CURVATURE PROPERTIES OF
THE CHERN CONNECTION OF TWISTOR SPACES

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ABSTRACT. The twistor space $\mathcal{Z}$ of an oriented Riemannian 4-manifold M admits a natural 1-parameter family of Riemannian metrics h_t compatible with the almost complex structures J_1 and J_2 introduced, respectively, by Atiyah, Hitchin and Singer, and Eells and Salamon. In this paper we compute the first Chern form of the almost Hermitian manifold $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$ and find the geometric conditions on M under which the curvature of its Chern connection D^n is of type $(1, 1)$. We also describe the twistor spaces of constant holomorphic sectional curvature with respect to D^n and show that the Nijenhuis tensor of J_2 is D^2-parallel provided the base manifold M is Einstein and self-dual.

1. Introduction. It is well known [14, 17] that every almost Hermitian manifold admits a unique connection for which the almost complex structure and the metric are parallel and the $(1,1)$-part of the torsion vanishes. It is usually called the Chern connection because, in the integrable case, it coincides with the Chern connection [7] of the tangent bundle considered as a Hermitian holomorphic bundle. This connection plays an important role in (almost) complex geometry since, by the Chern-Weil theory, the Chern classes of the manifold are directly related to its curvature.

Motivated by the recent works of Donaldson [11] and LeBrun [16], Apostolov and Dragichi [2] proposed to study the problem of existence of almost-Kähler structures of constant Hermitian scalar curvature and/or type $(1, 1)$ Ricci form of its Chern connection (from now on, we refer to it as the first Chern form). Our main purpose here is to show that the twistor spaces of self-dual Einstein 4-manifolds of negative scalar curvature admit such almost-Kähler structures.

Recall that the twistor space of an oriented Riemannian 4-manifold M is the 2-sphere bundle $\mathcal{Z}$ on M consisting of the unit (-1)-eigenvectors of the Hodge star operator acting on $\Lambda^2 TM$. The 6-manifold $\mathcal{Z}$ admits a natural 1-parameter family of Riemannian metrics h_t such that the natural projection $\pi : \mathcal{Z} \to M$ is a Riemannian submersion with totally geodesic fibres. These metrics are compatible with the almost-complex structures J_1 and J_2 on $\mathcal{Z}$ introduced, respectively, by Atiyah, Hitchin and Singer [3] and Eells and Salamon [12].

In Section 3, we show that the first Chern form of the almost Hermitian manifold $(\mathcal{Z}, h_t, J_2)$ always vanishes, which generalizes a result of Eells and Salamon [12] stating that the almost-complex structure J_2 has vanishing first Chern class. We obtain also an explicit formula for the first Chern form of $(\mathcal{Z}, h_t, J_1)$ in terms of the curvature of the base manifold M. In the case when M is self-dual, the latter formula has been actually given by Gauduchon [13].

In Section 4, we obtain the precise geometric conditions on M ensuring that the curvature of the Chern connection D^n of $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$, is of type $(1, 1)$. Note that, in many cases, this property of the curvature simplifies the computation of the Chern numbers, cf. e.g., [15]. We also study the problem when the connection D^n, $n = 1, 2$, has a constant holomorphic sectional curvature. The motivation behind this is the open question whether there are examples of non Kähler Hermitian manifolds whose Chern connection is of nonzero constant holomorphic sectional curvature, cf. [4, 5]. Proposition 3 shows that there are no twistorial examples of such manifolds.

In the last section, we prove that the Nijenhuis tensor of the almost-complex structure J_2 is D^2-parallel, provided that the base manifold M is Einstein and self-dual. Since in (real) dimension six, the Nijenhuis tensor can be identified via the metric with a section of the canonical bundle, the result strengthens the fact that $c_1(\mathcal{Z}, J_2) = 0$. If, in addition, M is of negative scalar curvature s, then the twistor space $(\mathcal{Z}, h_t, J_2)$, $t = -(12)/s$, is an almost- Kähler manifold with vanishing first Chern form, the curvature of its Chern connection is of type $(1, 1)$ and the Nijenhuis tensor of J_2 is parallel with respect to it. Finally, we note that the analogous statements for the twistor spaces of quaternionic-Kähler manifolds are also valid.

2. Preliminaries. Let M be a (connected) oriented Riemannian 4-manifold with metric g. Then g induces a metric on the bundle $\Lambda^2 TM$ of 2-vectors by the formula

$$g(X_1 \wedge X_2, X_3 \wedge X_4) = \frac{1}{2}[g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)].$$

The Riemannian connection of M determines a connection on the vector bundle $\Lambda^2 TM$ (both denoted by ∇), and the respective curvatures are related by

$$R(X, Y)(Z \wedge T) = R(X, Y)Z \wedge T + X \wedge R(Y, Z)T$$

for $X, Y, Z, T \in \chi(M)$; $\chi(M)$ stands for the Lie algebra of smooth vector fields on M. (For the curvature tensor R, we adopt the following definition $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$). The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T)$$

for all $X, Y, Z, T \in \chi(M)$. The Hodge star operator defines an endomorphism $*$ of $\Lambda^2 TM$ with $*^2 = \mathrm{Id}$. Hence,

$$\Lambda^2 TM = \Lambda_+^2 TM \oplus \Lambda_-^2 TM,$$

where $\Lambda_\pm^2 TM$ are the subbundles of $\Lambda^2 TM$ corresponding to the (± 1)-eigenvectors of $*$. Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM. Set

$$
\begin{aligned}
&s_1 = E_1 \wedge E_2 - E_3 \wedge E_4 \qquad \bar{s}_1 = E_1 \wedge E_2 + E_3 \wedge E_4 \\
(1) \quad &s_2 = E_1 \wedge E_3 - E_4 \wedge E_2 \qquad \bar{s}_2 = E_1 \wedge E_3 + E_4 \wedge E_2 \\
&s_3 = E_1 \wedge E_4 - E_2 \wedge E_3 \qquad \bar{s}_3 = E_1 \wedge E_4 + E_2 \wedge E_3.
\end{aligned}
$$

Then (s_1, s_2, s_3), respectively $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$, is a local oriented orthonormal frame of $\Lambda_-^2 TM$, respectively $\Lambda_+^2 TM$. The matrix of $\mathcal{R}$ with respect to the frame $(\bar{s}_i, s_i)$ of $\Lambda^2 TM$ has the form

$$\mathcal{R} = \begin{bmatrix} A & B \\ {}^t B & C \end{bmatrix},$$

where the 3×3-matrices A and C are symmetric and have equal traces. Let $\mathcal{B}$, $\mathcal{W}_+$ and $\mathcal{W}_-$ be the endomorphisms of $\Lambda^2 TM$ with matrices:

$$\mathcal{B} = \begin{bmatrix} 0 & B \\ {}^tB & 0 \end{bmatrix}, \ \mathcal{W}_+ = \begin{bmatrix} A - (s/6)I & 0 \\ 0 & 0 \end{bmatrix}, \ \mathcal{W}_- = \begin{bmatrix} 0 & 0 \\ 0 & C - (s/6)I \end{bmatrix},$$

where s is the scalar curvature and I is the unit 3×3-matrix. Then

$$\mathcal{R} = \frac{s}{6}\mathrm{Id} + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-$$

is the irreducible decomposition of $\mathcal{R}$ under the action of $SO(4)$ found by Singer and Thorpe [20]. Note that $\mathcal{B}$ and $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ represent the traceless Ricci tensor and the Weyl conformal tensor, respectively. The manifold M is called self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ ($\mathcal{W}_+ = 0$). It is Einstein exactly when $\mathcal{B} = 0$.

The twistor space of M is the subbundle $\mathcal{Z}$ of $\Lambda^2_- TM$ consisting of all unit vectors. The Riemannian connection ∇ of M gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. More precisely, let $\pi : \Lambda^2_- TM \to M$ be the natural projection. By definition, the vertical space at $\sigma \in \mathcal{Z}$ is $\mathcal{V}_\sigma = \mathrm{Ker}\,\pi_{*\sigma}$ ($T_\sigma \mathcal{Z}$ is always considered to be a subspace of $T_\sigma(\Lambda^2_- TM)$). Note that $\mathcal{V}_\sigma$ consists of those vectors of $T_\sigma \mathcal{Z}$ which are tangent to the fiber $\mathcal{Z}_p = \pi^{-1}(p) \cap \mathcal{Z}$, $p = \pi(\sigma)$, of $\mathcal{Z}$ through the point σ. Since $\mathcal{Z}_p$ is the unit sphere in the vector space $\Lambda^2_- T_pM$, $\mathcal{V}_\sigma$ is the orthogonal complement of σ in $\Lambda^2_- T_pM$. Let ξ be a local section of $\mathcal{Z}$ such that $\xi(p) = \sigma$. Since ξ has a constant length, $\nabla_X \xi \in \mathcal{V}_\sigma$ for all $X \in T_pM$. Given $X \in T_pM$, the vector $X^h_\sigma = \xi_* X - \nabla_X \xi \in T_\sigma \mathcal{Z}$ depends only on p and σ. By definition, the horizontal space at σ is $\mathcal{H}_\sigma = \{X^h_\sigma : X \in T_pM\}$. Note that the map $X \to X^h_\sigma$ is an isomorphism between T_pM and $\mathcal{H}_\sigma$ with inverse map $\pi_* \mid \mathcal{H}_\sigma$.

Let (U, x_1, x_2, x_3, x_4) be a local coordinate system of M, and let (E_1, E_2, E_3, E_4) be an oriented orthonormal frame of TM on U. If (s_1, s_2, s_3) is the local frame of $\Lambda^2_- TM$, defined by (1), then $\tilde{x}_i = x_i \circ \pi$, $y_j(\sigma) = g(\sigma, (s_j \circ \pi)(\sigma))$, $1 \le i \le 4$, $1 \le j \le 3$, are local coordinates of $\Lambda^2_- TM$ on $\pi^{-1}(U)$. For each vector field

$$X = \sum_{i=1}^4 X^i \frac{\partial}{\partial x_i}$$

on U, the horizontal lift X^h of X on $\pi^{-1}(U)$ is given by

$$(2) \qquad X^h = \sum_{i=1}^{4}(X^i \circ \pi)\frac{\partial}{\partial \tilde{x}_i} - \sum_{j,k=1}^{3} y_j g(\nabla_X s_j, s_k) \circ \pi \frac{\partial}{\partial y_k}.$$

Let $\sigma \in \mathcal{Z}$ and $\pi(\sigma) = p$. Using (2) and the standard identification $T_\sigma(\Lambda^2_- T_p M) \cong \Lambda^2_- T_p M$, one gets that

$$(3) \qquad\qquad [X^h, Y^h]_\sigma - [X,Y]^h_\sigma = R_p(X \wedge Y)\sigma$$

for all $X, Y \in \chi(U)$.

Each point $\sigma \in \mathcal{Z}$ defines a complex structure K_σ on $T_p M$ by

$$(4) \qquad\qquad g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y), \quad X, Y \in T_p M.$$

Note that K_σ is compatible with the metric g and the opposite orientation of M at p. The 2-vector 2σ is dual to the fundamental 2-form of K_σ.

Denote by $\times$ the usual vector product in the oriented 3-dimensional vector space $\Lambda^2_- T_p M$, $p \in M$. Then it is easily checked that

$$(5) \qquad\qquad g(R(a)b, c) = -g(\mathcal{R}(a), b \times c))$$

for $a \in \Lambda^2 T_p M$, $b, c \in \Lambda^2_- T_p M$ and

$$(6) \quad g(\sigma \times V, X \wedge K_\sigma Y) = g(\sigma \times V, K_\sigma X \wedge Y) = -g(V, X \wedge Y)$$

for $V \in \mathcal{V}_\sigma$, $X, Y \in T_p M$.

It is also easy to check that, for any $\sigma, \tau \in \mathcal{Z}$ with $\pi(\sigma) = \pi(\tau)$, we have

$$(7) \qquad\qquad K_\sigma \circ K_\tau = -g(\sigma, \tau)\mathrm{Id} - K_{\sigma \times \tau}.$$

Following [3] and [12], define two almost-complex structures J_1 and J_2 on $\mathcal{Z}$ by

$$J_n V = (-1)^n \sigma \times V \text{ for } V \in \mathcal{V}_\sigma$$
$$J_n X^h_\sigma = (K_\sigma X)^h_\sigma \text{ for } X \in T_p M, p = \pi(\sigma).$$

It is well-known [3] that J_1 is integrable, i.e., comes from a complex structure, if and only if M is self-dual. Unlike J_1, the almost-complex structure J_2 is never integrable [12].

Let h_t be the Riemannian metric on $\mathcal{Z}$ given by

$$h_t = \pi^* g + t g^v,$$

where $t > 0$, g is the metric of M and g^v is the restriction of the metric of $\Lambda^2 TM$ to the vertical distribution $\mathcal{V}$. Then $\pi : (Z, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibers, and the almost-complex structures J_1 and J_2 are compatible with the metrics h_t. Denote by D ($= D_t$) the Levi-Civita connection of $(\mathcal{Z}, h_t)$. Let σ be a point of $\mathcal{Z}$, X, Y vector fields on M near the point $\pi(\sigma)$ and A a vertical vector field near σ. It is not hard to see, cf. e.g., [8], that

$$(8) \qquad (D_{X^h} Y^h)_\sigma = (\nabla_X Y)^h_\sigma + \frac{1}{2} R(X \wedge Y)\sigma$$

$$(9) \qquad (D_A X^h)_\sigma = \mathcal{H}(D_{X^h} A)_\sigma = \frac{t}{2}(R(\sigma \times A)X)^h_\sigma.$$

3. The first Chern forms of twistor spaces. Given an almost-Hermitian manifold (N, g, J), denote by ∇ the Levi-Civita connection of g. Then the Chern connection $\widetilde{\nabla}$ of (N, g, J) is defined by, cf. e.g., [15, Theorem 6.1]:

$$
\begin{aligned}
(10) \qquad g(\tilde{\nabla}_X Y, Z) = {} & g(\nabla_X Y, Z) + \frac{1}{2} g((\nabla_X J)(JY), Z) \\
& + \frac{1}{4} g((\nabla_Z J)(JY) - (\nabla_Y J)(JZ) \\
& \qquad - (\nabla_{JZ} J)(Y) + (\nabla_{JY} J)(Z), X).
\end{aligned}
$$

It is one of the distinguished 1-parameter family of Hermitian connections defined by Gauduchon [14]:

$$
\begin{aligned}
(11) \qquad g(\tilde{\nabla}^u_X Y, Z) = {} & g(\nabla_X Y, Z) + \frac{1}{2} g((\nabla_X J)(JY), Z) \\
& + \frac{u}{4} g((\nabla_Z J)(JY) - (\nabla_Y J)(JZ) \\
& \qquad - (\nabla_{JZ} J)(Y) + (\nabla_{JY} J)(Z), X).
\end{aligned}
$$

The Chern connection corresponds to $u = 1$. Let $\Omega(X, Y) = g(JX, Y)$ be the Kähler form of (N, g, J) and $\delta\Omega$ the codifferential of Ω with respect to ∇. Denote by φ and ψ the 2-forms on N defined by

$$(12) \qquad \varphi(X, Y) = \mathrm{Trace}\,(Z \to g((\nabla_X J)(JZ), (\nabla_Y J)(Z))),$$
$$(13) \qquad \psi(X, Y) = \rho^*(X, JY),$$

where ρ^* is the $*$-Ricci tensor of (N, g, J). Recall that ρ^* is given by

$$\rho^*(X, Y) = \mathrm{Trace}\,(Z \to R(JZ, X)JY),$$

where R is the curvature tensor of ∇.

Lemma 1. *The first Chern form γ^u of the connection $\widetilde{\nabla}^u$ on an almost Hermitian manifold (N, g, J) is given by*

$$8\pi\gamma^u = -\varphi - 4\psi + 2u\,d\delta\Omega.$$

Proof. Denote by $\widehat{\nabla}$ the connection on N defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)(JY), \quad X, Y \in \chi(N).$$

Note that $\widehat{\nabla} g = 0$ and $\widehat{\nabla} J = 0$. Let S be the (1,2)-tensor field on N defined by
$$(14)$$
$$g(S(X, Y), Z) = \frac{1}{4}g((\nabla_Z J)(JY) - (\nabla_Y J)(JZ) - (\nabla_{JZ} J)(Y)$$
$$+ (\nabla_{JY} J)(Z), X).$$

Then
$$\widetilde{\nabla}^u_X Y = \widehat{\nabla}_X Y + uS(X, Y).$$

Below we consider only the case $u = 1$ since the general case follows in the same way. It is easy to check that S has the following properties:

$$(15) \qquad\qquad g(S(X, Y), Z) = -g(S(X, Z), Y),$$

$$(16) \qquad\qquad S(X, JY) = JS(X, Y),$$

$$(17) \quad S(X, JX) = -S(JX, X) = \frac{1}{4}((\nabla_X J)(X) + (\nabla_{JX} J)(JX)),$$

$$(18) \qquad\qquad g((\widehat{\nabla}_Y S)(JX, JX), X) = 0.$$

A straightforward computation shows that the curvature tensors R, $\widehat{R}$ and $\widetilde{R}$ of ∇, $\widehat{\nabla}$ and $\widetilde{\nabla}$ are related by

$$(19)$$
$$
\begin{aligned}
4\widehat{R}(X, Y, Z, W) = \ & 2R(X, Y, Z, W) + 2R(X, Y, JZ, JW) \\
& + g((\nabla_X J)(Z), (\nabla_Y J)(W)) \\
& - g((\nabla_X J)(W), (\nabla_Y J)(Z)),
\end{aligned}
$$

$$(20)$$
$$
\begin{aligned}
\widetilde{R}(X, Y, Z, W) = \ & \widehat{R}(X, Y, Z, W) - g((\widehat{\nabla}_X S)(Y, Z), W) \\
& + g((\widehat{\nabla}_Y S)(X, Z), W) + g(S(X, W), S(Y, Z)) \\
& - g(S(Y, W), S(X, Z)) - g(S(\widehat{T}(X, Y), Z), W),
\end{aligned}
$$

where $\widehat{T}$ is the torsion of $\widehat{\nabla}$.

Now fix a point $p \in N$, and choose an orthonormal frame $E_1, \dots, E_n$, $JE_1, \dots, JE_n$ near p such that $\widehat{\nabla} E_i \mid_p = 0$, $i = 1, \dots, n$. Then, using (15), (16) and (20), one gets:

$$
\begin{aligned}
4\pi\gamma(X, Y) = \ & \sum_{k=1}^{2n} \widetilde{R}(X, Y, E_k, JE_k) \\
= \ & \sum_{k=1}^{2n} [\widehat{R}(X, Y, E_k, JE_k) \\
& - X(g(S(Y, E_k), JE_k)) + Y(g(S(X, E_k), JE_k)) \\
& + g(S([X, Y,], E_k), JE_k)],
\end{aligned}
$$

at the point p.

Formula (19) together with the first Bianchi identity gives:

$$\sum_{k=1}^{2n} \widehat{R}(X, Y, E_k, JE_k) = -2\psi(X, Y) - \frac{1}{2}\varphi(X, Y).$$

Moreover, by (14), one has:

$$\sum_{k=1}^{2n} g(S(X, E_k), JE_k) = -\delta\Omega(X),$$

and the lemma follows from the above identities.

Now let M be an oriented Riemannian 4-manifold with twistor space $\mathcal{Z}$. Let $D(= D_t)$ be the Levi-Civita connection of $(\mathcal{Z}, h_t)$. Denote by $D^n(= D_t^n)$ the Chern connection of the almost-Hermitian manifold $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$, and by $\gamma_{t,n}$ its first Chern form. In the case when the base manifold M is self-dual an explicit formula for the first Chern form of D^1 has been given by Gauduchon [13]. Here we compute the first Chern forms $\gamma_{t,n}$, $n = 1, 2$, of the twistor space of an arbitrary oriented 4-manifold M. To do this, we shall use the following formulas for the covariant derivative of the almost-complex structure J_n with respect to the Levi-Civita connection D ([18]):

Lemma 2. *For any $\sigma \in \mathcal{Z}$, $A \in \mathcal{V}_\sigma$ and $X, Y \in T_pM$, $p = \pi(\sigma)$, one has*

$$h_t((D_{X^h}J_n)(Y^h), A)$$
$$= \frac{t}{2}[(-1)^n g(\mathcal{R}(A), X \wedge Y) - g(\mathcal{R}(\sigma \times A), X \wedge K_\sigma Y)]$$

$$h_t(D_A J_n)(X^h), Y^h)$$
$$= \frac{t}{2}g(\mathcal{R}(\sigma \times A), X \wedge K_\sigma Y + K_\sigma X \wedge Y) + 2g(A, X \wedge Y),$$

where K_σ is the complex structure on T_pM defined via (4). Moreover,

$$h_t(D_E J_n)(F), G) = 0,$$

whenever E, F, G are horizontal vectors or at least two of them are vertical vectors.

We shall also need the following formula for the $*$-Ricci tensor $\rho_{t,n}^*$ of $(\mathcal{Z}, h_t, J_n)$ [10]:

Lemma 3. Let $E, F \in T_\sigma \mathcal{Z}$ and $X = \pi_* E$, $Y = \pi_* F$, $A = \mathcal{V}E$, $B = \mathcal{V}F$. Then

$$
\begin{aligned}
\rho_{t,n}^*&(E, F) \\
&= [1 + (-1)^{n+1}]g(\mathcal{R}(\sigma), X \wedge K_\sigma Y) - \frac{t}{2}g(R(X \wedge K_\sigma Y)\sigma, R(\sigma)\sigma) \\
&\quad + \frac{t}{4}\mathrm{Trace}\,(Z \to g(R(X \wedge Z)\sigma, R(K_\sigma Z \wedge K_\sigma Y)\sigma)) \\
&\quad + \frac{t}{4}(-1)^{n+1}\mathrm{Trace}\,(\mathcal{V}_\sigma \ni C \to g(R(C)X, R(\sigma \times C)K_\sigma Y)) \\
&\quad + \frac{t}{2}(-1)^n g((\nabla_X \mathcal{R})(\sigma), B) + \frac{t}{2}g((\nabla_{K_\sigma Y} \mathcal{R})(\sigma), \sigma \times A) \\
&\quad + [1 + (-1)^{n+1}tg(\mathcal{R}(\sigma), \sigma)]g(A, B) \\
&\quad + (-1)^{n+1}\frac{t^2}{4}\mathrm{Trace}\,(Z \to g(R(\sigma \times A)K_\sigma Z, R(B)Z)).
\end{aligned}
$$

Now we are ready to prove the following

Proposition 1. The first Chern form $\gamma_{t,n}$ of the twistor space $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$, is given by

$$
2\pi\gamma_{t,n}(E, F) = [1 + (-1)^{n+1}][g(\mathcal{R}(\sigma), X \wedge Y) + g(A, \sigma \times B)],
$$

where $E, F \in T_\sigma \mathcal{Z}$ and $X = \pi_* E$, $Y = \pi_* F$, $A = \mathcal{V}E$, $B = \mathcal{V}F$.

Proof. Denote by $\varphi_{t,n}$ and $\psi_{t,n}$ the 2-forms on $\mathcal{Z}$ defined by (12) and (13), respectively. Let $\Omega_{t,n}$ be the Kähler form of $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$. By Lemma 1, we have

$$
8\pi\gamma_{t,n} = -\varphi_{t,n} - 4\psi_{t,n} + 2d\delta\Omega_{t,n}.
$$

Let U be an h_t-unit vertical vector at σ. Then, using Lemma 2, (5) and (6), one gets:

$$
\begin{aligned}
\varphi_{t,n}(E,F) = {} & t\operatorname{Trace}\left(Z \to g(R(X \wedge Z)\sigma, R(K_\sigma Z \wedge Y)\sigma)\right) \\
& + (-1)^{n+1} t\operatorname{Trace}\left(\mathcal{V}_\sigma \ni C \to g(R(C)X, R(\sigma \times C)Y)\right) \\
& + t^2 \operatorname{Trace}\left(Z \to g(R(\sigma \times A)K_\sigma Z, R(\sigma \times B)Z)\right) \\
& - 2tg(\mathcal{R}(\sigma \times A), B) + 2tg(\mathcal{R}(\sigma \times B), A) \\
& + 4g(\sigma \times A, B).
\end{aligned}
$$

Since $\psi_{t,n}(E,F) = \rho_{t,n}^*(E, J_n F)$, it follows from Lemma 3 that

$$
\begin{aligned}
4\pi\gamma_{t,n}(E,F) = {} & 2[1 + (-1)^{n+1}]\left[g(\mathcal{R}(\sigma), X \wedge Y) + g(A, \sigma \times B)\right] \\
& + t[g(\mathcal{R}(\sigma \times A), B) - g(\mathcal{R}(\sigma \times B), A) \\
& + 2g(\mathcal{R}(\sigma), \sigma)g(A, \sigma \times B) \\
& - g((\nabla_X \mathcal{R})(\sigma), \sigma \times B) \\
& + g((\nabla_Y \mathcal{R})(\sigma), \sigma \times A) \\
& - g(R(X \wedge Y)\sigma, R(\sigma)\sigma)] \\
& + d\delta\Omega_{t,n}(E,F).
\end{aligned}
\tag{21}
$$

It is easy to check by means of Lemma 2 and the identity (5) that the 1-form $\omega = -1/t\delta\Omega_{t,n}$ is given by $\omega(E) = g(\mathcal{V}E, R(\sigma)\sigma)$ for $E \in T_\sigma \mathcal{Z}$. Next we shall compute the differential of the form ω. Since $\sigma \to R(\sigma)\sigma$ is a vertical vector field on $\mathcal{Z}$, one has by (3):

$$
\begin{aligned}
(d\omega)_\sigma(X^h, Y^h) = -\omega_\sigma([X^h, Y^h]) = {} & -g(R(X \wedge Y)\sigma, R(\sigma)\sigma); \\
& X, Y \in \chi(M).
\end{aligned}
\tag{22}
$$

Now let s be a local section of $\mathcal{Z}$ such that $s(p) = \sigma$ and $\nabla s \mid_p = 0$. If B is a vertical vector field on $\mathcal{Z}$ and X is a vector field on M, it follows easily from (2) that

$$
[X^h, B]_\sigma = \nabla_{X_p}(B \circ s),
$$

where $B \circ s$ is considered as a section of $\Lambda_-^2 TM$. Then

$$
\begin{aligned}
(d\omega)_\sigma(X^h, B) & = s_*(X_p)(\omega(B)) - \omega_\sigma([X^h, B]) \\
& = X_p(g(B \circ s, R(s)s)) - g(\nabla_{X_p}(B \circ s), R(\sigma)\sigma)
\end{aligned}
$$

and, using (5), one gets:

$$(23) \qquad (d\omega)_\sigma(X^h, B) = -g((\nabla_{X_p}\mathcal{R})(\sigma), \sigma \times B_\sigma).$$

Finally, we will show that

$$(24) \qquad \begin{aligned} (d\omega)_\sigma(A, B) &= g(\mathcal{R}(\sigma \times A), B) - g(\mathcal{R}(\sigma \times B), A) \\ &\quad + 2g(\mathcal{R}(\sigma), \sigma)g(A, \sigma \times B) \end{aligned}$$

for any vertical vectors A and B at σ.

Let (s_1, s_2, s_3) be a local frame of $\Lambda^2_- TM$ defined by (1) such that $s_1(p) = \sigma$, and let $y_j(\tau) = g(\tau, (s_j \circ \pi)(\tau))$, $\tau \in \Lambda^2_- TM$, $1 \le j \le 3$. Set

$$U = -y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2}.$$

Then

$$J_1 U = y_1 y_3 \frac{\partial}{\partial y_1} + y_2 y_3 \frac{\partial}{\partial y_2} - (1 - y_3^2) \frac{\partial}{\partial y_3}$$

and $(U, J_1 U)$ is a local frame of the vertical bundle $\mathcal{V}$ near the point σ such that $[U, J_1 U]_\sigma = 0$. It is enough to check (24) for $A = U_\sigma$ and $B = J_1 U_\sigma$. Using (5), one gets:

$$\omega(U) = \sum_{j=1}^3 y_j [y_1 y_3 g(\mathcal{R}(s_j), s_1) \circ \pi + y_2 y_3 g(\mathcal{R}(s_j), s_2) \circ \pi$$

$$- (1 - y_3^2) g(\mathcal{R}(s_j), s_3) \circ \pi]$$

$$\omega(J_1 U) = \sum_{j=1}^3 y_j [y_2 g(\mathcal{R}(s_j), s_1) \circ \pi - y_1 g(\mathcal{R}(s_j), s_2) \circ \pi].$$

Then

$$\begin{aligned} (d\omega)_\sigma(U, J_1 U) &= \left(\frac{\partial}{\partial y_2}\right)_\sigma (\omega(J_1 U)) + \left(\frac{\partial}{\partial y_3}\right)_\sigma (\omega(U)) \\ &= -g(\mathcal{R}(s_3), s_3) - g(\mathcal{R}(s_2), s_2) + 2g(\mathcal{R}(s_1), s_1), \end{aligned}$$

which proves (24).

Now the proposition follows from (21)–(24). $\square$

4. Curvature properties of the Chern connection of twistor spaces. In this section we consider the problem when the curvature tensor $R_{t,n}$ of the Chern connection D^n of $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$, is of type $(1,1)$, i.e., $R_{t,n}(J_nE, J_nF)G = R_{t,n}(E, F)G$ for all $E, F, G \in T\mathcal{Z}$. We also study the problem when this connection has a constant holomorphic sectional curvature.

Proposition 2. (i) *The curvature tensor $R_{t,1}$ is of type $(1,1)$ if and only if the base manifold M is self-dual.*

(ii) *The curvature tensor $R_{t,2}$ is of type $(1,1)$ if and only if the base manifold M is Einstein and self-dual.*

Proof. (i) If $R_{t,1}$ is of type $(1,1)$, then $\gamma_{t,1}$ is an $(1,1)$-form with respect to J_1. This, together with Proposition 1 gives:

$$g(\mathcal{R}(\sigma), X \wedge Y - K_\sigma X \wedge K_\sigma Y) = 0$$

for all $\sigma \in \mathcal{Z}$ and $X, Y \in T_pM$, $p = \pi(\sigma)$. Since the 2-vectors of the form $X \wedge Y - K_\sigma X \wedge K_\sigma Y$ span the vertical space at σ, it is easy to see that the latter identity implies the self-duality of M.

Conversely, if M is self-dual, the almost-complex structure J_1 is integrable [**3**] and, as it is well-known, cf. e.g., [**15**, Lemma 2.1], the curvature of the Chern connection D^1 is of type $(1,1)$.

(ii) Given a point $\sigma \in \mathcal{Z}$ and $X, Y \in T_pM$, $p = \pi(\sigma)$, denote by $A(X, Y)$ and $B(X, Y)$ the vertical vectors at σ defined by

$$A(X, Y) = \frac{1}{4} R(X \wedge Y + K_\sigma X \wedge K_\sigma Y)\sigma$$
$$- \frac{1}{4t}(K_\sigma X \wedge Y + X \wedge K_\sigma Y)$$
$$B(X, Y) = \frac{1}{4} \sigma \times R(K_\sigma X \wedge Y - X \wedge K_\sigma Y)\sigma$$
$$+ \frac{1}{4t}(K_\sigma X \wedge Y + X \wedge K_\sigma Y),$$

where K_σ is the complex structure on T_pM corresponding to σ via (4). Using Lemma 2, formulas (5), (6), (8) and the identity $\sigma \times (K_\sigma X \wedge Y +$

$X \wedge K_\sigma Y) = X \wedge Y - K_\sigma X \wedge K_\sigma Y$, one gets from (10) that

$$D^2_{X^h} Y^h = (\nabla_X Y)^h + A(X,Y) + B(X,Y)$$
$$D^1_{X^h} Y^h = D^2_{X^h} Y^h - 2B(X,Y)$$
$$D^1_V X^h = D^2_V X^h$$

for $X, Y \in \chi(M)$ and $V \in \mathcal{V}$.

Now one obtains easily that

$$R_{t,1}(X^h, Y^h, Z^h, T^h)$$
$$= R_{t,2}(X^h, Y^h, Z^h, T^h)$$
$$+ 2t[g(A(X,Z), B(Y,T)) + g(A(Y,T), B(X,Z))$$
$$- g(A(Y,Z), B(X,T) - g(A(X,T), B(Y,Z))],$$

and

$$g(A(X,Z), B(Y,T)) - g(A(K_\sigma Y, T), B(K_\sigma X, Z))$$
$$= \frac{1}{8t}[g(\mathcal{R}(X \wedge K_\sigma Z + K_\sigma X \wedge Z), K_\sigma Y \wedge T - Y \wedge K_\sigma T)$$
$$- g(\mathcal{R}(X \wedge Z + K_\sigma X \wedge K_\sigma Z), Y \wedge T - K_\sigma Y \wedge K_\sigma T)].$$

These formulas, together with the first Bianchi identity, give:

$$(25) \quad R_{t,1}(X^h, Y^h, Z^h, T^h) - R_{t,1}(J_1 X^h, J_1 Y^h, Z^h, T^h)$$
$$= R_{t,2}(X^h, Y^h, Z^h, T^h) - R_{t,2}(J_2 X^h, J_2 Y^h, Z^h, T^h)$$
$$- \frac{1}{2} g(\mathcal{R}(X \wedge Y - K_\sigma X \wedge K_\sigma Y), Z \wedge T + K_\sigma Z \wedge K_\sigma T).$$

A similar computation gives

$$(26) \quad R_{t,1}(X^h, Y^h, V, J_1 V) - R_{t,1}(J_1 X^h, J_1 Y^h, V, J_1 V)$$
$$= -R_{t,2}(X^h, Y^h, V, J_2 V) + R_{t,2}(J_2 X^h, J_2 Y^h, V, J_2 V)$$
$$+ g(\mathcal{R}(X \wedge Y - K_\sigma X \wedge K_\sigma Y), \sigma),$$

for any h_t-unit vertical vector V at σ.

Now assume that $R_{t,2}$ is of type $(1,1)$ with respect to J_2. Then it follows from (25) and (26) that

$$\gamma_{t,1}(X^h, Y^h) - \gamma_{t,1}(J_1 X^h, J_1 Y^h) = 0,$$

which implies, as we have seen in the proof of (i), that M is self-dual. Hence, by (i), $R_{t,1}$ is of type (1,1) with respect to J_1, and the identity (25) becomes

$$g(\mathcal{R}(X \wedge Y - K_\sigma X \wedge K_\sigma Y, Z \wedge T + K_\sigma Z \wedge K_\sigma T) = 0$$

for $X, Y, Z, T \in \chi(M)$. Since M is self-dual, this implies $\mathcal{B} = 0$, i.e., M is Einstein.

Conversely, let M be Einstein and self-dual. Then the almost-Hermitian manifold $(\mathcal{Z}, h_t, J_2)$ is quasi-Kähler [**18**]. On the other hand, according to [**9**, Theorem, (i)], its Riemannian curvature tensor satisfies the identity $R(E, F, G, H) = R(JE, JF, JG, JH)$. Now it follows from [**15**, Theorem 6.2 (ii)] that the curvature tensor $R_{t,2}$ of the Chern connection D^2 is of type $(1, 1)$.

Next, we study the problem when the Chern connections D^1 and D^2 of a twistor space have constant holomorphic sectional curvatures.

Proposition 3. *The Chern connection D^1 of the almost-Hermitian manifold $(\mathcal{Z}, h_t, J_1)$ has a constant holomorphic sectional curvature κ if and only if $\kappa > 0$, the base manifold M is of constant sectional curvature κ and $t = 1/\kappa$.*

The holomorphic sectional curvature of the Chern connection D^2 of $(\mathcal{Z}, h_t, J_2)$ is never constant.

Proof. Let us note that, if (N, g, J) is an almost-Hermitian manifold, then the holomorphic sectional curvatures H and $\widetilde{H}$ of the Levi-Civita connection ∇ and the Chern connection $\widetilde{\nabla}$ are related by

(27)
$$\tilde{H}(X) = H(X) + \frac{1}{8}(\|(\nabla_X J)(X)\|^2 + \|(\nabla_{JX} J)(JX)\|^2)$$
$$+ \frac{3}{4}g((\nabla_X J)(X), (\nabla_{JX} J)(JX)).$$

This easily follows from (14)–(19).

Denote by $\widetilde{H}_{t,n}$ the holomorphic sectional curvature of the Chern connection D^n of $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$. Then, using the explicit formula

for the sectional curvature of $(\mathcal{Z}, h_t)$ given in [**8**, Proposition 3.5], formula (27) and Lemma 2, we obtain:

$$\widetilde{H}_{t,n}(X^h) = R(X, K_\sigma X, X, K_\sigma X) - \frac{t}{2} \|R(X \wedge K_\sigma X)\sigma\|_g^2,$$

where K_σ is the complex structure on $T_p M$, $p = \pi(\sigma)$, defined by (4).

Assume that $\widetilde{H}_{t,n} \equiv \kappa$. Then, for every $\sigma \in \mathcal{Z}$ and $X \in T_p M$, $p = \pi(\sigma)$, $\|X\| = 1$, one has:

$$(28) \qquad \kappa = R(X, K_\sigma X, X, K_\sigma X) - \frac{t}{2} \|R(X \wedge K_\sigma X)\sigma\|_g^2.$$

Let s_1, s_2, s_3 be local sections of $\mathcal{Z}$ defined by (1), and let

$$\sigma = \sum_{i=1}^{3} \lambda_i s_i, \qquad \sum_{i=1}^{3} \lambda_i^2 = 1.$$

Denote by K_i the complex structure on $T_p M$ determined by $s_i(p)$, and set

$$a_{ij} = g(\mathcal{R}(s_i), X \wedge K_j X), \qquad b_{ij} = g(\mathcal{R}(X \wedge K_i X), X \wedge K_j X).$$

Then

$$\|R(X \wedge K_\sigma X)\sigma\|_g^2 = \sum_{i=1}^{3} g(\mathcal{R}(\sigma \times s_i), X \wedge K_\sigma X)^2$$

$$= \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \lambda_j a_{ij} \right)^2 - \left(\sum_{i,j} \lambda_i \lambda_j a_{ij} \right)^2,$$

and

$$R(X, K_\sigma X, X, K_\sigma X) = \sum_{i,j=1}^{3} \lambda_i \lambda_j b_{ij}.$$

Varying $(\lambda_1, \lambda_2, \lambda_3)$ over the unit sphere S^2, one gets from (28) that

$$b_{ii} - \frac{t}{2} \sum_{k=1}^{3} a_{ki}^2 + \frac{t}{2} a_{ii}^2 = \kappa$$

$$b_{ii} + b_{jj} - \frac{t}{2} \sum_{k=1}^{3} (a_{ki}^2 + a_{kj}^2) + \frac{t}{2} (a_{ij} + a_{ji})^2 + t a_{ii} a_{jj} = 2\kappa$$

$$b_{ij} + b_{ji} - t \sum_{k=1}^{3} a_{ki} a_{kj} + t a_{ii}(a_{ij} + a_{ji}) = 0,$$

for $1 \leq i \neq j \leq 3$. These identities imply $a_{ii} = a_{jj}$ and $a_{ij} = -a_{ji}$ for $i \neq j$, i.e.,

$$g(\mathcal{R}(s_i), X \wedge K_i X) = g(\mathcal{R}(s_j), X \wedge K_j X)$$
$$g(\mathcal{R}(s_i), X \wedge K_j X) = -g(\mathcal{R}(s_j), X \wedge K_i X), \quad i \neq j.$$

Now, varying X over the unit sphere of $T_p M$, gives:

$$g(\mathcal{R}(s_i), s_j) = \delta_{ij} g(\mathcal{R}(s_1), s_1)$$
$$g(\mathcal{R}(s_i), \bar{s}_j) = 0, \quad 1 \leq i, j \leq 3.$$

Hence, M is Einstein and self-dual. Since $X \wedge K_\sigma X \in \mathbf{R}.\sigma \oplus \Lambda_+^2 T_p M$ for any $X \in T_p M$, it follows that $R(X \wedge K_\sigma X)\sigma = 0$ and (28) shows that M is of constant sectional curvature equal to κ. In this case, one obtains easily from [**8**, Proposition 3.5], Lemma 1 and Lemma 2 that the holomorphic sectional curvature $\widetilde{H}_{t,n}$ of D^n is given by

$$\widetilde{H}_{t,n}(E) = \kappa \|X\|^4 + t\|A\|^4 + \frac{(-1)^{n+1}}{4}(3 + (-1)^{n+1} + 4\kappa t)\|X\|^2 \|A\|^2,$$

where $X = \pi_* E$, $A = \mathcal{V}E$ and $\|E\|_{h_t}^2 = \|X\|^2 + t\|A\|^2 = 1$. Hence, for $n = 1$, the identity $\widetilde{H}_{t,n} \equiv \kappa$ is equivalent to $t = 1/\kappa$, while for $n = 2$, it is impossible. Thus, the proposition is proved.

Remark. Similar arguments show that the Levi-Civita connection of the almost-Hermitian manifold $(\mathcal{Z}, h_t, J_n)$ has a constant holomorphic sectional curvature κ only in the case when $n = 1$, M is of constant sectional curvature κ and $t = 1/\kappa$ ([**8**]).

5. Examples of twistor spaces with parallel Nijenhuis tensor. It is well known ([**9**, **18**]) that the twistor space $(\mathcal{Z}, h_t, J_2)$ of an Einstein, self-dual manifold M is a quasi-Kähler manifold satisfying the second Gray curvature condition. If $s > 0$ and $t = 6/s$, respectively $s < 0$ and $t = -(12)/s$ (s is the scalar curvature of M), then $(\mathcal{Z}, h_t, J_2)$ is nearly Kähler, respectively almost Kähler and, by results of [**6**, **19**], the Nijenhuis tensor of J_2 is parallel with respect to the Chern connection. In fact, this is true for any s and any t.

Proposition 4. *Let M be an Einstein and self-dual 4-manifold with twistor space $\mathcal{Z}$. Then the Nijenhuis tensor of the almost-complex*

structure J_2 is parallel with respect to the Chern connection of the almost-Hermitian manifold $(\mathcal{Z}, h_t, J_2)$.

Proof. Denote by N the Nijenhuis tensor of the almost-complex structure J_2. Let σ be a point of $\mathcal{Z}$, X, Y, Z vector fields on M near the point $p = \pi(\sigma)$, and A, B vertical vector fields near σ.

The identity

$$N(E, F)$$
$$= -J_2(D_E J_2)(F) + J_2(D_F J_2)(E) - (D_{J_2 F} J_2)(E) + (D_{J_2 E} J_2)(F)$$

and Lemma 2 imply the following formula:
(29)
$$N(X^h, Y^h)_\sigma = -\frac{s}{3}(X \wedge K_\sigma Y + K_\sigma X \wedge Y); \ N(X^h, A)_\sigma = 2(K_{\sigma \times A} X)^h_\sigma.$$

As to the Chern connection D^2 of $(\mathcal{Z}, h_t, J_2)$, formulas (8), (9), (10) and Lemma 2 give:

(30)
$$D^2_{X^h} Y^h = (\nabla_X Y)^h; \ (D^2_A X^h)_\sigma = \frac{1}{2}(K_{\sigma \times A} X)^h;$$
$$D^2_{X^h} A = D_{X^h} A - \frac{ts}{24}(K_{\sigma \times A} X)^h_\sigma = \mathcal{V}D_{X^h} A = [X^h, A].$$

Let ξ be a section of $\mathcal{Z}$ near p such that $\xi(p) = \sigma$ and $\nabla \xi|_p = 0$. Then it is easy to see that, at the point p, $\nabla K_\xi = 0$ and $\mathcal{V}D_{X^h} A = \nabla_X (A \circ \xi)$ for any vertical vector field A where $A \circ \xi$ is considered as a section of $\Lambda^2_- TM$. Now the identities (29) and (30) imply that

$$(D^2_{Z^h} N)(X^h, Y^h)_\sigma = 0.$$

Let E_1, E_2, E_3, E_4 be an oriented orthonormal frame of TM near p such that $\nabla E_i|_p = 0$, $1 \le i \le 4$, and $s_1(p) = \sigma$ where (s_1, s_2, s_3) is the local frame of $\Lambda^2_- TM$ defined by (1). Then, by (29), we have

$$N(X^h, A) = -4 \sum_{i=1}^{4} g(J_2 A, (X \wedge E_i) \circ \pi) E^h_i.$$

Let us also note that

$$Z_\sigma^h g\big(J_2 A, (X \wedge E_i) \circ \pi\big) = Z_p g\big((J_2 A) \circ \xi, X \wedge E_i\big)$$
$$= g\big(J_2 \nabla_{Z_p}(A \circ \xi), X \wedge E_i\big)$$
$$+ g\big(J_2 A_\sigma, \nabla_{Z_p} X \wedge E_i\big).$$

Now it is clear that

$$(D_{Z^h}^2 N)(X^h, A)_\sigma = 0.$$

We have also
$$(D_{Z^h}^2 N)(A, B)_\sigma = 0,$$
in view of (30) and the fact that N vanishes for any two vertical vectors.

We shall further use the notations introduced at the end of the proof of Proposition 1.

The fibers of $\mathcal{Z}$ are totally geodesic submanifolds, Kählerian with respect to J_2, so the Chern connection D^2 coincides with the Levi-Civita connection D of h_t for vertical vectors. Since $D_U U$ and $D_U J_1 U$ are vertical vectors and $[U, J_1 U]_\sigma = 0$, it follows from the standard formula for the Levi-Civita connection that

$$(31) \qquad\qquad (D_U U)_\sigma = (D_U J_1 U)_\sigma = 0.$$

Hence,

$$D_{U_\sigma}^2 N(X^h, Y^h)$$
$$= U_\sigma\big(g(N(X^h, Y^h), U)U_\sigma + U_\sigma\big(g(N(X^h, Y^h), J_1 U)\big)J_1 U_\sigma.$$

By (6) and (30), we have

$$g(N(X^h, Y^h), U) = \frac{2s}{3} g(J_1 U, X \wedge Y)$$
$$= \frac{2s}{3}(1 - y_3)^{-1/2}\big(y_1 y_3 g(s_1, X \wedge Y) \circ \pi$$
$$+ y_2 y_3 g(s_2, X \wedge Y) \circ \pi$$
$$- (1 - y_3^2)g(s_3, X \wedge Y) \circ \pi\big),$$

and

$$g(N(X^h, Y^h), J_1 U) = -\frac{2s}{3} g(U, X \wedge Y)$$
$$= \frac{2s}{3}(1 - y_3)^{-1/2}(y_2 g(s_1, X \wedge Y) \circ \pi$$
$$- y_1 g(s_2, X \wedge Y) \circ \pi).$$

It follows that

$$D_{U_\sigma} N(X^h, Y^h) = -\frac{2s}{3} g(s_1, X \wedge Y)_p s_3(p).$$

Using (7), (29) and (30), we easily obtain

$$N(D_U^2 X^h, Y^h)_\sigma = \frac{s}{6} g(X, Y)_p s_2(p) - \frac{s}{3} g(s_1, X \wedge Y)_p s_3(p),$$
$$N(X^h, D_U^2 Y^h)_\sigma = -\frac{s}{6} g(X, Y)_p s_2(p) - \frac{s}{3} g(s_1, X \wedge Y)_p s_3(p).$$

It follows that
$$(D_U^2 N)(X^h, Y^h)_\sigma = 0.$$

Similarly,
$$(D_{J_1 U}^2 N)(X^h, Y^h)_\sigma = 0.$$

Therefore,
$$(D_A^2 N)(X^h, Y^h)_\sigma = 0,$$

for any vertical vector A at σ.

By (29) and (30), we obtain

$$D_{U_\sigma}^2 N(X^h, U) = -X_\sigma^h \text{ and } D_{U_\sigma}^2 N(X^h, J_1 U) = (K_\sigma X)_\sigma^h.$$

Taking also into account (7), we get

$$N(D_{U_\sigma}^2 X^h, U) = -X_\sigma^h \text{ and } N(D_{U_\sigma}^2 X^h, J_1 U) = (K_\sigma X)_\sigma^h.$$

Then, by (31), we have

$$(D_U^2 N)(X^h, U)_\sigma = (D_U^1 N)(X^h, J_1 U)_\sigma = 0.$$

Similarly, we get

$$(D^2_{J_1 U} N)(X^h, U)_\sigma = (D^1_{J_1 U} N)(X^h, J_1 U)_\sigma = 0.$$

Therefore,

$$(D^2_A N)(X^h, B)_\sigma = 0.$$

Finally, let A, B, C be vertical vectors at σ. Since, for vertical vectors, D^1 coincides with the Levi-Civita connection of the fiber through σ, we have

$$(D^2_A N)(B, C)_\sigma = 0.$$

Remark. Identity (30) shows that the Chern connection D^2 actually does not depend on t when the base manifold is Einstein and self-dual. Then Proposition 4 for $s \neq 0$ follows also from the results in [**6, 19**] mentioned in the beginning of this section. Propositions 1 and 4 can be extended to the twistor spaces of quaternionic-Kähler manifolds by means of the formulas in [**1**].

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