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LOCAL CONNECTEDNESS AND RETRACTIONS IN HYPERSPACES

JANUSZ J. CHARATONIK AND PATRICIA PELLICER-COVARRUBIAS

ABSTRACT. Given a metric continuum X, let 2^X denote the hyperspace of all nonempty closed subsets of X; further, let $C(X)$ (and $C_2(X)$) be the hyperspace of all members of 2^X which are connected (composed of at most two components, respectively). For a point $p \in X$ we consider mappings $\varphi_p : C(X) \to C_2(X)$ and $\psi_p : 2^X \to 2^X$ both defined by $A \mapsto A \cup \{p\}$. Relations between local connectedness of X at p and conditions under which these mappings are deformation retractions or strong deformation retractions (under a special convention for φ_p) are studied. The conditions are related to contractibility of the corresponding hyperspaces and to the existence of a special local base of the point p.

1. Introduction. Throughout this paper a *continuum* means a compact, connected metric space. If X is a continuum, 2^X will denote the hyperspace of nonempty closed subsets of X, which is assumed to be equipped with the Hausdorff metric, see [**2**, Definitions 1.5, 1.6 and 2.1, pages 6, 11, respectively]. We also utilize three special subspaces of 2^X. Namely, for a point $p \in X$, we let 2^X_p denote the hyperspace consisting of all members of 2^X that contain p. Further, $C(X)$ means the hyperspace of subcontinua of X, and $C_2(X)$ stands for the hyperspace composed of all members of 2^X which have at most two components.

Results obtained in this article are related to certain ones obtained earlier by the second named author in [**6**]. Namely we investigate connections between local connectivity of a continuum X at a point $p \in X$ (also connectedness im kleinen of X at p) and properties of some mappings between hyperspaces of X. More precisely, for a continuum X and a point $p \in X$, we consider two mappings: $\psi_p : 2^X \to 2^X_p$ given by $\psi_p(A) = A \cup \{p\}$ and $\varphi_p : C(X) \to C_2(X)$ defined similarly by

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$\varphi_p(A) = A \cup \{p\}$. Among other results we show that if ψ_p is a strong deformation retraction, then X is connected im kleinen at p (but it need not be locally connected at this point).

2. Preliminaries. The symbol $\mathbf{N}$ stands for the set of all positive integers. All considered spaces are assumed to be metric, and all mappings are continuous. By a *continuum* we mean a compact connected space. The symbols $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ stand for the closure and the interior of the set A in the space X, respectively.

Let Y and Z be spaces. A mapping from $Y \times [0,1]$ into Z is called a *homotopy*. For a homotopy $h : Y \times [0,1] \to Z$ and any $t \in [0,1]$ we let $h_t : Y \to Z$ denote the mapping defined by $h_t(y) = h(y,t)$ for all $y \in Y$. We say that two mappings $f, g : Y \to Z$ are *homotopic* provided that there is a homotopy $h : Y \times [0,1] \to Z$ such that $h_0 = f$ and $h_1 = g$, in which case we say that h is a *homotopy joining* f *to* g.

Let X be a space, and let A be a subspace of X. Recall that a mapping $r : X \to A$ is called a *retraction* if the restriction $r|_A$ is the identity. Further, r is called a *deformation retraction* if r is a retraction homotopic to the identity on X, i.e., if there is a homotopy $h : X \times [0,1] \to X$ such that for each $x \in X$ we have $h(x,0) = x$ and $h(x,1) = r(x)$. Finally, r is called a *strong deformation retraction* provided that there exists a homotopy $h : X \times [0,1] \to X$ such that for each $x \in X$ we have $h(x,0) = x$, $h(x,1) = r(x)$ and $h(x,t) = x$ for every $x \in A$ and $t \in [0,1]$. In these cases, A is called a *retract*, a *deformation retract* and a *strong deformation retract* of X, respectively.

If Z is a space and $Y \subset Z$, then Y is said to be *contractible in* Z provided that there is a homotopy $h : Y \times [0,1] \to Z$ such that h_0 is the inclusion mapping of Y into Z and h_1 is a constant mapping of Y into Z. A space X is said to be *contractible* provided that the identity on X is homotopic to a constant mapping, see e.g., [**2**, page 155].

Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric, see e.g., [**5**, (0.1), page 1; (0.12), page 10]. We denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X; further, we let $C_2(X)$ denote the hyperspace of all elements of 2^X having at most two components, both equipped with the inherited topology (thus induced by the Hausdorff metric).

It is known that 2^X, $C(X)$ and $C_2(X)$ are arcwise connected continua (for 2^X see [**5**, (1.13), page 65]; for $C(X)$ and $C_2(X)$, see [**3**, Theorem 3.1, page 240]).

Further, for $p \in X$, define $2^X_p = \{A \in 2^X : p \in A\}$ and $C(p, X) = \{A \in C(X) : p \in A\}$.

For each nonempty closed subset $\mathcal{A}$ of 2^X, denote by $\cup\mathcal{A}$ the union of all elements of $\mathcal{A}$. The following known lemmas (see [**5**, Lemma 1.48, page 100; Lemma 1.49, page 102]; compare [**2**, Exercise 11.5, page 91]) will be utilized in proofs.

Lemma 2.1. *For each continuum X the union function $\cup : 2^{2^X} \to 2^X$ given by $\cup(\mathcal{A}) = \cup\{A \in 2^X : A \in \mathcal{A}\}$ is a surjective mapping.*

Lemma 2.2. *Let X be a continuum, and let $\mathcal{A}$ be a subcontinuum of 2^X. If $\mathcal{A} \cap C(X) \neq \varnothing$, then $\cup\mathcal{A}$ is a subcontinuum of X.*

A collection $\mathcal{F}$ of subsets of a space is called a *nest* if for any $F_1, F_2 \in \mathcal{F}$ we have either $F_1 \subset F_2$ or $F_2 \subset F_1$.

For a continuum X and a point $p \in X$, define $\psi_p : 2^X \to 2^X_p$ by $\psi_p(A) = A \cup \{p\}$ for each $A \in 2^X$, see [**6**, page 277]. Note that ψ_p is a retraction. Similarly, for a continuum X and $p \in X$, define $\varphi_p : C(X) \to C_2(X)$ by $\varphi_p(A) = A \cup \{p\}$ for each $A \in C(X)$. In this case we cannot ask whether φ_p is a retraction, because $C_2(X)$ is not a subspace of $C(X)$. However, we establish the following convention, see [**6**, page 279].

Since $\varphi_p|_{C(p,X)}$ is the identity mapping on $C(p, X)$, we shall say that φ_p is a *retraction in* $C_2(X)$. Similarly, we shall say that φ_p is a *deformation retraction in* $C_2(X)$ if there exists a homotopy $G : C(X) \times [0, 1] \to C_2(X)$ joining φ_p to the identity mapping in $C(X)$. Finally, we will say that φ_p is a *strong deformation retraction in* $C_2(X)$ provided that the homotopy G satisfies the condition $G(A, t) = A$ whenever $A \in C(p, X)$ and $t \in [0, 1]$.

3. Local interrelations. Let a continuum X and a point $p \in X$ be given. Equivalences between local connectedness of the continuum X

and properties of the mappings ψ_p and φ_p are shown in [**6**, Corollary 4.13, page 289]. The result is the following.

Proposition 3.1. *For a continuum X the following conditions are equivalent*:

(3.1.1) *X is locally connected*;

(3.1.2) *for every $p \in X$, ψ_p is a strong deformation retraction*;

(3.1.3) *for every $p \in X$, φ_p is a strong deformation retraction in $C_2(X)$.*

Since the above mentioned equivalences concern the *global* situation, when X is considered to be locally connected at each of its points, it is natural to investigate the *local* situation, when the properties are considered at a certain fixed point p of X. We start our investigation with recalling two characterizations which are needed to show the next result. The characterizations are given in [**6**, Corollaries 4.12, page 289 and 4.11, page 288, respectively].

Proposition 3.2. *Let a continuum X and a point $p \in X$ be given.*

(3.2.1) *The mapping $\psi_p : 2^X \to 2^X_p$ is a strong deformation retraction if and only if the hyperspace 2^X is contractible and there exists a local basis of closed, connected, nested neighborhoods $\{K_n : n \in \mathbf{N}\}$ of p such that $2^{K_{n+1}}$ is contractible in 2^{K_n} for each $n \in \mathbf{N}$.*

(3.2.2) *The mapping $\varphi_p : C(X) \to C_2(X)$ is a strong deformation retraction in $C_2(X)$ if and only if the hyperspace $C(X)$ is contractible and there exists a local basis of closed, connected, nested neighborhoods $\{K_n : n \in \mathbf{N}\}$ of p such that $C(K_{n+1})$ is contractible in $C_2(K_n)$ for each $n \in \mathbf{N}$.*

Now we are ready to prove the following theorem.

Theorem 3.3. *Let X be a continuum, and let $p \in X$. The following two conditions are equivalent.*

(3.3.1) *ψ_p is a strong deformation retraction*;

(3.3.2) *φ_p is a strong deformation retraction in $C_2(X)$.*

Proof. Assume (3.3.1). By (3.2.1) the hyperspace 2^X is contractible. This is equivalent to the contractibility of the hyperspace $C(X)$ according to [**2**, Theorem 20.1, page 164]. Again by (3.2.1), there exists a local basis of closed, connected, nested neighborhoods $\{K_n : n \in \mathbf{N}\}$ of p such that $2^{K_{n+1}}$ is contractible in 2^{K_n} for each $n \in \mathbf{N}$. Thus there is a homotopy $G : 2^{K_{n+1}} \times [0,1] \to 2^{K_n}$ such that $G(A,0) = A$ and $G(A,1) = K_n$ for each $A \in 2^{K_{n+1}}$. Define a homotopy $H : C(K_{n+1}) \times [0,1] \to C_2(K_n)$ by $H(A,t) = \cup\{G(A,s) : s \in [0,t]\}$ for each $A \in C(K_{n+1})$ and $t \in [0,1]$. Applying Lemmas 2.1 and 2.2 we see that H is well defined and continuous. Further, note that $H(A,0) = \cup\{G(A,0)\} = \cup\{A\} = A$ and $H(A,1) = \cup\{G(A,s) : s \in [0,1]\} \supset G(A,1) = K_n$, whence it follows that $H(A,1) = K_n$. Therefore the homotopy H shows that $C(K_{n+1})$ is contractible in $C_2(K_n)$ for each $n \in \mathbf{N}$. Hence by equivalence (3.2.2) it follows that (3.3.2) holds, as required.

Conversely, assume (3.3.2). By (3.2.2) the hyperspace $C(X)$ is contractible, which is equivalent to the contractibility of the hyperspace 2^X according to [**2**, Theorem 20.1, page 164]. Again by (3.2.2), there exists a local basis of closed, connected, nested neighborhoods $\{K_n : n \in \mathbf{N}\}$ of p such that $C(K_{n+1})$ is contractible in $C_2(K_n)$ for each $n \in \mathbf{N}$. Thus there is a homotopy $H : C(K_{n+1}) \times [0,1] \to C_2(K_n)$ such that $H(A,0) = A$ and $H(A,1) = K_n$ for each $A \in C(K_{n+1})$. Define a homotopy $G : 2^{K_{n+1}} \times [0,1] \to 2^{K_n}$ by $G(A,t) = \cup\{H(\{a\},t) : a \in A\}$ for each $A \in 2^{K_{n+1}}$ and $t \in [0,1]$. By compactness of A, applying Lemma 2.1 we see that G is well defined and continuous. Further, note that $G(A,0) = \cup\{H(\{a\},0) : a \in A\} = \cup\{\{a\} : a \in A\} = A$ and $G(A,1) = \cup\{H(\{a\},1) : a \in A\} = \cup\{K_n : a \in A\} = K_n$ for each $n \in \mathbf{N}$. Thus, G is a contraction which shows that $2^{K_{n+1}}$ is contractible in 2^{K_n}. Hence, by equivalence (3.2.1) it follows that (3.3.1) holds, as required. The proof is complete. $\square$

As can be observed from the proof of Proposition 3.1, given in [**6**, page 289], conditions (3.3.1) and (3.3.2) are related not only to local connectedness of X at p but also to connectedness im kleinen of X at p. Recall, see for example [**2**, page 83], that a space X is said to be:

–*locally connected at a point* $p \in X$ provided that there exists in X a local basis of connected open sets at p;

–connected im kleinen at a point $p \in X$ provided that there exists in X a local basis of connected sets at p (note that if the considered space X is compact and Hausdorff, then X is connected im kleinen at p if and only if there exists in X a local basis at p composed of continua, see [**4**, page 120]).

It is obvious that if a space X is locally connected at a point, then it is connected im kleinen at this point; the converse is false even for continua, see for example [**2**, page 83 and Figure 22, page 84]. To see some other relations, consider two (known) examples.

Example 3.4. There exists a continuum X such that

(3.4.1) the hyperspaces 2^X and $C(X)$ are not contractible;

(3.4.2) for no point $p \in X$ the mapping ψ_p is a strong deformation retraction;

(3.4.3) for no point $p \in X$ the mapping φ_p is a strong deformation retraction in $C_2(X)$.

Proof. Let X be the one-point union of two harmonic fans having the accumulation point of the sets of their end points in common only, see [**2**, Figure 25, page 158]. Then (3.4.1) holds, see [**2**, Exercise 19.12, page 161] and compare [**2**, Theorem 20.1, page 164]. Therefore, (3.4.2) and (3.4.3) are consequences of (3.2.1) and (3.2.2), respectively. $\square$

Remark 3.5. Since the continuum X of Example 3.4 contains points p at which X is locally connected, it follows that local connectedness of a continuum at a point p (and thus connectedness im kleinen at p) does not imply conditions (3.3.1) and (3.3.2).

Recall that an arcwise connected and hereditarily unicoherent continuum is called a *dendroid*. A dendroid X is said to be *smooth at a point* $q \in X$ provided that there is a point $q \in X$ such that for each point $x \in X$ and for each sequence of points $\{x_n : n \in \mathbf{N}\}$ converging to x, the sequence of arcs $\{qx_n : n \in \mathbf{N}\}$ converges to the arc qx. A dendroid X is said to be *smooth* provided that it is smooth at some point $q \in X$.

Example 3.6. There exists a continuum X such that

(3.6.1) X is a smooth dendroid;

(3.6.2) X is contractible;

(3.6.3) the hyperspaces 2^X and $C(X)$ are contractible;

(3.6.4) there exists a point $p \in X$ having a local basis of closed, connected, nested neighborhoods $\{K_n : n \in \mathbf{N}\}$ of p such that $2^{K_{n+1}}$ is contractible in 2^{K_n} for each $n \in \mathbf{N}$;

(3.6.5) the mapping $\psi_p : 2^X \to 2^X_p$ is a strong deformation retraction;

(3.6.6) X is connected im kleinen at p while it is not locally connected at p.

Proof. Let X be the continuum pictured in [**2**, Figure 22, page 84], let p be the point furthest to the left (as denoted in the Figure), and let q be the point furthest to the right. Then X is smooth at q, see [**2**, page 194], whence (3.6.1) follows. This implies (3.6.2) since each smooth dendroid is contractible, [**1**, Corollary 12, page 311]. Further, (3.6.2) implies contractibility of the hyperspace 2^X and $C(X)$ by [**2**, Corollary 20.2, page 166]. So (3.6.3) is shown.

For each $n \in \mathbf{N}$ let F_n denote the nth harmonic fan contained in X, and let p_n be the vertex of F_n. Thus $p_1 = q$, $p = \lim p_n$, and

$$X = \{p\} \cup \bigcup \{F_n : n \in \mathbf{N}\} = \mathrm{cl}\bigg(\bigcup \{F_n : n \in \mathbf{N}\} \bigg).$$

For each $n \in \mathbf{N}$ define $K_n = \{p\} \cup \bigcup \{F_m : m \geq n\}$. Thus K_n is a continuum containing p in its interior, and $K_{n+1} \subset K_n$ for each n. Therefore, the family $\{K_n : n \in \mathbf{N}\}$ forms a local basis of closed, connected, nested neighborhoods of p. Observe further that each K_n is homeomorphic to the whole continuum X. It follows by (3.6.3) that 2^{K_n} is contractible for each n, and therefore $2^{K_{n+1}}$ is contractible in 2^{K_n}. Hence (3.6.4) is proved.

Now (3.6.5) is a consequence of (3.6.3) and (3.6.4) according to (3.2.1). And finally (3.6.6) is a well-known fact. The proof is complete. $\square$

In the next theorem all possible implications between the considered conditions are discussed.

Theorem 3.7. *Let X be a continuum, and let $p \in X$. Consider the following conditions.*

(3.7.1) *X is locally connected at p.*

(3.7.2) *X is connected im kleinen at p.*

(3.3.1) *ψ_p is a strong deformation retraction.*

(3.3.2) *φ_p is a strong deformation retraction in $C_2(X)$.*

Then the only implications between these conditions are:

$$\begin{array}{ccc} (3.7.1) & \Longrightarrow & (3.7.2) \\ & & \Uparrow \\ (3.3.1) & \Longleftrightarrow & (3.3.2) \end{array}$$

Proof. Indeed, the implication from (3.7.1) to (3.7.2) is obvious. The one from (3.3.1) (or from (3.3.2)) to (3.7.2) is a consequence of (3.2.2) because the existence of a local basis at p composed of continua implies connectedness im kleinen of X at p. Finally, the equivalence between (3.3.1) and (3.3.2) is Theorem 3.3.

Condition (3.7.2) does not imply (3.7.1) by the well-known example, see [**2**, page 83 and Figure 22, page 84]. Neither (3.7.1) nor (3.7.2) implies (3.3.1) and/or (3.3.2) by Remark 3.5. Finally, (3.3.1) or (3.3.2) does not imply (3.7.1) by Example 3.6. □

In connection with the implication from (3.3.2) to (3.7.2) (see the diagram in Theorem 3.7 above) one can ask if condition (3.3.2) (that φ_p is a *strong* deformation retraction in $C_2(X)$) can be relaxed to the one of being a deformation retraction in $C_2(X)$ only. The answer is negative, as can be seen from the next example.

Example 3.8. There exist a continuum X and a point $p \in X$ such that:

(3.8.1) the hyperspaces $C(X)$ and $C_2(X)$ are contractible;

(3.8.2) for each point $x \in X$ the mapping φ_x is a deformation retraction in $C_2(X)$;

(3.8.3) the mapping φ_p is not a strong deformation retraction in $C_2(X)$.

Proof. Let X be the circle with a spiral, that is, $X = \mathbf{S}^1 \cup S$, where $\mathbf{S}^1$ is the unit circle in the plane, and S is the spiral given in polar coordinates (ρ, θ) by

$$S = \left\{(\rho, \theta) : \rho = 1 + \frac{1}{1+\theta} \quad \text{and} \quad \theta \geq 0\right\},$$

see [**2**, Figure 14, page 51]. Thus, S approximates $\mathbf{S}^1$. It is known that $C(X)$ is homeomorphic to the cone over X, [**2**, Example 7.1, page 53], so it is contractible, whence $C_2(X)$ is contractible, too, see [**3**, Theorem 3.7, page 241]. Thus, (3.8.1) is shown. Now (3.8.2) follows from (3.8.1) by [**6**, Theorem 4.1, page 280]. To see that φ_p is not a strong deformation retraction in $C_2(X)$, take an arbitrary point $p \in \mathbf{S}^1$ and observe that there is no local basis of closed connected neighborhoods of p. Thus, (3.8.3) is a consequence of [**6**, Corollary 4.11, page 288]. □

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Instituto de Matemáticas, UNAM, Ciudad Universitaria, México, D.F., 04510 México
Email address: jjc@matem.unam.mx

Departamento de Matemáticas, Facultad de Ciencias, UNAM, Ciudad Universitaria, 04510 México, D.F., México
Email address: paty@ciencias.unam.mx