ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 1, 2009

AN EXISTENCE RESULT FOR SOLUTIONS OF NONLINEAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS FOR HIGHER-ORDER p-LAPLACIAN DIFFERENTIAL EQUATIONS

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ABSTRACT. An existence result for solutions of nonlinear two-point boundary value problems of p-Laplacian differential equations is proved. The theorem obtained is different from those known since we don't apply Green's functions of the corresponding problem, and the methods to obtain the a priori bounds of solutions are different enough from those known. An example that cannot be solved by known results is given to illustrate our theorem.

1. Introduction. In [**10**], Erbe and Tang studied the existence of positive solutions of the following Sturm-Liouville boundary value problem consisting of the second order differential equation and the so called Sturm-Liouville boundary conditions

$$\begin{cases} x''(t) = f(t, x(t)) & 0 < t < 1, \\ \alpha x(0) - \beta x'(0) = 0 & \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{1}$$

where f is continuous and nonnegative, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $\delta \geq 0$ with $\alpha\delta+\gamma\delta+\alpha\beta > 0$. This problem comes from the situation involving nonlinear elliptic problems in annular regions, see [**10**]. The authors in [**10, 11**] proved that the above problem has at least one positive solution under the following assumptions:

(B) The function f is continuous and positive on $[0,1] \times [0,+\infty)$ and

$$f_0 = \lim_{y \to 0^+} \max_{t \in [0,1]} \frac{f(t,y)}{y} = 0, \qquad f_\infty = \lim_{y \to \infty} \min_{t \in [0,1]} \frac{f(t,y)}{y} = \infty,$$

i.e., f is super-linear at both endpoints $x = 0$ and $x = \infty$; or

2000 AMS *Mathematics subject classification.* Primary 54B20, 54F15.

The author is supported by the National Natural Sciences Foundation of P.R. China, the Science Foundation of Hunan Province (06JJ50008), the Natural Science foundation of Guangdong province (7004569) and the Science foundation of Hainan province (80402).
Received by the editors on June 20, 2006.
DOI:10.1216/RMJ-2009-39-1-147

$$f_0 = \lim_{y\to 0^+} \min_{t\in[0,1]} \frac{f(t,y)}{y} = \infty, \qquad f_\infty = \lim_{y\to\infty} \max_{t\in[0,1]} \frac{f(t,y)}{y} = 0,$$

i.e., f is sub-linear at both endpoints $x = 0$ and $x = \infty$.

In textbook [**2**], the problem with nonlinear boundary conditions

$$\begin{cases} x''(t) = f(t, x(t)) & 0 < t < 1, \\ x(0) = 0 & x'(1) + \psi(x(1)) = 0 \end{cases}$$

is considered. Using upper and lower solution methods, the authors established an existence result for solutions of the above-mentioned problem.

There are many papers concerned with the problem for p-Laplacian differential equations with linear boundary conditions

$$\begin{cases} [\phi(x'(t))]' = f(t, x(t)) & 0 < t < 1, \\ \alpha x(0) - \beta x'(0) = 0 & \gamma x(1) + \delta x'(1) = 0 \end{cases} \tag{2}$$

or with nonlinear boundary conditions

$$\begin{cases} [\phi(x'(t))]' = f(t, x(t)) & 0 < t < 1, \\ \alpha x(0) - \beta B_0(x'(0)) = 0 & \gamma x(1) + \delta B_1(x'(1)) = 0, \end{cases} \tag{3}$$

for example [**4**–**8, 13**–**16, 18, 19, 22**–**24**]; the authors in these papers established existence results for positive solutions or solutions of the above-mentioned problems by using fixed point theorems in cones of Banach spaces or upper and lower solution methods and the monotone iterative technique. However, the following assumptions were supposed:

(A) There is a constant $M > 0$ so that

$$0 \le B_i(x) \le Mx \text{ for all } x \ge 0, \quad i = 0, 1.$$

For the higher order case, BVPs for nonlinear differential equations have received much attention in obtaining conditions on nonlinearities for which there are either at least one, at least two or at least three positive solutions since they can arise in many applications; one may see Chyan and Henderson [**9**]. The right focal boundary value problem for higher order differential equations

$$\begin{cases} (-)^{n-p}x^{(n)}(t) = f(t, x(t), \dots, x^{(n-1)}(t)) & 0 < t < 1, \\ x^{(i)}(0) = 0 \text{ for } i = 0, 1, \dots, p, \ x^{(i)}(1) = 0, \ i = p+1, \dots, n-1 \end{cases}$$

and its special cases have been studied by many authors [**4, 11, 16**] by using Krasnoselski's fixed point theorem in cones. In [**1**], the authors proved that the above problem has solutions if the nonlinear function f is at most linear growth:

(C) There are nonnegative numbers a_i and L so that

$$|f(t, x_0, \dots, x_{n-1})| \leq L + \sum_{i=0}^{n-1} a_i |x_i|.$$

Recently, Qi, in [**20**], investigated the following Stourm-Liouville boundary value problem for higher-order differential equations in Banach spaces

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) = \theta & 0 < t < 1, \\ x^{(i)}(0) = \theta \text{ for } i = 0, 1, \dots, n-3, \\ \alpha x^{(n-2)}(0) - \beta x^{(n-1)}(0) = \theta, \\ \gamma x^{(n-2)}(1) + \delta x^{(n-1)}(1) = \theta, \end{cases} \tag{4}$$

where α, β, δ, $\gamma \geq 0$. He proved the existence of positive solutions under the assumption $\Delta = \beta\delta + \delta\alpha + \alpha\gamma > 0$ and the assumptions imposed on f as follows:

(D) either

$$\frac{||f(t, x_0, \dots, x_{n-2})||}{\sum_{i=0}^{n-2} ||x_i||} \longrightarrow 0, \qquad \sum_{i=0}^{n-2} ||x_i|| \longrightarrow 0;$$

$$\frac{||f(t, x_0, \dots, x_{n-2})||}{\sum_{i=0}^{n-2} ||x_i||} \longrightarrow +\infty, \quad \sum_{i=0}^{n-2} ||x_i|| \longrightarrow \infty;$$

or

$$\frac{||f(t, x_0, \dots, x_{n-2})||}{\sum_{i=0}^{n-2} ||x_i||} \longrightarrow +\infty, \quad \sum_{i=0}^{n-2} ||x_i|| \longrightarrow 0;$$

$$\frac{||f(t, x_0, \dots, x_{n-2})||}{\sum_{i=0}^{n-2} ||x_i||} \longrightarrow 0, \qquad \sum_{i=0}^{n-2} ||x_i|| \longrightarrow \infty.$$

In [**3, 17, 25**], Agawarl, Lian and Wong studied BVP

$$(5)\quad \begin{cases} [\phi(x^{(n-1)}(t)]' + f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) = 0 \quad 0 < t < 1, \\ x^{(i)}(0) = 0 \text{ for } i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) - B_0(x^{(n-1)}(0)) = 0, \\ x^{(n-2)}(1) + B_1(x^{(n-1)}(1)) = 0 \end{cases}$$

and BVP

$$(6)\quad \begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) = 0 \quad 0 < t < 1, \\ x^{(i)}(0) = 0 \text{ for } i = 0, 1, \dots, n-3, \\ \alpha x^{(n-2)}(0) - \beta x^{(n-1)}(0) = 0, \\ \gamma x^{(n-2)}(1) + \delta x^{(n-1)}(1) = 0. \end{cases}$$

In the results, in which sufficient conditions that guarantee the existence of at least one positive solution are established, in paper [**17**], it is also supposed that B_0 and B_1 in (5) satisfy condition (A). For (6), the following existence result is established by using the upper and lower solution method.

Theorem [**25**]. *Suppose that*

(H) *there exists a function* $g \in ([0,1] \times [0,\infty)^{n-1}; [0,\infty))$ *which satisfies*

$$f(t, 0, 0, \dots, 0) \geq 0 \textit{ on } [0,1] \ (f \textit{ maybe has a negative value for } u_i \neq 0),$$
$$g(t, |u_1|, |u_2|, \dots, |u_{n-1}|) \geq f(t, u_1, u_2, \dots, u_{n-1}) \textit{ on } [0,1] \times R^{n-1}$$

and one of the following:

(i) $\max g_0 = A_1 \in [0, D_1)$ *and* $\min g_\infty = A_2 \in ((D_2)/M, \infty]$,

(ii) $\min g_0 = A_3 \in ((D_2)/M, \infty]$ *and* $\max g_\infty = A_4 \in [0, D_1)$,

(iii) *there exist two nonnegative functions* $h \in C([0,\infty)^{n-1}; [0,\infty))$, *increasing with respect to* $u_{n-1} \in [0,\infty)$, *and* $q \in C([0,1]; [0,\infty))$ *such that*

$$\begin{cases} g(t, u_1, u_2, \dots, u_{n-1}) := q(t) h(u_1, u_2, \dots, u_{n-1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{on } [0,1] \times [0,\infty)^{n-1}, \\ \sup\limits_{u_{n-1} \in (0,\infty)} \ \min\limits_{(u_1, \dots, u_{n-2}) \in [0,\infty)} \dfrac{u_{n-1}}{Q h(u_1, \dots, u_{n-1})} > 1, \end{cases}$$

where

$$\max g_0 := \lim_{u_1,u_2,\dots,u_{n-1}\to 0^+} \max_{t\in[0,1]} \frac{g(t,u_1,u_2,\dots,u_{n-1})}{u_{n-1}},$$
$$\min g_0 := \lim_{u_1,u_2,\dots,u_{n-1}\to 0^+} \min_{t\in[1/2,3/4]} \frac{g(t,u_1,u_2,\dots,u_{n-1})}{u_{n-1}},$$
$$\max g_\infty := \lim_{u_1,u_2,\dots,u_{n-1}\to \infty} \max_{t\in[0,1]} \frac{g(t,u_1,u_2,\dots,u_{n-1})}{u_{n-1}},$$
$$\min g_\infty := \lim_{u_1,u_2,\dots,u_{n-1}\to \infty} \min_{t\in[1/2,3/4]} \frac{g(t,u_1,u_2,\dots,u_{n-1})}{u_{n-1}},$$

$$\left(\int_0^1 k(s,s)\,ds\right)^{-1} := D_1 = \frac{6\rho}{6\delta\beta+3\gamma\beta+\alpha\gamma+3\alpha\delta},$$
$$\left(\int_{1/2}^{3/4} k\left(\frac{1}{2},s\right)ds\right)^{-1} := D_2 = \frac{64\rho}{16\beta\delta+6\beta\gamma+3\alpha\gamma+8\alpha\delta}$$

and

$$Q := \max_{t\in[0,1]} \int_0^1 k(t,s)q(s)\,ds.$$

Then BVP (6) *has at least one nonnegative solution.*

We note that the nonlinearity f of the equation in the above-mentioned papers only depends on $t, x, x', \dots, x^{(n-2)}$ and the growth conditions imposed on f are at most linear growth. In Lian and Wong's theorem mentioned above, it is not easy to check the existence of $\max g_0$, $\min g_0$, $\max g_\infty$ and $\min g_\infty$; on the other hand, if one of them does not exist, problem (6) cannot be solved, and problem (5) cannot be solved if B_0, B_1 don't satisfy condition (A).

Motivated and inspired by the above papers, in this paper, we are concerned with the following higher-order differential equation with p-Laplacian operator

$$[\phi(x^{(n-1)}(t))]' = f(t,x(t),x'(t),\dots,x^{(n-2)}(t)), \quad 0<t<1, \tag{7}$$

subject to the nonlinear two-point boundary value conditions

$$\begin{cases} x^{(i)}(0)=0 \text{ for } i=0,1,\dots,n-3,\\ x^{(n-2)}(0)-B_0(x^{(n-1)}(0))=0,\\ B_1(x^{(n-2)}(1))+x^{(n-1)}(1)=0, \end{cases} \tag{8}$$

where $\phi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi(0) = 0$ with $p > 1$ and the inverse function of ϕ is denoted by ψ.

Our purpose for this paper is to establish existence results for solutions of problems (7) and (8). Our results are new since

(i) we allow that f depends on t, x, $x', \dots, x^{(n-2)}$ and that the degree of variables of f are greater than 1 if f is a polynomial;

(ii) the conditions imposed on B_0 and B_1 are weaker than known ones since we allow that B_0 and B_1 are superlinear or sublinear and even need not be increasing and satisfy condition (A);

(iii) the method of the proof, which is different from the known ones since Green's functions of the corresponding problems and assumptions (A), (B), (C), (H) and (D) are not used, are considerably technical;

(iv) the results here are easy to check.

We will use the classical Banach space $C^k[0,1]$ with the norm

$$||x|| = \max\{||x||_\infty, \dots, |x^{(k)}||_\infty\},$$

and the Sobolev space $W^{n-1,n-2}(0,1)$ defined by

$$W^{n-1,n-2} = \{x : [0,1] \longrightarrow R,\ x, \cdots, x^{(n-2)} \text{ are absolutely continuous on } [0,1],\ x^{(n-1)} \in L^1[0,1]\}.$$

It is easy to transform BVPs (7) and (8) to the system

$$\begin{cases} x^{(n-1)}(t) = \psi(y(t)), \\ y'(t) = f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) \quad 0 < t < 1, \end{cases} \tag{9}$$

$$\begin{cases} x^{(i)}(0) = 0 \text{ for } i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) - B_0(\psi(y(0))) = 0, \\ y(1) + \psi(B_1(x^{(n-2)}(1))) = 0. \end{cases} \tag{10}$$

Let $X = C^{n-2}[0,1] \times C^0[0,1]$ and $Y = L^1[0,1] \times L^1[0,1] \times R^2$. X is endowed with the norm

$$||x|| = \max\left\{||x_1||_\infty, \dots, ||x_1^{(n-2)}||_\infty,\ ||x_2||_\infty\right\}$$

for all $x = (x_1, x_2) \in X$ and Y is endowed with the norm

$$||y||_1 = \max\left\{\int_0^1 |y_1(s)|\,ds,\int_0^1 |y_2(s)|\,ds,\ |a_1|,\ |a_2|\right\}$$

for $y=(y_1,y_2,a_1,a_2)\in Y$. Then X and Y are Banach spaces.

For BVPs (9) and (10), let

$$D(L)=\{(x_1,x_2)\in X,\ (x_1^{(n-1)},x_2')\in Y,\ x_1^{(i)}(0)=0,\ i=0,\dots,n-3\}.$$

Define the linear operator $L: X\cap D(L)\to Y$ and the nonlinear operator $N: X\to Y$ by

$$L\begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix} x_1^{(n-1)}(t)\\ x_2'(t)\\ x_1^{(n-2)}(0)\\ x_2(1)\end{pmatrix}\quad \text{for } x=(x_1,x_2)\in X\cap D(L),$$

$$N\begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix} \psi(x_2(t))\\ f(t,x_1(t),x_1'(t),\dots,x_1^{(n-2)}(t))\\ B_0(\psi(x_2(0))),\\ -\psi(B_1(x_1^{(n-2)}(1)))\end{pmatrix},$$

for $x=(x_1,x_2)\in X$, respectively. It is easy to know that (x,y) is a solution of BVPs (9) and (10) if and only if (x_1,x_2) is a solution of the operator equation $L(x_1,x_2)=N(x_1,x_2)$, and x_1 is a solution of BVPs (7) and (8) if (x_1,x_2) is a solution of BVPs (9) and (10).

This paper is organized as follows. In Section 2, we present an existence result for the solutions of problems (7) and (8). We also give an example to illustrate the main result. In Section 3, the proof of the main result is given.

2. Main result and an example. In this section, we first present sufficient conditions for the existence of solutions of BVPs (7) and (8). Then an example will be given to illustrate the main theorem.

We set the following assumptions that may be used in the main results.

(A_1) There exist continuous functions $h:[0,1]\times R^{n-1}\to R$, and continuous functions $g_i:[0,1]\times R\to R (i=0,1,\dots,n-2)$ and positive numbers $\overline{\beta}$ and m such that f satisfies

$$f(t, x_0, x_1, \dots, x_{n-2}) = h(t, x_0, x_1, \dots, x_{n-2}) + \sum_{i=0}^{n-2} g_i(t, x_i),$$

and also that h satisfies

$$x_{n-2}h(t, x_0, x_1, \dots, x_{n-2}) \geq \overline{\beta}|x_{n-2}|^{m+1}$$

for all $t \in [0, 1]$ and $(x_0, x_1, \dots, x_{n-2}) \in R^{n-1}$ and

$$\lim_{|x|\to\infty} \sup_{t\in[0,1]} \frac{|g_i(t, x)|}{|x|^m} = r_i \in [0, +\infty) \quad \text{for } i = 0, 1, \dots, n-2.$$

(B_1) $f(t, \bullet, \dots, \bullet)$ is continuous and $f(\bullet, x_0, \dots, x_{n-2}) \in L^1[0, 1]$;

(B_2) $\phi(x) = |x|^{p-2}x$ with its inverse function $\psi(x) = |x|^{q-2}x$, where $1/p + 1/q = 1$;

(B_3) B_0 and B_1 are continuous and satisfy $xB_i(x) \geq 0$ for all $x \in R$.

Remark. Conditions (A_1) are imposed on the nonlinearity f; they are different from known ones imposed on the nonlinearity in BVPs. The growth conditions imposed on f are allowed to be super-linear (the degrees of phases variables are allowed to be greater than 1 if it is a polynomial). For functions B_0 and B_1, we don't need assumption (A), which is a linear condition.

Theorem L. *Suppose that* (A_1), (B_1), (B_2) *and* (B_3) *hold. Then* BVP*s* (7) *and* (8) *have at least one solution provided*

$$\sum_{i=0}^{n-3} \frac{r_i}{[(n-i-3)!]^m} + r_{n-2} < \overline{\beta}. \tag{11}$$

The proofs of theorems will be given in Section 3. Now, we present an example to illustrate Theorem L.

Example 2.1. Consider the following problem

(12)
$$\begin{cases} [\phi(x^{(4)}(t))]' = (2 + x^2(t) + 2[x'(t)]^2)[x''(t)]^3 + \sum_{i=0}^{2} a_i [x^{(i)}(t)]^3 + r(t), \\ x(0) = x'(0) = 0, \ x''(0) - 2[x'''(0)]^3 = x'''(1) + 8[x''(1)]^5 = 0. \end{cases}$$

Corresponding to BVPs (7) and (8), $f(t, x_0, x_1, x_2) = (2+x_0^2+2x_1^2)x_2^3 + \sum_{i=0}^{2} a_i x_i^3 + r(t)$. Choose $h(t, x_0, x_1, x_2) = (2 + x_0^2 + 2x_1^2)x_2^3$ and $g_i(t, x_i) = a_i x_i^3$ for $i = 0, 1, 2$. Choose $\overline{\beta} = 2$ and $m = 3$. By Theorem 2.1, it is easy to check that, for each $r \in L^1[0, 1]$, problem (12) has at least one solution if $\sum_{i=0}^{2} |a_i| < 2$.

3. Proof of Theorem L. In this section, we prove Theorem L presented in Section 2. This will be done by using the following fixed point theorem.

Lemma GM [**12, 21**]. *Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with $\operatorname{Ker} L = \{0\}$ and $N : X \to Y$ is L-compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is an open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in (0, 1)$, then there is at least one $x \in \Omega$ so that $Lx = Nx$.*

Proof of Theorem L. From the definitions of X, Y, $D(L)$ and the operators L and N as defined in Section 1, it is easy to show that $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with $\operatorname{Ker} L = \{0\}$ and $N : X \to Y$ is L-compact on any open bounded subset of X. Let

$$\Omega_1 = \{(x, y) \in D(L) : L(x, y) = \lambda N(x, y) \text{ for } \lambda \in [0, 1]\}. \tag{12}$$

We first prove that Ω_1 is bounded. For $(x, y) \in \Omega_1$, we have

$$\begin{cases} x^{(n-1)}(t) = \lambda \psi(y(t)), \\ y'(t) = \lambda f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) \quad 0 < t < 1, \\ x^{(i)}(0) = 0 \text{ for } i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) = \lambda B_0(\psi(y(0))), \\ y(1) = -\lambda \psi(B_1(x^{(n-2)}(1))). \end{cases} \tag{13}$$

We get

$$\left[\phi\left(\frac{x^{(n-1)}(t)}{\lambda}\right)\right]' = \lambda f(t, x(t), x'(t), \dots, x^{(n-2)}(t)).$$

So

$$\begin{aligned}(14)\quad &\left[\phi\left(x^{(n-1)}(t)\right)\right]' x^{(n-2)}(t)\\ &\qquad = \phi(\lambda)\lambda f(t, x(t), x'(t), \dots, x^{(n-2)}(t))x^{(n-2)}(t).\end{aligned}$$

Integrating (14) from 0 to 1, we get from (B_2) and (B_3) that

$$\begin{aligned}\phi(\lambda)\lambda \int_0^1 & f(s, x(s), \dots, x^{(n-2)}(s))x^{(n-2)}(s)\, ds\\ &= \phi(x^{(n-1)}(1))x^{(n-2)}(1) - \phi(x^{(n-1)}(0))x^{(n-2)}(0)\\ &\quad - \int_0^1 \phi(x^{(n-1)}(s))x^{(n-1)}(s)\, ds\\ &= -\phi(\lambda)\lambda\psi(B_1(x^{(n-2)}(1)))x^{(n-2)}(1)\\ &\quad - \phi(\lambda)\lambda y(0)B_0(\psi(y(0)))\\ &\quad - \int_0^1 \phi(x^{(n-1)}(s))x^{(n-1)}(s)\, ds\\ &\le 0.\end{aligned}$$

(A_1) implies that

$$\int_0^1 h(s, x(s), \dots, x^{(n-2)}(s))x^{(n-2)}(s)\, ds + \sum_{i=0}^{n-2}\int_0^1 g_i(s, x^{(i)}(s))x^{(n-2)}(s)\, ds \le 0.$$

Hence,

$$\begin{aligned}\overline{\beta}\int_0^1 |x^{(n-2)}(s)|^{m+1}\, ds &\le -\sum_{i=0}^{n-2}\int_0^1 g_i(s, x^{(i)}(s))x^{(n-2)}(s)\, ds\\ &\le \sum_{i=0}^{n-2}\int_0^1 |g_i(s, x^{(i)}(s))||x^{(n-2)}(s)|\, ds.\end{aligned}$$

From (11), pick $\varepsilon > 0$ so that

$$\overline{\beta} > \sum_{i=0}^{n-3} \frac{r_i + \varepsilon}{[(n-i-3)!]^m} + r_{n-2} + \varepsilon.$$

For such $\varepsilon > 0$, there is a $\delta > 0$ so that

$$|g_i(t,x)| \le (r_i + \varepsilon)|x|^m \text{ for } |x| > \delta \text{ and } t \in [0,1], \quad i = 0, \dots, n-2. \tag{15}$$

Let, for $i = 0, \dots, n-2$,

$$\begin{aligned} \Delta_{1,i} &= \{t : t \in [0,1],\ |x^{(i)}(t)| \le \delta\}, \\ \Delta_{2,i} &= \{t : t \in [0,1],\ |x^{(i)}(t)| > \delta\}, \\ g_{\delta,i} &= \max_{t\in[0,1],|x|\le\delta} |g_i(t,x)|. \end{aligned} \tag{16}$$

We note, for $i = 0, \dots, n-3$, that

$$\begin{aligned} |x^{(i)}(t)| &= \left| \int_0^t \frac{(t-s)^{n-3-i}}{(n-3-i)!} x^{(n-2)}(s)\, ds \right| \\ &\le \frac{1}{(n-3-i)!} \int_0^1 |x^{(n-2)}(s)|\, ds. \end{aligned}$$

Then we get

$$\begin{aligned} &\overline{\beta} \int_0^1 |x^{(n-2)}(s)|^{m+1}\, ds \\ &\quad \le \sum_{i=0}^{n-2} \int_{\Delta_{1,i}} |g_i(s, x^{(i)}(s))||x^{(n-2)}(s)|\, ds \\ &\qquad + \sum_{i=0}^{n-2} \int_{\Delta_{2,i}} |g_i(s, x^{(i)}(s))||x^{(n-2)}(s)|\, ds \\ &\quad \le \sum_{i=0}^{n-2} g_{\delta,i} \left(\int_0^1 |x^{(n-2)}(s)|^{m+1}\, ds \right)^{1/(m+1)} \\ &\qquad + (r_{n-2} + \varepsilon) \int_0^1 |x^{(n-2)}(s)|^{m+1}\, ds \\ &\qquad + \sum_{i=0}^{n-3} \frac{r_i + \varepsilon}{[(n-3-i)!]^m} \int_0^1 |x^{(n-2)}(s)|^{m+1}\, ds. \end{aligned}$$

One has

$$\left(\overline{\beta}-\sum_{i=0}^{n-3}\frac{r_i+\varepsilon}{[(n-3-i)!]^m}-(r_{n-2}+\varepsilon)\right)\int_0^1|x^{(n-2)}(s)|^{m+1}\,ds \leq \sum_{i=0}^{n-2}g_{\delta,i}\left(\int_0^1|x^{(n-2)}(s)|^{m+1}\,ds\right)^{1/(m+1)}.$$

It is easy to see from the definition of ε that there is an $M>0$ such that

$$\int_0^1|x^{(n-2)}(s)|^{m+1}\,ds\leq M.$$

Hence, for $i=0,\dots,n-3$, we get

$$||x^{(i)}||_\infty \leq \frac{1}{(n-3-i)!}\int_0^1|x^{(n-2)}(s)|\,ds\leq\frac{1}{(n-3-i)!}M^{1/(m+1)}.$$

Now, we consider $||x^{(n-2)}||_\infty$. It follows from the above inequality that there is a $t_0\in[0,1]$ such that $|x^{(n-2)}(t_0)|\leq M^{1/(m+1)}$.

For $t\leq t_0$, we get from (14) that

$$\begin{aligned}\phi(\lambda)\lambda\int_0^t f(s,x(s),\dots,x^{(n-2)}(s))x^{(n-2)}(s)\,ds &= \phi(x^{(n-1)}(t))x^{(n-2)}(t)-\phi(x^{(n-1)}(0))x^{(n-2)}(0)\\ &\quad -\int_0^t\phi(x^{(n-1)}(s))x^{(n-1)}(s)\,ds\\ &\leq \phi(x^{(n-1)}(t))x^{(n-2)}(t).\end{aligned}$$

Thus, we have

$$x^{(n-1)}(t)\psi(x^{(n-2)}(t)) \geq \psi\left(\phi(\lambda)\lambda\int_0^t f(s,x(s),\dots,x^{(n-2)}(s))x^{(n-2)}(s)\,ds\right).$$

It follows that

$$\int_t^{t_0}x^{(n-1)}(s)\psi(x^{(n-2)}(s))\,ds \geq \int_t^{t_0}\psi\left(\phi(\lambda)\lambda\int_0^s f(u,x(u),\dots,x^{(n-2)}(u))x^{(n-2)}(u)\,du\right)\,ds.$$

Hence,

$$
\begin{aligned}
&\frac{1}{q}|x^{(n-2)}(t)|^q \\
&\le \frac{1}{q}|x^{(n-2)}(t_0)|^q - \int_t^{t_0} \psi\left(\phi(\lambda)\lambda\int_0^s f(u, x(u), \dots,\right. \\
&\qquad\qquad \left. x^{(n-2)}(u))x^{(n-2)}(u)\,du\right)\,ds \\
&\le \frac{1}{q}M^{q/(m+1)} - \int_t^{t_0}\psi\biggl(\phi(\lambda)\lambda\int_0^s\biggl(h(u, x(u), \dots, x^{(n-2)}(u)) \\
&\qquad\qquad\qquad + \sum_{i=0}^{n-2} g_i(u, x^{(i)}(u))\biggr)x^{(n-2)}(u)\,du\biggr)\,ds \\
&\le \frac{1}{q}M^{q/(m+1)} \\
&\quad - \int_t^{t_0}\psi\biggl(\phi(\lambda)\lambda\biggl(\int_0^s \overline{\beta}|x^{(n-2)}(u)|^{m+1}\,du \\
&\qquad\qquad\qquad + \sum_{i=0}^{n-2}\int_0^s g_i(u, x^{(i)}(u))x^{(n-2)}(u)\,du\biggr)\biggr)\,ds \\
&\le \frac{1}{q}M^{q/(m+1)} \\
&\quad - \int_t^{t_0}\psi\biggl(\phi(\lambda)\lambda\int_0^s\biggl(\sum_{i=0}^{n-2} g_i(u, x^{(i)}(u))\biggr)x^{(n-2)}(u)\,du\biggr)\,ds \\
&\le \frac{1}{q}M^{q/(m+1)} + \psi\biggl(\int_0^1\sum_{i=0}^{n-2}|g_i(u, x^{(i)}(u))||x^{(n-2)}(u)|\,du\biggr) \\
&\le \frac{1}{q}M^{q/(m+1)} + \psi\biggl(\sum_{i=0}^{n-2} g_{\delta,i}\int_0^1|x^{(n-2)}(s)|\,ds \\
&\qquad\qquad\qquad + \sum_{i=0}^{n-3}\frac{r_i+\varepsilon}{[(n-3-i)!]^m}\biggl(\int_0^1|x^{(n-2)}(u)|\,du\biggr)^{m+1} \\
&\qquad\qquad\qquad + (r_{n-2}+\varepsilon)\int_0^1|x^{(n-2)}(s)|^{m+1}\,ds\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{q}M^{q/(m+1)} + \psi\bigg(\sum_{i=0}^{n-2} g_{\delta,i}\bigg(\int_0^1 |x^{(n-2)}(s)|^{m+1}\,ds\bigg)^{1/(m+1)} \\
&\qquad\qquad\qquad + \sum_{i=0}^{n-3} \frac{r_i+\varepsilon}{[(n-3-i)!]^m} \int_0^1 |x^{(n-2)}(u)|^{m+1}\,du \\
&\qquad\qquad\qquad + (r_{n-2}+\varepsilon)\int_0^1 |x^{(n-2)}(s)|^{m+1}\,ds\bigg) \\
&\le \frac{1}{q}M^{q/(m+1)} \\
&\quad + \psi\bigg(\sum_{i=0}^{n-2} g_{\delta,i} M^{1/(m+1)} + \sum_{i=0}^{n-3} \frac{r_i+\varepsilon}{[(n-3-i)!]^m} M + (r_{n-2}+\varepsilon)M\bigg).
\end{aligned}
$$

For $t \ge t_0$, we have

$$
\begin{aligned}
\phi(\lambda)\lambda \int_t^1 & f(s,x(s),\dots,x^{(n-2)}(s))x^{(n-2)}(s)\,ds \\
&= \phi(x^{(n-1)}(1))x^{(n-2)}(1) - \phi(x^{(n-1)}(t))x^{(n-2)}(t) \\
&\quad - \int_t^1 \phi(x^{(n-1)}(s))x^{(n-1)}(s)\,ds \\
&\le -\phi(x^{(n-1)}(t))x^{(n-2)}(t).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
x^{(n-1)}&(t)\psi(x^{(n-2)}(t)) \\
&\le -\psi\left(\phi(\lambda)\lambda\int_t^1 f(u,x(u),\dots,x^{(n-2)}(u))x^{(n-2)}(u)\,du\right).
\end{aligned}
$$

We get

$$
\begin{aligned}
\frac{1}{q}|x^{(n-2)}(t)|^q \le \frac{1}{q}&|x^{(n-2)}(t_0)|^q \\
&- \int_{t_0}^t \psi\left(\phi(\lambda)\lambda\int_s^1 f(u,x(u),\right. \\
&\qquad\qquad \left.\dots,x^{(n-2)}(u))x^{(n-2)}(u)\,du\right)\,ds.
\end{aligned}
$$

Similar to the above argument, we can get

$$\begin{aligned}\frac{1}{q}|x^{(n-2)}(t)|^q \le \frac{1}{q}M^{1/(m+1)} &+ \sum_{i=0}^{n-2} g_{\delta,i}M^{1/(m+1)}\\ &+ \sum_{i=0}^{n-3}\frac{r_i+\varepsilon}{[(n-3-i)!]^m}M + (r_{n-2}+\varepsilon)M.\end{aligned}$$

It follows from the above discussion that there is a constant $M_1 > 0$ so that $||x^{(n-2)}||_\infty \le M_1$. Hence,

$$||x^{(i)}||_\infty \le \frac{1}{(n-3-i)!}\,M_1, \quad i = 0, \dots, n-3.$$

Now, we consider $||y||_\infty$. It follows from (13) that $|y(1)| \le \psi(B_1(M_1))$. So (13) implies that

$$\begin{aligned}|y(t)| &= \left|y(1) - \lambda\int_t^1 f(s, x(s), \dots, x^{(n-2)}(s)\,ds\right|\\ &\le \psi(B_1(M_1)) + \int_0^1 |f(s, x(s), \dots, x^{(n-2)}(s)|\,ds\\ &\le \psi(B_1(M_1)) + \int_0^1 \max_{\substack{|x_i|\le(1/(n-3-i)!)M_1\\ i=0,\dots,n-3\\ |x_{n-2}|\le M_1}} |f(t, x_0, \dots, x_{n-2})|\,dt\\ &=: M_2.\end{aligned}$$

It follows that

$$||(x,y)|| \le \max\left\{M_2, \frac{1}{(n-3-i)!}M_1,\ i = 0, \dots, n-3, M_1\right\} =: M_3.$$

Let $\Omega = \{(x,y) \in X : ||x|| < M_3 + 1\}$. Then $L(x,y) \ne \lambda N(x,y)$ for $\lambda \in [0,1]$ and $(x,y) \in D(L) \cap \partial\Omega$. It follows from Lemma GM that $L(x,y) = N(x,y)$ has at least one solution in Ω. Then BVPs (7) and (8) have at least one solution $x + x_0$. The proof is complete. $\square$

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