\}$ a sequence of asymptotically nonexpansive self-mappings on D with Lipschitz-constant sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\mathcal{F} \neq \varnothing$.*

Given any $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) T_n x_n, \quad n \in \mathbf{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions (4.3)–(4.4). Assume that the family $\{I - T_n\}_{n \in \mathbf{N}}$ is demi-closed at zero and $P_{\mathcal{F}}$ is the sunny nonexpansive retraction from D onto $\mathcal{F}$. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}} u$.

Proof. First, we show that $\{x_n\}$ satisfies condition $(\mathcal{D})$ for $\{T_n\}$. Since $\{x_n\}$ is bounded, by Step 1 of Theorem 4.3 and the reflexivity of X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z \in D$. Since $x_{n_k} \rightharpoonup z$ and $x_n - T_n x_n \to 0$ as $n \to \infty$ by Theorem 4.1, it follows from the demi-closedness of $\{T_n\}$ that $z \in \mathcal{F}$. Thus,

$$z_n - T_n z_n \to 0 \Longrightarrow w_w(z_n) \subset \mathcal{F},$$

i.e, condition $(\mathcal{D})$ is satisfied. Since $P_{\mathcal{F}}$ is the sunny nonexpansive retraction from D onto $\mathcal{F}$, it follows from Lemma 2.3 that

$$\langle x - Px, J_\varphi(z - Px) \rangle \leq 0. \tag{4.11}$$

From (4.11) and the weak continuity of J_φ, we have that

$$\begin{aligned}\limsup_{n\to\infty} \langle u - P_{\mathcal{F}} u, J_\varphi(x_n - P_{\mathcal{F}} u) \rangle &= \lim_{k\to\infty} \langle u - P_{\mathcal{F}} u, J_\varphi(x_{n_k} - P_{\mathcal{F}} u) \rangle \\ &= \langle u - P_{\mathcal{F}} u, J_\varphi(z - P_{\mathcal{F}} u) \rangle \leq 0.\end{aligned}$$

Hence, Theorem 4.6 follows from Theorem 4.3. $\square$

As an immediate consequence of Theorem 4.6, we have the following result which extends Theorem 1.1 from Hilbert spaces to Banach spaces and also from nonexpansive mappings to a much larger class of asymptotically nonexpansive mappings.

Theorem 4.7. *Let X be a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex subset of X and $T : D \to D$ an asymptotically nonexpansive mapping with Lipschitz-constant sequence $\{k_n\}$ for the iterates,*

$\{T^n\}$, of T, such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $F(T)\neq\varnothing$. Assume that $P_{F(T)}$ is the sunny nonexpansive retraction from D onto $F(T)$. Given $u, x_1\in D$, define a sequence $\{x_n\}$ in D by

$$x_{n+1} := \alpha_n u + (1-\alpha_n)T^n x_n, \quad n\in\mathbf{N}, \tag{4.12}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying conditions (4.3)–(4.4) with $\varepsilon_n = \|T^n x_n - T^{n+1}x_n\|$. Then $\{x_n\}$ converges strongly to $P_{F(T)}u$.

Proof. Note that Theorem 4.2 implies that $\{x_n\}$ has the AF point property for T. Since X is reflexive and $I-T$ is demi-closed at zero, it follows that

$$x_n - Tx_n \longrightarrow 0 \Longrightarrow w_w(x_n)\subset F(T).$$

Thus, condition $(\mathcal{D})$ is satisfied. Therefore, the result follows from Theorem 4.6. □

Combining Theorems 3.1 and 4.7, we obtain the following result.

Theorem 4.8. *Let X be a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex bounded subset of X. Let $T: D\to D$ be uniformly asymptotically regular and asymptotically nonexpansive with Lipschitz-constant sequence $\{k_n\}$ for the iterates, $\{T^n\}$, of T, such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$. Given $u, x_1\in D$, define a sequence $\{x_n\}$ in D by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T^n x_n, \quad n\in\mathbf{N}, \tag{4.13}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying conditions (4.3)–(4.4). Then $\{x_n\}$ converges strongly to $P_{F(T)}u$, where $P_{F(T)}$ is the sunny nonexpansive retraction from D onto $F(T)$.

Proof. Since T is uniformly asymptotically regular, it follows from Theorem 3.1 that $F(T)$ is nonempty and a sunny nonexpansive retract of D. Let $P_{F(T)}$ be the sunny nonexpansive retraction from D onto $F(T)$. Hence the conclusion follows from Theorem 4.7. □

Remark 4.1. Theorem 4.8 is similar to Theorem 2.3 of Schu [**24**], which requires property (A). In our approach the convergence of the sequence defined by (3.1) is not used to prove the convergence of the iterative sequence $\{x_n\}$ defined in (4.13).

The following result shows that Wittmann's result, see Theorem 3.1, is valid in Banach spaces under certain conditions.

Corollary 4.4. *Let X be a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ. Let D be a nonempty closed convex subset of X and $T : D \to D$ a nonexpansive mapping such that $F(T) \neq \varnothing$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ satisfying*:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \left|1 - \frac{\alpha_n}{\alpha_{n+1}}\right| = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Given $u, x_1 \in D$, define a sequence $\{x_n\}$ in D by

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) T x_n, \quad n \in \mathbf{N}. \tag{4.14}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}u$, where $P_{F(T)}$ is the sunny nonexpansive retraction from D onto $F(T)$.

Proof. It follows from Corollary 4.1 that $\|x_n - Tx_n\| \to 0$, i.e., $\{x_n\}$ satisfies the AF point property for T. Moreover, from Corollary 3.1, there exists the sunny nonexpansive retraction $P_{F(T)}$ from D onto $F(T)$. Hence, $\{x_n\}$ converges strongly to $P_{F(T)}u$. □

Corollary 4.4 is similar to Theorem 10 of [**25**], which requires a star-shaped domain and property (H). Our results do not carry over to star-shaped domains.

Remark 4.2. In the case of a Hilbert space, see [**8, 15**].

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