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SEGAL ALGEBRAS ON HERMITIAN GROUPS

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ABSTRACT. Let G be a locally compact group. We call a Segal algebra $S^1(G)$ quasi-Hermitian if $\sigma_{S^1(G)}(\tilde{f}*f) \subseteq [0,\infty)$, for all $f \in S^1(G)$. We show that, when G is a $[SIN]$-group, then the following are equivalent: (i) G is Hermitian. (ii) Any Segal algebra on G is quasi-Hermitian. (iii) There exits a quasi-Hermitian Segal algebra on G.

1. Introduction. For a Banach algebra A and $a \in A$, $\sigma_A(a)$ always stands for the spectrum of a in A. Throughout this paper, G denotes a locally compact group. We recall that a linear subspace $S^1(G)$ of $L^1(G)$ is called a Segal algebra if it satisfies the following conditions:

(S1) $S^1(G)$ is dense in $L^1(G)$;

(S2) For $f \in S^1(G)$, we have $L_a f \in S^1(G)$, $a \in G$, where L_a denotes the left translation operator;

(S3) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$ such that $\|L_a f\|_S = \|f\|_S$ and $\|f\|_{L^1(G)} \leq \|f\|_S$, $f \in S^1(G)$, $a \in G$;

(S4) For each $f \in S^1(G)$, the mapping $a \mapsto L_a f$ of G into $S^1(G)$ is continuous.

We are now going to look at the case when G is a Hermitian group. We recall that a Banach $*$-algebra A is said to be Hermitian if $\sigma_A(x) \subseteq \mathbf{R}$ for any self-adjoint element $x \in A$. According to the Shirali-Ford theorem, A is Hermitian if and only if A is symmetric, that is, $\sigma_A(x^*x) \subseteq [0,\infty)$, for all $x \in A$ [**1**, Section 41, Theorem 5, page 226]. And a locally compact group G is said to be Hermitian if $L^1(G)$ is Hermitian.

It should be mentioned that, under the convolution in $L^1(G)$, $(S^1(G), \|\cdot\|_S)$ is a dense left ideal of $L^1(G)$ automatically [**4**, Sec-

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tion 4, Proposition 1, page 19]. In view of the fact that $S^1(G)$ is a left ideal of $L^1(G)$, we can introduce the following new definition:

Definition 1.1. We call a Segal algebra $S^1(G)$ (not necessarily invariant under the involution of $L^1(G)$) quasi-Hermitian if $\sigma_{S^1(G)}(\tilde{f} * f) \subseteq [0, \infty)$, for all $f \in S^1(G)$, where "$*$" denotes the convolution product in $L^1(G)$ and "$\sim$" denotes the involution of $L^1(G)$.

In this paper, we introduce a new class of locally compact groups $[Seg]$ which contains the class of $[SIN]$-groups. Moreover, we shall show that when G is in $[Seg]$, then the following are equivalent:

(i) G is Hermitian.

(ii) Any Segal algebra on G is quasi-Hermitian.

(iii) There exists a quasi-Hermitian Segal algebra on G.

2. Main result. For our purpose, we shall denote $[Seg]$ the class of locally compact groups G for which every Segal algebra $S^1(G)$ on G satisfies the condition: *given any* $f \in L^1(G)$ *and any* $\varepsilon > 0$, *there exists an* $h \in S^1(G)$ *such that* $\|f - h\|_{L^1(G)} < \varepsilon$ *and* $f * h = h * f$.

Recall that a locally compact group is called a $[SIN]$-group if every neighborhood of the identity e contains a compact invariant neighborhood of e. Kotzmann and Rindler characterized the class of $[SIN]$-groups by the existence of central approximate identity for a Segal algebra [**2**, Theorem 1]. More precisely, given a Segal algebra $S^1(G)$, they show that G is a $[SIN]$-group if and only if $S^1(G)$ has an approximate identity contained in the center of $S^1(G)$. Furthermore, in this case we can choose this approximate identity which is bounded in $L^1(G)$.

Lemma 2.1. *If* G *is a* $[SIN]$*-group, then* $G \in [Seg]$.

Proof. Let $\varepsilon > 0$ and $f \in L^1(G)$. By Kotzmann and Rindler's result, $S^1(G)$ has a central approximate identity $(e_\alpha)_{\alpha\in\Lambda}$ with $\sup_{\alpha\in\Lambda} \|e_\alpha\|_{L^1(G)} \leq m$, for some $m \geq 1$. Choose $g \in S^1(G)$ such that $\|f - g\|_{L^1(G)} < \varepsilon/(3m)$. And take some e_β, $(\beta \in \Lambda)$ satisfying

$\|g - g * e_\beta\|_{S^1(G)} < \varepsilon/3$. Then we have $\|f - f * e_\beta\|_{L^1(G)} < \varepsilon$. Since $S^1(G)$ is a left ideal, $f * e_\beta \in S^1(G)$. Obviously, $(e_\alpha)_{\alpha\in\Lambda}$ is contained in the center of $L^1(G)$ because $S^1(G)$ is dense in $L^1(G)$. Then $f * e_\beta$ is the required element. $\square$

Before going to prove the main result, we have to make use the following simple lemma.

Lemma 2.2. *Let $(A, \|\cdot\|_A)$ be a Banach algebra which contains a Banach algebra $(B, \|\cdot\|_B)$ as a left ideal. Then we have*

$$\sigma_A(b) \cup \{0\} = \sigma_B(b) \cup \{0\}.$$

Suppose further that if $\overline{B}^{\|\cdot\|_A} = A$, then we have $\sigma_A(b) = \sigma_B(b)$, for all $b \in B$.

In particular, for any locally compact group G, we have

$$\sigma_{S^1(G)}(f) = \sigma_{L^1(G)}(f).$$

Proof. For any Banach algebra C and any element $x \in C$, we recall the fact that for nonzero $\lambda \notin \sigma_C(x)$ if and only if $\lambda^{-1}x$ is quasi-regular in C, i.e., there is an element $y \in C$ such that $\lambda^{-1}x + y = (\lambda^{-1}x)y = y(\lambda^{-1}x)$ [**1**, Section 5, Lemma 2]. Hence, it is enough to show that if an element $b \in B$ is quasi-regular in A, then b is also quasi-regular in B. In fact if an element $b \in B$ is quasi-regular in A, then there is an element $a \in A$ such that $a + b = ab = ba$. Since B is a left ideal of A, a is contained in B, that is, b is also a quasi-regular element in B. Hence, we have $\sigma_A(b) \cup \{0\} = \sigma_B(b) \cup \{0\}$. Therefore, when B is dense in A, then $\sigma_A(b) = \sigma_B(b)$ for all $b \in B$. $\square$

Theorem 2.3. *Suppose that $G \in [Seg]$. Then the following are equivalent*:

(i) *G is Hermitian.*

(ii) *Any Segal algebra on G is quasi-Hermitian.*

(iii) *There exists a quasi-Hermitian Segal algebra on G.*

In particular, when G is a $[SIN]$-group, then the above conditions are equivalent.

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 2.3 immediately.

(ii) $\Rightarrow$ (iii) is obvious.

It remains to show that (iii) $\Rightarrow$ (i). Let $S^1(G)$ be a quasi-Hermitian Segal algebra on G. Suppose that (i) does not hold. Then there is a self-adjoint element $f \in L^1(G)$ such that the imaginary unit $i \in \sigma_{L^1(G)}(f)$ [**3**, Theorem 9.8.2, page 1009]. Then $-1 \in \sigma_{L^1(G)}(f * f)$. On the other hand, there is a $\delta > 0$ satisfying $d(-1, \sigma_{L^1(G)}(g)) \stackrel{\text{def}}{=} \inf\{|\lambda + 1| | \lambda \in \sigma_{L^1(G)}(g)\} < 1/2$, whenever $g \in L^1(G)$ with $\|f * f - g\|_{L^1(G)} < \delta$ and $(f * f) * g = g * (f * f)$ [**1**, Section 6, Proposition 18, page 26]. For $G \in [Seg]$, there exists an element $h \in S^1(G)$ such that $\|f * f - \tilde{h} * h\|_{L^1(G)} < \delta$ and $f * h = h * f$. Hence, we have $d(-1, \sigma_{L^1(G)}(\tilde{h} * h)) < 1/2$. But $S^1(G)$ is quasi-Hermitian and from Lemma 2.3 we have $\sigma_{L^1(G)}(\tilde{h} * h) = \sigma_{S^1(G)}(\tilde{h} * h) \subseteq [0, \infty)$. This is a contradiction. Therefore, G is Hermitian. Now the last assertion follows from Lemma 2.1 at once. $\square$

A locally compact group is called an $[IN]$-group if it has a compact invariant neighborhood of the identity. We do not know whether any $[IN]$-group belongs to $[Seg]$. But it is known that if we let K be the intersection of all compact invariant neighborhoods of the identity, then K is a compact normal subgroup of G and G/K is a $[SIN]$-group [**3**, Section 12.6.16, page 1460]. We have the following result.

Corollary 2.4. *Let G be an $[IN]$-group. And let K be the compact normal subgroup of G such that G/K is a $[SIN]$-group,* as above. *If there exists a quasi-Hermitian Segal algebra on G, then G/K is Hermitian.*

Proof. Let $S^1(G)$ be a quasi-Hermitian Segal algebra on G. Write $S^1(G/K)$ the image of $S^1(G)$ under the canonical mapping from $L^1(G)$ onto $L^1(G/K)$. Then $S^1(G/K)$ is the quotient of $S^1(G)$, and it is a Segal algebra on G/K under the quotient norm [**4**, Section 13, Theorem 1, page 62]. Hence, $S^1(G/K)$ is also a quasi-Hermitian Segal algebra on G/K. Using Theorem 2.4, G/K is Hermitian. $\square$

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