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FOUR BIVARIATE DISTRIBUTIONS WITH GAMMA TYPE MARGINALS

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ABSTRACT. Four new bivariate distributions with gamma type marginals are introduced. Various representations are derived for their joint densities, product moments, conditional densities and conditional moments. Some of these representations involve special functions such as the complementary incomplete gamma and Whittaker functions. Construction of multivariate generalizations is discussed. Finally, an application to rainfall data from Florida is provided.

1. Introduction. There have only been a few bivariate gamma distributions proposed in the statistics literature, see Chapter 48 in Kotz et al. [**6**] for a good review (see also Gupta [**3**] and Gupta and Wong [**4**]). These distributions have attracted useful applications in several areas; for example, in the modeling of rainfall at two nearby rain gauges [**5**], data obtained from rainmaking experiments [**8, 9**], the dependence between annual streamflow and areal precipitation [**1**], wind gust data [**11**] and the dependence between rainfall and runoff [**7**]. s They have also attracted applications in reliability theory, renewal processes and stochastic routing problems.

The aim of this paper is to construct four new bivariate distributions with gamma type marginals and to study their properties. We derive various representations for the joint densities, product moments, conditional densities and conditional moments associated with each bivariate distribution. We provide an application to rainfall data from Florida and discuss ways to construct multivariate generalizations.

The calculations of this paper make use of several special functions, including the incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} \exp(-t) \, dt,$$

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the complementary incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t)\, dt,$$

the Whittaker function defined by

$$W_{\lambda,\mu}(a) = \frac{a^{\mu+1/2} \exp(-a/2)}{\Gamma(\mu - \lambda + 1/2)} \int_0^\infty t^{\mu-\lambda-1/2} (1+t)^{\mu-\lambda-1/2} \exp(-at)\, dt,$$

the confluent hypergeometric function of one variable defined by

$${}_1F_1(\alpha; \beta; x) = \sum_{k=0}^\infty \frac{(\alpha)_k}{(\beta)_k} \frac{x^k}{k!},$$

the confluent hypergeometric function of two variables defined by

$$\Phi(\alpha, \beta, \gamma, x, y) = \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{(\alpha)_{j+k} (\beta)_k}{(\gamma)_{j+k}\, j!k!} x^j y^k,$$

the Gauss hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^\infty \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},$$

and the generalized hypergeometric function defined by

$${}_2F_2(\alpha, \beta; \gamma, \delta; x) = \sum_{k=0}^\infty \frac{(\alpha)_k (\beta)_k}{(\gamma)_k (\delta)_k} \frac{x^k}{k!},$$

where $(c)_k = c(c+1)\cdots(c+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Prudnikov [**10**] and Gradshteyn and Ryzhik [**2**].

2. Construction I. The basis for the construction of the first two bivariate distributions is the following characterization of gamma and beta distributions due to Yeo and Milne [**12**].

Lemma 1 [**12**]. *Suppose that U and V are independent, absolutely continuous and nonnegative random variables such that U has bounded support. Then, for any $a > 0$ and $b > 0$, any two of the following three conditions imply the third*:

(i) *UV is gamma distributed with shape parameter a and scale parameter $1/\mu$, where $0 < \mu < \infty$;*

(ii) *U is beta distributed with shape parameters a and b;*

(iii) *V is gamma distributed with shape parameter $a + b$ and scale parameter $1/\mu$.*

An obvious way to generate a bivariate gamma from this lemma is to consider the joint distribution of $X = UV$ and V. The joint pdf of U and V is:

$$f(u, v) = \frac{u^{a-1}(1-u)^{b-1}}{B(a,b)} \frac{v^{a+b-1}\exp(-v/\mu)}{\mu^{a+b}\Gamma(a+b)},$$

and thus the joint pdf of X and V becomes:

$$f(x, v) = \frac{x^{a-1}(v-x)^{b-1}\exp(-v/\mu)}{\mu^{a+b}\Gamma(a)\Gamma(b)} \tag{1}$$

for $x \leq v$ and $v > 0$. Unfortunately, the pdf (1) corresponds to a known bivariate gamma distribution–McKay's bivariate gamma distribution, see [**6**, Section 48.2.1] for details.

Take U, V and W to be independent, absolutely continuous and nonnegative random variables. Then two new bivariate gamma distributions can be constructed as follows:

1. Assume that W is beta distributed with shape parameters a and b. Assume further that U and V are gamma distributed with common shape parameter c and scale parameters $1/\mu_1$ and $1/\mu_2$, respectively, where $c = a + b$. Define

$$X = UW, \qquad Y = VW. \tag{2}$$

Then, by Lemma 1, X and Y will be gamma distributed with common shape parameter a and scale parameters $1/\mu_1$ and $1/\mu_2$, respectively.

However, they will be correlated so that (X, Y) will have a bivariate gamma distribution over $(0, \infty) \times (0, \infty)$.

2. Assume that U and V are beta distributed with shape parameters (a_1, b_1) and (a_2, b_2), respectively, where $a_1 + b_1 = a_2 + b_2 = c$, say. Assume further that W is gamma distributed with shape parameter c and scale parameter $1/\mu$. Define

$$X = UW, \qquad Y = VW. \tag{3}$$

Then, by Lemma 1, X and Y will be gamma distributed with common scale parameter $1/\mu$ and shape parameters a_1 and a_2, respectively. However, they will be correlated so that (X, Y) will again have a bivariate gamma distribution over $(0, \infty) \times (0, \infty)$.

Theorem 1 states that the joint pdf of (X, Y) for the first construct can be expressed in terms of Whittaker function.

Theorem 1. *Let U, V and W be independent random variables with W beta distributed with shape parameters a and b and U and V gamma distributed with common shape parameter c and scale parameters $1/\mu_1$ and $1/\mu_2$, respectively, where $c = a + b$. Let X and Y be as in (2). Then the joint pdf of X and Y is given by*

$$\begin{aligned} f(x, y) &= C\Gamma(b)(xy)^{c-1}\left(\frac{x}{\mu_1} + \frac{y}{\mu_2}\right)^{(a-1/2)-c} \\ &\quad \times \exp\left\{-\frac{1}{2}\left(\frac{x}{\mu_1} + \frac{y}{\mu_2}\right)\right\} W_{c-b+(1-a/2),c-(a/2)}\left(\frac{x}{\mu_1} + \frac{y}{\mu_2}\right) \end{aligned} \tag{4}$$

for $x > 0$ and $y > 0$, where the constant C is given by

$$\frac{1}{C} = (\mu_1\mu_2)^c\,\Gamma(c)\Gamma(a)\Gamma(b).$$

Proof. The joint pdf of U, V and W is:

$$f(u, v, w) = C(uv)^{c-1}w^{a-1}(1 - w)^{b-1}\exp\left\{-\left(\frac{u}{\mu_1} + \frac{v}{\mu_2}\right)\right\}$$

from which the joint pdf of X, Y and W becomes:

$$f(x, y, w) = C(xy)^{c-1} w^{a-2c-1} (1-w)^{b-1} \exp\left\{ -\frac{1}{w} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right) \right\}.$$

Integrating over $0 < w < 1$, one obtains

$$f(x, y) = C(xy)^{c-1} I(x, y), \tag{5}$$

where $I(x, y)$ denotes the integral

$$I(x, y) = \int_0^1 w^{a-2c-1} (1-w)^{b-1} \exp\left\{ -\frac{1}{w} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right) \right\} dw. \tag{6}$$

Substituting $u = 1/w - 1$ and then using the definition of Whittaker function, one can write

(7)

$$\begin{aligned} I(x, y) = {} & \Gamma(b) \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right)^{(a-1/2)-c} \\ & \times \exp\left\{ -\frac{1}{2} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right) \right\} W_{c-b+(1-a/2), c-(a/2)} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right). \end{aligned}$$

The result in (4) follows by substituting (7) into (5). □

Figure 1 illustrates the shape of the joint pdf (4) for selected values of (a, b) with $\mu_1 = \mu_2 = 1$.

In the particular case $b = 1$, (4) can be simplified to an expression involving the complementary incomplete gamma function, as stated by the following corollary.

Corollary 1. *If* $b = 1$, *then the joint pdf* (4) *reduces to the simpler form*:

$$f(x, y) = C(xy)^{c-1} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right)^{a-2c} \Gamma\left(2c - a, \frac{x}{\mu_1} + \frac{y}{\mu_2} \right).$$

Proof. Note the integral $I(x, y)$ in (6) can be expressed in terms of the complementary incomplete gamma function if $b = 1$. □

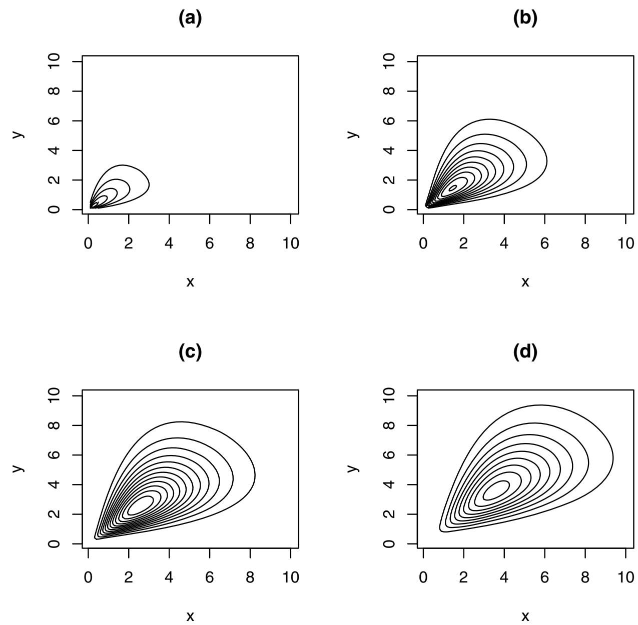

FIGURE 1. Contours of the joint pdf (4) with $(a,b)=(2,2)$ for (a), $(a,b)=(3,3)$ for (b), $(a,b)=(4,4)$ for (c) and $(a,b)=(5,5)$ for (d). It is assumed throughout that $\mu_1=\mu_2=1$.

Theorem 2 states that the joint pdf of (X,Y) for the second construct (3) can be expressed as an infinite sum of the Whittaker functions.

Theorem 2. *Let U, V and W be independent random variables with W gamma distributed with shape parameter c and scale parameter $1/\mu$ and U and V beta distributed with shape parameters (a_1,b_1) and (a_2,b_2), respectively, where $a_1+b_1=a_2+b_2=c$, say. Let X and Y be as in (3). Then the joint pdf of X and Y is given by*

$$
\begin{aligned}
f(x,y) &= C\Gamma(b_1)\Gamma(b_2)\,\mu^{(b_1+b_2-c+1)/2}x^{(a_1+b_2-3)/2}y^{a_2-1}\\
&\quad\times\exp\left(-\frac{x}{2\mu}\right)\\
&\quad\times\sum_{j=0}^{\infty}\frac{(-1)^j(\mu x)^{-j/2}y^j}{j!\Gamma(b_2-j)}W_{(b_2-b_1-c-j-1)/2,(b_1+b_2-c-j)/2}\left(\frac{x}{\mu}\right)
\end{aligned}
\tag{8}
$$

for $x \geq y > 0$, *where the constant* C *is given by*

$$\frac{1}{C} = \mu^c \Gamma(c) B\left(a_1, b_1\right) B\left(a_2, b_2\right).$$

The corresponding expression for $0 < x \leq y$ *can be obtained from* (8) *by symmetry, i.e., interchange* x *with* y, a_1 *with* a_2 *and* b_1 *with* b_2.

Proof. The joint pdf of U, V and W is:

$$f(u, v, w) = C u^{a_1-1} v^{a_2-1} (1-u)^{b_1-1} (1-v)^{b_2-1} w^{c-1} \exp\left(-\frac{w}{\mu}\right)$$

from which the joint pdf of X, Y and W becomes:

$$f(x, y, w) = C x^{a_1-1} y^{a_2-1} w^{1-c} (w-x)^{b_1-1} (w-y)^{b_2-1} \exp\left(-\frac{w}{\mu}\right) \tag{9}$$

for $w \geq \max(x, y)$. The integration of (9) over $\max(x, y) \leq w < \infty$ is not easy. However, using the series representation

$$(1+z)^\alpha = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)} \frac{z^j}{j!},$$

one can write

$$f(x, y) = C\Gamma\left(b_2\right) x^{a_1-1} y^{a_2-1} \sum_{j=0}^{\infty} \frac{(-y)^j I_j(x)}{j! \Gamma\left(b_2-j\right)} \tag{10}$$

for $x \geq y > 0$, where $I_j(x)$ denotes the integral

$$I_j(x) = \int_x^{\infty} w^{b_2-c-j} (w-x)^{b_1-1} \exp\left(-\frac{w}{\mu}\right) dw.$$

Substituting $u = w/x - 1$ and then using the definition of the Whittaker function, one can express

$$\begin{aligned} I_j(x) = {} & \Gamma\left(b_1\right) \mu^{(b_1+b_2-c-j+1)/2} x^{(b_1+b_2-c-j-1)/2} \\ & \times \exp\left(-\frac{x}{2\mu}\right) W_{(b_2-b_1-c-j-1)/2, (b_1+b_2-c-j)/2}\left(\frac{x}{\mu}\right). \end{aligned} \tag{11}$$

The result in (8) follows by substituting (11) into (10). □

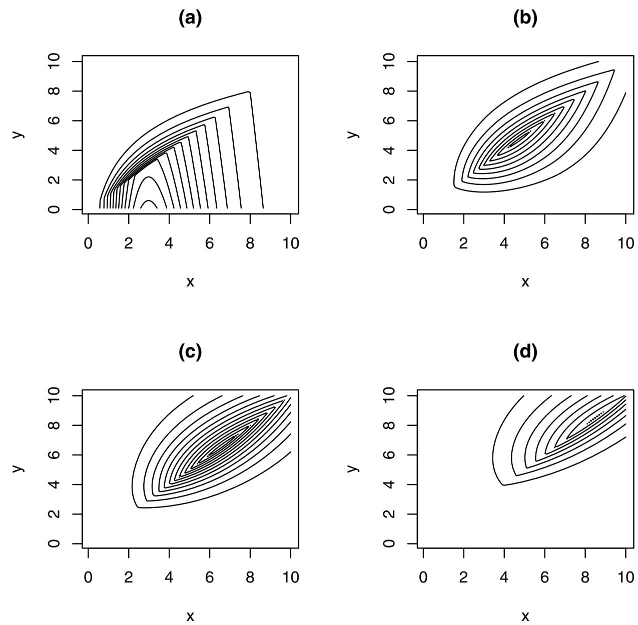

FIGURE 2. Contours of the joint pdf (8) with $(a_1, a_2, c) = (4, 1, 8)$ for (a), $(a_1, a_2, c) = (4, 3, 8)$ for (b), $(a_1, a_2, c) = (4, 5, 8)$ for (c) and $(a_1, a_2, c) = (4, 7, 8)$ for (d). It is assumed throughout that $\mu = 1$.

Figure 2 illustrates the shape of the joint pdf (8) for selected values of (a_1, a_2, c) with $\mu = 1$.

Corollary 2. *If $b_1 = 1$, then the joint pdf (8) reduces to the simpler form*:

$$f(x, y) = C\Gamma(b_2)\,\mu^{1-a_2} x^{a_1-1} y^{a_2-1} \times \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(b_2 - j)} \left(\frac{y}{\mu}\right)^j \Gamma\left(1 + b_2 - c - j, \frac{x}{\mu}\right).$$

On the other hand, if $b_2 = 1$, then (8) reduces to

$$f(x, y) = C\Gamma(b_1)\,\mu^{(b_1-c+2)/2} W_{(2-b_1-c)/2,(1-a_1)/2}\left(\frac{x}{\mu}\right) \times x^{(a_1/2)-1} y^{a_2-1} \exp\left(-\frac{x}{2\mu}\right).$$

If both $b_1 = 1$ *and* $b_2 = 1$, *then* (8) *reduces to*

$$f(x, y) = C\mu^{2-c} x^{a_1 - 1} y^{a_2 - 1} \Gamma\left(2 - c, \frac{x}{\mu}\right).$$

Proof. As in Corollary 1. □

Remark 1. The product moments of the two distributions introduced above can be expressed in terms of elementary functions. Since $E(X^m Y^n) = E((UW)^m (VW)^n) = E(U^m) E(V^n) E(W^{m+n})$ from (2), the product moments of (4) are given by

$$E\left(X^m Y^n\right) = \frac{\mu_1^m \mu_2^n \Gamma(c+m)\Gamma(c+n) B(a+m+n, b)}{\Gamma(a)\Gamma(b)\Gamma(c)} \tag{12}$$

for $m \geq 1$ and $n \geq 1$. In particular,

$$\mathrm{Cov}\left(X, Y\right) = \frac{\mu_1 \mu_2 a b}{a+b+1}$$

and

$$\mathrm{Corr}\left(X, Y\right) = \frac{b}{a+b+1}. \tag{13}$$

Since $E(X^m Y^n) = E((UW)^m (VW)^n) = E(U^m) E(V^n) E(W^{m+n})$ from (3), the product moments of (8) are given by

$$E\left(X^m Y^n\right) = \frac{\mu^{m+n} \Gamma(m+n+c) B\left(m+a_1, b_1\right) B\left(n+a_2, b_2\right)}{\Gamma(c) B\left(a_1, b_1\right) B\left(a_2, b_2\right)} \tag{14}$$

for $m \geq 1$ and $n \geq 1$. In particular,

$$\mathrm{Cov}\left(X, Y\right) = \frac{\mu^2 a_1 a_2}{c}$$

and

$$\mathrm{Corr}\left(X, Y\right) = \frac{\sqrt{a_1 a_2}}{c}. \tag{15}$$

Remark 2. The conditional distributions corresponding to (4) and (8) can be easily calculated. For the pdf (4), the conditional pdf of Y given $X = x$ is given by

$$f(y \mid x) = C\mu_1^a \Gamma(a)\Gamma(b) x^b y^{c-1} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right)^{(a-1/2)-c} \times \exp\left\{ \frac{1}{2} \left(\frac{x}{\mu_1} - \frac{y}{\mu_2} \right) \right\} W_{c-b+(1-a/2),c-(a/2)} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2} \right).$$

The corresponding pdf of X given $Y = y$ is obtained by interchanging x with y and μ_1 with μ_2. For the pdf (8), the conditional pdf of Y given $X = x$ is given by

(16)

$$f(y \mid x) = C\Gamma(a_1)\Gamma(b_1)\Gamma(b_2)\mu^{(a_1+b_2+1)/2} x^{(b_2-1-a_1)/2} y^{a_2-1} \times \exp\left(\frac{x}{2\mu} \right) \sum_{j=0}^{\infty} \frac{(-1)^j (\mu x)^{-j/2} y^j}{j! \Gamma(b_2 - j)} \times W_{(b_2-b_1-c-j-1)/2,(b_1+b_2-c-j)/2} \left(\frac{x}{\mu} \right)$$

for $x \geq y > 0$, and by

(17)

$$f(y \mid x) = C\mu^{a_1}\Gamma(a_1)\Gamma(b_1)\Gamma(b_2)\mu^{(b_1+b_2-c+1)/2} y^{(a_2+b_1-3)/2} \times \exp\left(\frac{x}{\mu} - \frac{y}{2\mu} \right) \times \sum_{j=0}^{\infty} \frac{(-1)^j (\mu y)^{-j/2} x^j}{j! \Gamma(b_1 - j)} W_{(b_2-b_1-c-j-1)/2,(b_1+b_2-c-j)/2} \left(\frac{y}{\mu} \right)$$

for $y \geq x > 0$. The corresponding pdf of X given $Y = y$ is obtained by interchanging x with y, a_1 with a_2 and b_1 with b_2. The conditional moments can be obtained by integrating the above pdfs by using equation (2.19.5.10) in Prudnikov [**10**]. The resulting forms are complicated and involve multiple sums of the generalized hypergeometric function.

It is natural to ask how the pdfs (1), (4) and (8) can be generalized to the multivariate case. Lemma 1 can be applied in several ways to generate multivariate gamma distributions. Some of these are:

1. Assume that W is a beta distributed random variable with shape parameters a and b. Assume further that U_j, $j = 1, 2, \dots, p$ are gamma distributed independent random variables (and independent of W) with common shape parameter c and scale parameters $1/\mu_j$, $j = 1, 2, \dots, p$, where $c = a + b$.

2. Assume that U_j, $j = 1, 2, \dots, p$ are beta distributed random variables with shape parameters (a_j, b_j), $j = 1, 2, \dots, p$, where $a_j + b_j = c$ (say) for $j = 1, 2, \dots, p$. Assume further that W is a gamma distributed random variable (independent of U_j, $j = 1, 2, \dots, p$) with shape parameter c and scale parameter $1/\mu$.

In both these cases, by Lemma 1, $(U_1 W, U_2 W, \dots, U_p W)$ will have a p-variate gamma distribution over $(0, \infty)^p$.

The following generalization of Lemma 1 provided by Yeo and Milne [**12**] provides other ways to generate multivariate gammas.

Lemma 2 [**12**]. *Suppose for a fixed integer $p \geq 2$ that $X_1, X_2, \dots, X_p$ are independent and identically distributed* (iid) *nonnegative random variables which are independent of another nonnegative random variable X with bounded support, and that*

$$Y = X (X_1 + X_2 + \cdots + X_p).$$

Then the two following conditions are equivalent.

(i) *Y has the same distribution as each of $X_1, X_2, \dots, X_p$ and belongs to the class of distributions whose characteristic function is of the form*

$$\phi(t) = 1 - A|t| \{1 + o(t)\}$$

as $t \to 0$, where A is a real constant.

(ii) *X is beta distributed with shape parameters* 1 *and* $p - 1$.

One can generate several multivariate gammas by taking $X_1, X_2, \dots,$ X_p to be iid gamma distributed.

3. Construction II. Here, we construct the first bivariate distribution which has gamma and beta distributions as its marginals. The basis for this construction is the well-known characterization that a beta random variable with parameters (a, b) can be represented as $X/(X+Y)$, where X and Y are independent gamma random variables with parameters $(a, 1/\lambda)$ and $(b, 1/\lambda)$, respectively. We define random variables U and V by

$$U = X, \qquad V = \frac{X}{X+Y}. \tag{18}$$

Then, U will be gamma distributed with parameters $(a, 1/\lambda)$ and V will be beta distributed with parameters (a, b). However, they will be correlated so that (U, V) will have a bivariate distribution over $(0, \infty) \times (0, 1)$ with gamma and beta marginals.

Remark 3. Let X and Y be independent gamma random variables with parameters $(a, 1/\lambda)$ and $(b, 1/\lambda)$, respectively. If U and V are as in (18), then the joint pdf of (U, V) is

$$f(u, v) = \frac{u^{a+b-1} v^{-(1+b)} (1-v)^{b-1} \exp\{-u/(\lambda v)\}}{\lambda^{a+b} \Gamma(a) \Gamma(b)} \tag{19}$$

for $0 < u < \infty$ and $0 < v < 1$.

If $a = 1$, then (19) reduces to a bivariate pdf with exponential and beta marginals. If, on the other hand, $b = 1$, then (19) reduces to a bivariate pdf with gamma and power function marginals.

The first derivatives of (19) with respect to u and v are

$$\frac{\partial \log f}{\partial u} = \frac{a+b-1}{u} - \frac{1}{\lambda v}$$

and

$$\frac{\partial \log f}{\partial v} = \frac{1-b}{1-v} - \frac{1+b}{v} + \frac{u}{\lambda v^2},$$

respectively. Thus, if $a > 2$ and $b > 1$, then it follows that (19) has a single mode at

$$(u_0, v_0) = \left(\frac{(a+b-1)(a-2)\lambda}{a+b-3}, \frac{a-2}{a+b-3} \right).$$

On the other hand, if $a \leq 2$ or $b \leq 1$, then

$$\frac{\partial \log f}{\partial u} > 0 \Longleftrightarrow u < (a+b-1)\lambda v$$

and

$$\frac{\partial \log f}{\partial v} > 0 \Longleftrightarrow u > \frac{\lambda v(1+b-2v)}{1-v}.$$

Some particular values of (19) can be computed as

$$f\left(\frac{(a+b-1)(a-2)\lambda}{a+b-3}, \frac{a-2}{a+b-3}\right) = \frac{(a-2)^{a-2}(b-1)^{b-1}(a+b-1)^{a+b-1}\exp(1-a-b)}{\lambda(a+b-3)^{a+b-3}\Gamma(a)\Gamma(b)},$$

$$f\left((a+b-1)\lambda v, v\right) = \frac{(a+b-1)^{a+b-1}v^{a-2}(1-v)^{b-1}\exp(1-a-b)}{\lambda\Gamma(a)\Gamma(b)},$$

and

$$f\left(\frac{\lambda v(1+b-2v)}{1-v}, v\right) = \frac{v^{a-2}(1+b-2v)^{a+b-1}\exp\left\{(2v-1-b)/(1-v)\right\}}{(1-v)^a\lambda\Gamma(a)\Gamma(b)}.$$

Figure 3 illustrates the shape of (19) for some selected values of (a, b) and $\lambda = 1$. It can be seen how changing (a, b) makes the dependence between U and V inflated or asymmetric.

Theorem 3 provides the joint cdf of (U, V) for the construct (18).

Theorem 3. *The joint cdf of U and V corresponding to* (19) *is given by*

$$F(u,v) = \frac{1}{(a+b)\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\left(u/(\lambda v)\right)^{b-k}}{\Gamma(1+b-k)} \times {}_2F_2\left(b-k, a+b; a+b+1, b-k+1; -\frac{u}{\lambda v}\right). \tag{20}$$

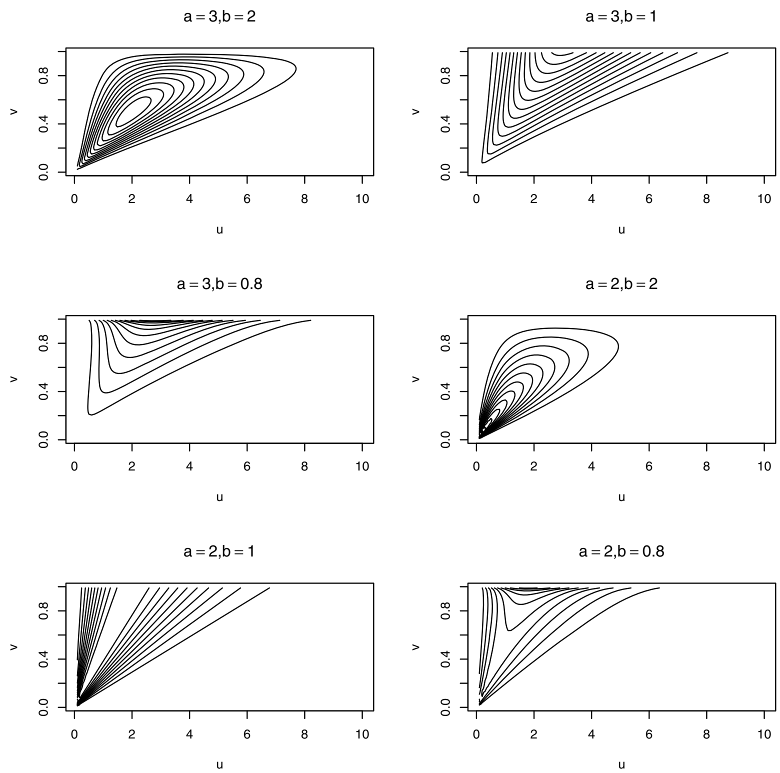

FIGURE 3.

Proof. One can write

$$F(u,v) = \frac{1}{\lambda^{a+b}\Gamma(a)\Gamma(b)} \int_0^u\int_0^v x^{a+b-1}y^{-(1+b)}(1-y)^{b-1}\exp\{-x/(\lambda y)\}\,dy\,dx.$$

By setting $z = x/(\lambda y)$ and using the definition of the incomplete gamma function, the above can be reduced to:

$$F(u,v) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^v y^{a-1}(1-y)^{b-1}\gamma\left(a+b, u/(\lambda y)\right) dy. \tag{21}$$

This integral is difficult to calculate. However, using the series representation

$$(1+z)^{\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)} \frac{z^j}{j!},$$

(21) can be expanded as

$$F(u,v) = \frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k I_k}{\Gamma(b-k)}, \tag{22}$$

where I_k denotes the integral

$$I_k = \int_0^v y^{a+k-1} \gamma\left(a+b, u/(\lambda y)\right) dy. \tag{23}$$

By an application of equation (2.10.2.2) in Prudnikov et al. [**10**], (23) can be evaluated as

(24)

$$I_k = \frac{u^{a+b}}{(a+b)(k-b)\lambda^{a+b} v^{b-k}} {}_2F_2\left(b-k, a+b; a+b+1, b-k+1; -\frac{u}{\lambda v}\right).$$

The result in (20) follows by combining (22) and (24). □

Remark 4. The product moments of (19) can be expressed in terms of elementary functions. Standard double integration shows that

$$\mathrm{E}\left(U^m V^n\right) = \frac{\lambda^m \Gamma(m+a+b)\Gamma(m+n+a)}{\Gamma(a)\Gamma(m+n+a+b)} \tag{25}$$

for $m \geq 1$ and $n \geq 1$. In particular,

$$\mathrm{Cov}\ (U,V) = \frac{\lambda ab}{(a+b)(1+a+b)}$$

and

$$\mathrm{Corr}\ (U,V) = \frac{\sqrt{b}}{\sqrt{1+a+b}}.$$

Remark 5. The conditional pdfs and cdfs of (19) follow easily. For the pdf (19), the conditional pdf of V given $U = u$ is given by

$$f(v \mid u)=\frac{u^{b}\exp(u/\lambda)v^{-(1+b)}(1-v)^{b-1}\exp\{-u/(\lambda v)\}}{\lambda^{b}\Gamma(b)} \tag{26}$$

for $0<v<1$. The conditional pdf of U given $V=v$ is given by

$$f(u \mid v)=\frac{u^{a+b-1}\exp\{-u/(\lambda v)\}}{(\lambda v)^{a+b}\Gamma(a+b)} \tag{27}$$

for $0<u<\infty$. The corresponding conditional cdfs are

$$F(v \mid u)=\frac{1}{\Gamma(b)}\Gamma\left(b,\frac{u(1-v)}{\lambda v}\right) \tag{28}$$

and

$$F(u \mid v)=\frac{1}{\Gamma(a+b)}\gamma\left(a+b,\frac{u}{\lambda v}\right). \tag{29}$$

Remark 6. The moments of the conditional distributions are as follows. For the pdf (19), using equation (3.383.5) in [**2**], the nth conditional moment of V given $U=u$ can be expressed as

$$\mathrm{E}\left(V^{n} \mid u\right)=\left(\frac{u}{\lambda}\right)^{b}\Psi\left(b,b+1-n,\frac{u}{\lambda}\right) \tag{30}$$

for $n\geq 1$. Some particular values of (30) are

$$\mathrm{E}\left(V \mid u\right)=\left(\frac{u}{\lambda}\right)^{b}\Psi\left(b,b,\frac{u}{\lambda}\right)$$

and

$$\mathrm{Var}\ (V \mid u)=\left(\frac{u}{\lambda}\right)^{b}\left\{\Psi\left(b,b-1,\frac{u}{\lambda}\right)-\left(\frac{u}{\lambda}\right)^{b}\Psi^{2}\left(b,b,\frac{u}{\lambda}\right)\right\}.$$

For the pdf (19), using standard integration, the mth conditional moment of U given $V=v$ can be expressed as

$$\text{(31)} \qquad \mathrm{E}\left(U^m \mid v\right) = \frac{(\lambda v)^m \Gamma\left(a+b+m\right)}{\Gamma\left(a+b\right)}$$

for $m \geq 1$. Some particular values of (31) are

$$\begin{aligned} \mathrm{E}\left(U \mid v\right) &= \lambda v(a+b), \\ \mathrm{Var}\ \left(U \mid v\right) &= (\lambda v)^2 (a+b), \\ \mathrm{Skewness}\ \left(U \mid v\right) &= \frac{2}{\sqrt{a+b}}, \end{aligned}$$

and

$$\mathrm{Kurtosis}\ \left(U \mid v\right) = \frac{3(a+b+2)}{a+b}.$$

4. Construction III. If U is a chi-squared random variable with degrees of freedom ν, then the positive square root $V = \sqrt{U}$ is said to have the chi distribution with degrees of freedom ν. Its pdf is given by

$$\text{(32)} \qquad f(v) = \frac{v^{\nu-1} \exp\left(-v^2/2\right)}{2^{\nu/2-1}\Gamma(\nu/2)}$$

for $v > 0$ and $\nu > 0$. Here, we construct a new bivariate chi distribution. The basis for this construction is the well-known reproductive property of the chi-squared random variables given by the following lemma.

Lemma 3. *If U and V are independent chi-squared random variables with degrees of freedom α and β, respectively, then the sum $U+V$ is also a chi-squared random variable with degrees of freedom $\alpha+\beta$.*

Take U, V and W to be independent chi-squared random variables with degrees of freedom a, b and c, respectively. Then the new bivariate chi distribution can be constructed as follows:

$$\text{(33)} \qquad X = \sqrt{U+W}, \qquad Y = \sqrt{V+W}.$$

By Lemma 3, X and Y will be chi distributed with degrees of freedom $a+c$ and $b+c$, respectively. However, they will be correlated so that (X,Y) will have a bivariate chi distribution over $(0,\infty)\times(0,\infty)$.

Theorem 4 provides expressions for the joint pdf of (X,Y) in terms of the confluent hypergeometric functions in one and two variables.

Theorem 4. *Let U, V and W be independent chi-squared random variables with degrees of freedom a, b and c, respectively. Let X and Y be as in* (33)*. Then, the joint pdf of X and Y is given by*

(34)
$$f(x,y)=\begin{cases} C_1 x^{a+c-1}y^{b-1}\exp(-x^2+y^2/2) & \text{if } x<y \\ \quad \times\Phi((c/2),1-(b/2),(a+c/2),(x^2/y^2),(x^2/2)), & \\ C_2 x^{a-1}y^{b+c-1}\exp(-x^2+y^2/2) & \text{if } x>y \\ \Phi((c/2),1-(a/2),(b+c/2),(y^2/x^2),(y^2/2)), & \end{cases}$$

where the constants C_1 and C_2 are given by

$$\frac{1}{C_1}=2^{(a+b+c/2)-2}\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{a+c}{2}\right) \tag{35}$$

and

$$\frac{1}{C_2}=2^{(a+b+c/2)-2}\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b+c}{2}\right), \tag{36}$$

respectively. An alternative form for (34) *is*

$$f(x,y)=\begin{cases} Cx^{a+c-1}y^{b-1}\exp(-x^2+y^2/2)\sum_{k=0}^{\infty} & \text{if } x<y \\ ((-1)^k\Gamma((c/2)+k)/(\Gamma((a+c/2)+k)))(x^2/y^2)^k & \\ \quad \times {}_1F_1((c/2)+k;(a+c/2)+k;(x^2/2)), & \\ Cx^{a-1}y^{b+c-1}\exp(-x^2+y^2/2)\sum_{k=0}^{\infty} & \text{if } x>y \\ ((-1)^k\Gamma((c/2)+k)/\Gamma((b+c/2)+k))(y^2/x^2)^k & \\ \quad \times {}_1F_1((c/2)+k;(b+c/2)+k;(y^2/2)), & \end{cases} \tag{37}$$

where the constant C is given by

$$\frac{1}{C}=2^{(a+b+c/2)-2}\Gamma\left(\frac{c}{2}\right).$$

Proof. The joint pdf of U, V and W in (33) is:

$$f(u,v,w) = \frac{u^{(a/2)-1}v^{(b/2)-1}w^{(c/2)-1}\exp(-u+v+w/2)}{2^{(a+b+c/2)}\Gamma(a/2)\Gamma(b/2)\Gamma(c/2)}$$

from which it follows that the joint pdf of (X, Y, W) is:

$$\begin{aligned}(38)\quad & f(x,y,w)\\ &= \frac{xy(x^2-w)^{(a/2)-1}(y^2-w)^{(b/2)-1}w^{(c/2)-1}\exp(-x^2+y^2-w/2)}{2^{(a+b+c/2)-2}\Gamma(a/2)\Gamma(b/2)\Gamma(c/2)}.\end{aligned}$$

The joint pdf $f(x,y)$ is the integral of (38) over possible values of w, i.e.,

$$f(x,y) = \frac{xyI(x,y)}{2^{(a+b+c/2)-2}\Gamma(a/2)\Gamma(b/2)\Gamma(c/2)}, \tag{39}$$

where $I(x,y)$ denotes the integral

$$\begin{aligned} I(x,y) = \int_0^{\min(x^2,y^2)} & (x^2-w)^{(a/2)-1}(y^2-w)^{(b/2)-1}w^{(c/2)-1}\\ & \times \exp\left(-\frac{x^2+y^2-w}{2}\right)dw. \end{aligned} \tag{40}$$

If $x < y$, then, setting $t = w/x^2$, (40) reduces to

(41)

$$\begin{aligned} I(x,y) = x^{a+c-2}y^{b-2}\int_0^1 (1-t)^{(a/2)-1}\left(1-\frac{x^2t}{y^2}\right)^{(b/2)-1} & t^{(c/2)-1}\\ & \times \exp\left(\frac{x^2t}{2}\right)dt. \end{aligned}$$

Using equation (3.385) in [**2**], (41) can be expressed in terms of the confluent hypergeometric function of two variables as:

$$I(x,y) = x^{a+c-2}y^{b-2}B\left(\frac{a}{2},\frac{c}{2}\right)\Phi\left(\frac{c}{2},1-\frac{b}{2},\frac{a+c}{2},\frac{x^2}{y^2},\frac{x^2}{2}\right). \tag{42}$$

The first result in (34) follows by substituting (42) into (39). The second result in (34) can be derived similarly. Now let us prove (37). Using the series expansion

$$(1+z)^{\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)} \frac{z^j}{j!},$$

the integral (41) can be expanded as

$$I(x,y) = x^{a+c-2} y^{b-2} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b/2)}{\Gamma(b/2-k)} \left(\frac{x^2}{y^2}\right)^k J(k), \tag{43}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^1 t^{(c/2)+k-1} (1-t)^{(a/2)-1} \exp\left(\frac{x^2 t}{2}\right) dt. \tag{44}$$

Using equation (3.383.1) in [**2**], (44) can be expressed in terms of the confluent hypergeometric function as:

$$J(k) = B\left(\frac{a}{2}, \frac{c}{2}+k\right) {}_1F_1\left(\frac{c}{2}+k; \frac{a+c}{2}+k; \frac{x^2}{2}\right). \tag{45}$$

The first result in (37) follows by combining (43) and (45) and substituting into (39). The second result in (37) can be derived similarly. □

Figure 4 illustrates the shape of the joint pdf (34) for selected values of (a, b, c). It can be seen how changing the parameter values makes the dependence between X and Y inflated or asymmetric.

Remark 7. Using the definition of the confluent hypergeometric function of two variables and equation (6.455.1) in [**2**], the product moments of (34) can be expressed as

$$E\left(X^m Y^n\right) = 2C_1 \delta(a,b,m,n) + 2C_2 \delta(b,a,n,m)$$

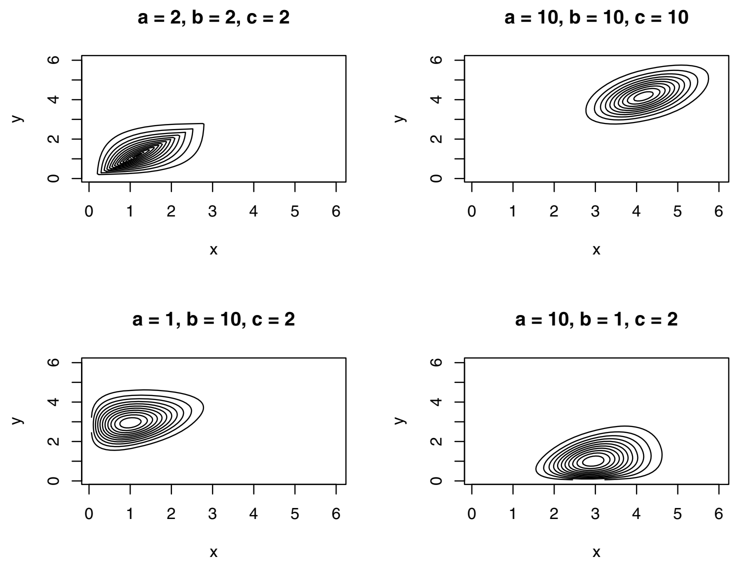

FIGURE 4. Contours of the joint pdf (34) for selected values of (a, b, c).

for $m \geq 1$ and $n \geq 1$, where C_1 and C_2 are given by (35) and (36), respectively, and $\delta(a, b, m, n)$ takes the form

$$\begin{aligned}\delta(a, b, m, n) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(c/2)_{j+k}(1-(b/2))_k}{(a+c/2)_{j+k} j! k! (m+a+c+j)} \\ & \times \Gamma\left(\frac{m+n+a+b+c+j+k}{2}\right) \\ & \times {}_2F_1\left(1, \frac{m+n+a+b+c+j+k}{2}; \right. \\ & \qquad \left. \frac{m+a+c+j+2}{2}; \frac{1}{2}\right).\end{aligned}$$

Remark 8. The conditional pdfs and moments follow easily from those given above. For the pdf (34), the conditional pdf of Y given $X = x$ is given by

$$
(46)\quad f(y \mid x) = \begin{cases} D_1 y^{b-1} \exp(-y^2/2) & \\ \quad \times \Phi((c/2), 1-(b/2), (a+c/2), (x^2/y^2), (x^2/2)) & \text{if } x < y, \\ D_2 x^{-c} y^{b+c-1} \exp(-y^2/2) & \\ \Phi((c/2), 1-(a/2), (b+c/2), (y^2/x^2), (y^2/2)) & \text{if } x > y, \end{cases}
$$

where the constants D_1 and D_2 are given by

$$
\frac{1}{D_1} = 2^{(b/2)-1} \Gamma\left(\frac{b}{2}\right) \tag{47}
$$

and

$$
\frac{1}{D_2} = \frac{2^{(b/2)-1} \Gamma(a/2) \Gamma(b+c/2)}{\Gamma(a+c/2)}, \tag{48}
$$

respectively. The corresponding conditional pdf of X given $Y = y$ can be obtained by interchanging x with y, a with b and m with n. Using the definitions of the confluent hypergeometric function of two variables and the complementary incomplete gamma function, the nth conditional moment of Y given $X = x$ can be expressed as

$$
E\left(Y^n \mid X = x\right) = D_1 E_1 + D_2 E_2,
$$

where D_1 and D_2 are given by (47) and (48), respectively, and E_1 and E_2 are given by

$$
E_1 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c/2)_{j+k} (1-(b/2))_k}{(a+c/2)_{j+k} j! k!} 2^{(n+b-2i-2j-2/2)} x^{2(i+j)} \times \Gamma\left(\frac{n+b-2i}{2}, \frac{x^2}{2}\right)
$$

and

$$
E_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c/2)_{j+k} (1-(a/2))_k}{(b+c/2)_{j+k} j! k!} 2^{(n+b+c+2i-2/2)} x^{-(2i+c)} \times \gamma\left(\frac{n+b+c+2i+2j}{2}, \frac{x^2}{2}\right),
$$

respectively. The corresponding mth conditional moment of X given $Y = y$ can be obtained by interchanging x with y, a with b and m with n.

Lemma 3 can be applied in several ways to generate multivariate chi distributions. If U_j, $j = 1, 2, \ldots, p$ are chi-squared distributed independent random variables with degrees of freedom a_j, then the following will have p-variate chi distributions over $(0, \infty)^p$:

1. $(\sqrt{U_1 + U_2 + \cdots + U_p}, \sqrt{U_1}, \sqrt{U_2}, \ldots, \sqrt{U_{i-1}}, \sqrt{U_{i+1}}, \ldots, \sqrt{U_p})$ for $i = 1, \ldots, p-1$.

2. $(\sqrt{U_1}, \sqrt{U_1 + U_2}, \ldots, \sqrt{U_1 + \cdots + U_p})$.

3. $(\sqrt{U_1 + W}, \sqrt{U_2 + W}, \ldots, \sqrt{U_p + W})$, where W is a chi-squared random variable with degrees of freedom b and is independent of U_j, $j = 1, 2, \ldots, p$.

The ideas presented here will be the subject of a future investigation.

TABLE 1. Locations of the stations.

Location	Station No.	Years of Data	Latitude	Longitude
Clermont	83	1901–2002	28° 32′ N	81° 46′ W
Brooksville	89	1901–2002	28° 33′ N	82° 23′ W
Orlando	91	1901–2001	28° 32′ N	81° 22′ W
Bartow	142	1901–2003	27° 53′ N	81° 50′ W
Avon Park	146	1902–2002	27° 35′ N	81° 30′ W
Arcadia	148	1907–2002	27° 12′ N	81° 51′ W
Kissimmee	160	1901–2002	28° 17′ N	81° 24′ W
Inverness	164	1901–2002	28° 50′ N	82° 19′ W
Plant City	259	1901–2003	28° 01′ N	82° 06′ W
Tarpon Springs	298	1901–2002	28° 08′ N	82° 45′ W
Tampa Intl. Airport	299	1901–2003	27° 58′ N	82° 31′ W
St. Leo	306	1902–2003	28° 20′ N	82° 15′ W
Gainesville	310	1901–2000	29° 39′ N	82° 19′ W
Ocala	333	1901–2002	29° 11′ N	82° 08′ W

5. Application. In this section, we illustrate an application of the results above to extreme rainfall data. The data consists of annual maximum daily rainfall for the years from 1901 to 2003 for 14 locations in west central Florida. The data were obtained from the Department of

Meteorology in Tallahassee, Florida. Table 1 gives the station number, years of data, latitude and the longitude for the 14 locations.

It is of interest to know how the extreme rainfall at anyone of these stations relates to the others. This can be studied by looking at the joint distribution of (X, Y) when X and Y denote the extreme rainfall at the respective stations.

There are $7 \times 13 = 91$ pairs of stations. We fitted (8) to describe the joint distribution of extreme rainfall for each pair. We chose (8) in preference to (4), (19) and (34) because of the highest number of parameters. The data for each pair were taken to be the annual maximum daily rainfall values for the years common to the two stations.

The fitting of (8) was performed by the method of moments. Estimation by the method of maximum likelihood appears intractable because of the presence of the Whittaker functions in (8). Given the simplicity of the product moments in Remark 1, the method of moments is the obvious choice for estimation. Suppose $\{(x_i, y_i), i = 1, 2, \ldots, n\}$ is a random sample with sample means $(\bar{x}, \bar{y})$, sample variances (s_x^2, s_y^2) and sample correlation coefficient r. Consider the transformation $(\tilde{x}_i, \tilde{y}_i) = (\bar{x} x_i / s_x^2, \bar{y} y_i / s_y^2)$ so that its marginals could be assumed to be gamma distributed with unit scale parameters, i.e., $\mu = 1$. If the sample $\{(\tilde{x}_i, \tilde{y}_i)\}$ arises from (8), then the method of moments estimators of a_1, a_2 and c are

$$\widehat{a_1} = \frac{\bar{x}^2}{s_x^2}, \qquad \widehat{a_2} = \frac{\bar{y}^2}{s_y^2} \quad \text{and} \quad \hat{c} = \frac{\sqrt{\widehat{a_1}\widehat{a_2}}}{r},$$

respectively. Table 2 gives the fitted parameter estimates for the 91 pairs of stations. Using the fitted parameter estimates, we plotted the contours of the joint pdf (8) for each pair–to examine the relative change of extreme rainfall among the 14 stations.

TABLE 2. Parameter estimates of (8) for each pair of stations.

Station (X)	Station (Y)	$\widehat{a_1}$	$\widehat{a_2}$	$\hat{c}$
Clermont	Brooksville	3.218846	2.834433	4.040199
Clermont	Orlando	3.242782	3.301213	4.738214
Clermont	Bartow	3.668596	3.730267	5.6373
Clermont	Avon-Park	3.041417	3.228340	6.323879
Clermont	Arcadia	3.093167	2.827528	6.855383
Clermont	Kissimmee	2.852649	2.902481	4.72639

TABLE 2. (Continued).

Station (X)	Station (Y)	$\hat{a_1}$	$\hat{a_2}$	$\hat{c}$
Clermont	Inverness	4.35823	2.512238	8.623658
Clermont	Plant-City	3.574229	3.503682	6.71207
Clermont	Tarpon-Springs	3.200771	3.032279	5.384406
Clermont	Tampa-Intl-Airport	3.210878	3.105294	6.709737
Clermont	St-Leo	3.210878	2.674914	4.582917
Clermont	Gainesville	3.200771	7.820751	32.83452
Clermont	Ocala	3.105414	4.288232	7.32462
Brooksville	Orlando	2.558395	3.085072	3.884535
Brooksville	Bartow	2.730809	2.265939	3.514642
Brooksville	Avon-Park	2.726731	3.212586	5.731151
Brooksville	Arcadia	2.548018	2.544151	4.472605
Brooksville	Kissimmee	2.434280	2.811075	4.743419
Brooksville	Inverness	3.385246	2.507584	6.543513
Brooksville	Plant-City	2.719422	3.2042	4.79122
Brooksville	Tarpon-Springs	2.584794	2.907684	4.242484
Brooksville	Tampa-Intl-Airport	2.573945	2.268199	3.777553
Brooksville	St-Leo	2.573945	2.433441	3.415164
Brooksville	Gainesville	2.584794	8.054324	27.24378
Brooksville	Ocala	2.834384	4.266868	6.694927
Orlando	Bartow	3.853775	2.219964	3.681571
Orlando	Avon-Park	3.294072	3.211248	4.858484
Orlando	Arcadia	3.004763	2.531245	4.27543
Orlando	Kissimmee	2.846413	2.762606	4.377054
Orlando	Inverness	3.448624	2.520210	6.114125
Orlando	Plant-City	3.917996	3.197551	6.1983
Orlando	Tarpon-Springs	3.058132	2.826874	4.180394
Orlando	Tampa-Intl-Airport	3.10071	2.243076	3.650189
Orlando	St-Leo	3.10071	2.342843	3.913142
Orlando	Gainesville	3.058132	7.928737	30.76442
Orlando	Ocala	3.505751	4.236783	7.01554
Bartow	Avon-Park	3.489156	3.776601	6.409493
Bartow	Arcadia	2.204778	2.634857	4.100867
Bartow	Kissimmee	2.037106	2.9251	4.051984
Bartow	Inverness	2.479450	2.625725	6.668833
Bartow	Plant-City	2.240396	3.167257	4.070862
Bartow	Tarpon-Springs	2.255005	2.957736	4.012826

TABLE 2. (Continued).

Station (X)	Station (Y)	$\hat{a_1}$	$\hat{a_2}$	$\hat{c}$
Bartow	Tampa-Intl-Airport	2.265939	2.287860	3.007684
Bartow	St-Leo	2.265939	2.535290	3.419411
Bartow	Gainesville	2.255005	7.888075	111.9936
Bartow	Ocala	4.116757	4.729147	10.67784
Avon-Park	Arcadia	3.211288	2.762028	6.104219
Avon-Park	Kissimmee	3.231778	2.881294	6.454619
Avon-Park	Inverness	3.734841	2.641319	9.147827
Avon-Park	Plant-City	3.793597	3.602429	15.09799
Avon-Park	Tarpon-Springs	3.256159	2.928265	5.515781
Avon-Park	Tampa-Intl-Airport	3.256159	2.945651	5.872365
Avon-Park	St-Leo	3.256159	2.653244	5.204906
Avon-Park	Gainesville	3.256159	8.097768	39.91536
Avon-Park	Ocala	3.302372	4.044312	7.459265
Arcadia	Kissimmee	2.446936	2.701772	4.887486
Arcadia	Inverness	2.59692	2.514762	9.529713
Arcadia	Plant-City	2.684088	3.248164	5.291207
Arcadia	Tarpon-Springs	2.520882	2.868216	5.916902
Arcadia	Tampa-Intl-Airport	2.520882	2.205464	3.804544
Arcadia	St-Leo	2.520882	2.46637	4.605786
Arcadia	Gainesville	2.520882	7.768185	55.68855
Arcadia	Ocala	2.952989	4.272891	10.61715
Kissimmee	Inverness	2.926808	2.898133	6.00001
Kissimmee	Plant-City	2.981039	3.296744	6.712081
Kissimmee	Tarpon-Springs	2.801933	2.733992	5.72304
Kissimmee	Tampa-Intl-Airport	2.825697	2.152856	4.910383
Kissimmee	St-Leo	2.825697	2.321331	3.982177
Kissimmee	Gainesville	2.801933	7.391583	43.41384
Kissimmee	Ocala	2.961115	3.896707	7.132644
Inverness	Plant-City	2.685083	3.352062	18.19546
Inverness	Tarpon-Springs	2.507584	3.238387	8.939196
Inverness	Tampa-Intl-Airport	2.507584	2.380957	6.201889
Inverness	St-Leo	2.507584	2.523381	5.645889
Inverness	Gainesville	2.507584	7.9631	91.93246
Inverness	Ocala	2.44509	4.542143	10.64130
Plant-City	Tarpon-Springs	3.217778	2.955461	6.149223
Plant-City	Tampa-Intl-Airport	3.2042	2.276896	4.72709

TABLE 2. (Continued).

Station (X)	Station (Y)	$\hat{a_1}$	$\hat{a_2}$	$\hat{c}$
Plant-City	St-Leo	3.2042	2.499223	4.533339
Plant-City	Gainesville	3.217778	7.777853	30.44846
Plant-City	Ocala	3.71176	4.69162	21.69946
Tarpon-Springs	Tampa-Intl-Airport	2.888101	2.273758	3.790318
Tarpon-Springs	St-Leo	2.888101	2.411552	3.854949
Tarpon-Springs	Gainesville	2.888101	7.930197	33.20254
Tarpon-Springs	Ocala	3.183238	4.195801	8.470409
Tampa-Intl-Airport	St-Leo	2.288052	2.407778	3.14861
Tampa-Intl-Airport	Gainesville	2.273758	7.930197	35.10345
Tampa-Intl-Airport	Ocala	3.185001	4.255038	10.81237
St-Leo	Gainesville	2.411552	7.930197	26.19865
St-Leo	Ocala	2.77196	4.255038	7.331554
Gainesville	Ocala	8.478297	4.195801	33.17598

For illustrative purposes, we discuss here the results for just 2 of the 91 pairs: Clermont and Plant City (Pair 1) and Clermont and Gainesville (Pair 2). The fitted joint contours of (8) for these two pairs are shown in Figure 5. It is evident from the fitted joint contours that the quality of the fits for both pairs is reasonable. The plot for pair 1 appears asymmetric with the thicker tail on the left and that for pair 2 appears asymmetric with the thicker tail on the right. This suggests that Plant City receives relatively more extreme rainfall compared to Clermont and that Clermont receives relatively more extreme rainfall compared to Gainesville.

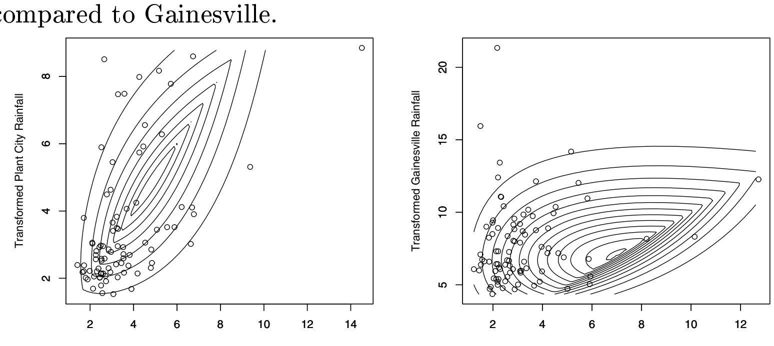

FIGURE 5. Contour plots of the fitted pdf (8) for data from Pair 1 (left) and data from Pair 2 (right).

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