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ANALYSIS OF AN ECO-EPIDEMIOLOGICAL MODEL WITH TIME DELAY

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ABSTRACT. An eco-epidemiological model with disease in the predator is studied in this paper. Maturity and digestion delay are considered. It is assumed that the infected predator can't prey. The existence of disease-free and endemic equilibria are obtained. The conditions of locally asymptotic stability of equilibria are obtained by discussing eigenvalue equations of the equilibria. Furthermore, global asymptotic stability of disease-free equilibria and the conditions under which the model hold a Hopf bifurcation are discussed, and a numerical simulation is given.

1. Introduction. Recent research of the epidemiology model based on the species dynamics model has already obtained many results. But most of them are sing-species models with an epidemic. We all know that in nature, a species does not exist alone. There are interactions between different species, which has an important influence on species permanence and the spreading of diseases. So it is necessary to discuss epidemiological models related to several species. But little attention had been paid to this area of research. Furthermore, more and more practical considerations have been added to the models as research has progressed, such as time delay and impulse, which has produced models with more practical significance.

Paper [**7**] discussed a predator-prey model with disease in the prey. The conditions of global stability of equilibriums, Hopf bifurcation and permanence of the system are analyzed. Existence of the Hopf bifurcation was investigated. Paper [**4**] studied an eco-epidemiological model with delays. It was assumed that only prey can spread the disease and the predator mainly eats the infected prey. The invariance of nonnegativity, nature of boundary equilibria and global stability are

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analyzed. The authors showed that a Hopf bifurcation can occur as the delays increased. Papers [**6, 8**] discussed an eco-epidemiological model with disease in the predator. The boundedness of solution and sufficient conditions of locally asymptotically stable equilibria were studied; furthermore, conditions of global stability of the equilibria were also obtained. But the time delay elements were not considered in the model.

In this paper we research an eco-epidemiological model with disease in the predator and consider the maturity and digestion delays and that the prey population grows according to a logistic law: the infected predator doesn't have the ability to prey. Stability of the equilibria and the condition when Hopf bifurcation occurs will be analyzed. These aspects haven't appeared in other papers.

2. Foundation of the model. Let $X = X(t)$ denote the total population density of the prey. The total predator population is composed of two population classes: one is the class of susceptible predator, denoted by $S = S(t)$ and the other is the class of the infected prey, denoted by $I = I(t)$.

We make the following assumptions: (a) the prey population grows according to a logistic law; (b) the disease spreads among the predator population only. The prey are not converted into predator immediately after being caught but need a period of time to be digested. And only the mature susceptible predator has the ability to prey.

From the above assumptions, the model equations are

$$\begin{cases} \dfrac{dX}{dt} = rX(1-(X/K)) - pXS(t-\overline{\tau}_1) \\ \dfrac{dS}{dt} = -cS - \beta SI + kpSX(t-\overline{\tau}_2) \\ \dfrac{dI}{dt} = \beta SI - dI \end{cases} \tag{1}$$

where r is the intrinsic growth rate, K is the environment carrying capacity, p is the predation coefficient, k is the coefficient for converting prey into predator ($0 < k \leq 1$), and c and d are the death rate of the susceptible predator and infected predator, respectively. Incidence is assumed to be the simple mass action incidence βSI, where β is called the transmission coefficient. $\overline{\tau}_1$ and $\overline{\tau}_2$ denote the maturity delay and the digestion delay, respectively. All the coefficients are positive constants.

For the sake of simplicity, we place in dimensionless form the model equations (1) by rescaling the variables on the carrying capacity value K, i.e.,

$$x = \frac{X}{K}, s = \frac{S}{K}, i = \frac{I}{K}$$

and then using as dimensionless time, $\omega = \beta K t$. This leads to the dimensionless equations

$$\begin{cases} \frac{dx}{d\omega} = ax(1-x) - bxs(\omega - \tau_1) \\ \frac{ds}{d\omega} = -gs - si + kbsx(\omega - \tau_2) \\ \frac{di}{d\omega} = si - hi, \end{cases} \tag{2}$$

where $a = r/(\beta K)$, $b = p/\beta$, $g = c/(\beta K)$, $h = d/(\beta K)$, $\tau_1 = \beta K \overline{\tau}_1$ and $\tau_2 = \beta K \overline{\tau}_2$.

The initial conditions are

$$(\varphi_1(\theta), \psi_1(\theta), \psi_2(\theta)) \in C_+ = C[(-\tau, 0), R^3_+]$$

$$\varphi_1(0) > 0, \quad \psi_i(0) > 0, \quad i = 1, 2, \tau \max\{\tau_1, \tau_2\}, \tag{3}$$

where $R^3_+ = \{(x, s, i) \in R^3 : x \geq 0, s \geq 0, i \geq 0\}$. For convenience, in the following we replace ω by t for dimensionless time.

It is easy to see that any solutions of system (2) satisfy $(x(t), s(t), i(t)) \in R^3_+$, $t \geq 0$, and R^3_+ is positively invariant.

3. Analysis of the model.

Lemma 3.1. *Assume that the initial condition of system* (2) *satisfies* $\varphi_1(\theta) \geq 1$, $\theta \in [-\tau, 0]$. *Then either* (i) $x(t) \geq 1$ *for all* $t \geq 0$ *and therefore as* $t \to +\propto$, $(x(t), s(t), i(t) \to E_1 = (1, 0, 0)$ *or* (ii) *there exists* a $t_1 > 0$ *such that* $x(t) < 1$ *for all* $t > t_1$. *If* $\varphi_1(\theta) < 1, \theta \in [-\tau, 0]$, *then* $x(t) < 1$ *for all* $t \geq 0$.

Proof. First we consider $x(t) \geq 1$ for all $t \geq 0$. From the first equation (2) we get

$$\frac{dx}{dt} = ax(1-x) - bxs(t-\tau_1) \leq 0.$$

Let $\lim_{t\to+\propto} x(t) = \eta$. If $\eta > 1$, then by the Barbálat lemma [**2**, page 4, Lemma 1.2.2], we have

$$\begin{aligned} 0 &= \lim_{t\to+\propto} \frac{dx}{dt} = \lim_{t\to+\propto}[ax(1-x) - bxs(t-\tau_1)] \\ &= \lim_{t\to+\propto}[a\eta(1-\eta) - b\eta s(t-\tau_1)] \\ &\leq \lim_{t\to+\propto} a\eta(1-\eta) < 0. \end{aligned}$$

The contradiction shows that $\eta = 1$, i.e.,

$$\lim_{t\to+\propto} x(t) = 1. \tag{4}$$

$x(t)$is differentiable and $x'(t)$ uniformly continuous for $t > 0$, so all the assumptions of the Barbá lat lemma hold true and, therefore

$$\lim_{t\to+\propto} \frac{dx}{dt} = 0. \tag{5}$$

From (4) , we get $\lim_{t\to+\propto} dx/dt = \lim_{t\to+\propto} -bs(t-\tau_1) = 0$. Thus, $\lim_{t\to+\propto} s(t) = 0$. From the third equation of (2), if $\lim_{t\to+\propto} s(t) = 0$, then for any given $\varepsilon > 0$ sufficiently small and satisfying $\varepsilon < h$, there is a $T_\varepsilon > \tau$ such that, for $t > T_\varepsilon$, we have $di/dt = \varepsilon i - hi$. Then, by the comparison theorem, $\lim_{t\to+\propto} i(t) = 0$. This completes case (i).

Assume that assumption (i) is violated. Then there exists a $t_0 > 0$ where $x(t_0) = 1$ for the first time. Then $dx/dt|_{t=t_0} = -bs(t_0-\tau_1) < 0$. This implies that once a solution $x(t)$ has entered into the interval $(0,1)$, then it remains bounded there for all $t > t_0$, i.e., (ii) holds true.

Finally, if $\varphi_1(\theta) < 1, \theta \in [-\tau, 0]$, applying the previous argument it follows that $x(t) < 1$ for all $t \geq 0$. This completes the proof. □

3.1. Equilibria and stability. System (2) has the following nonnegative equilibria, namely,

$$\begin{aligned} E_0 &= (0,0,0), & E_1 &= (1,0,0), \\ E_2 &= \left(\frac{g}{kb}, \frac{a}{b}\left(1-\frac{g}{kb}\right), 0\right), & E_3 &= (x^*, s^*, i^*), \end{aligned}$$

where $x^* = 1-(bh)/a$, $s^* = h$ and $i^* = kb(1-[(bh)/a]) - g$. Of course, E_2 exists if $g/b < k$ and E_3 exists if $k > (g/b)(a/(a-bh))$. It is clear that the existence of E_3 implies the existence of E_2.

Let $\overline{E} = \{\overline{x}, \overline{s}, \overline{i}\}$ be any arbitrary equilibrium. Then the characteristic equation about $\overline{E}$ is

$$\begin{vmatrix} a-2a\overline{x}-b\overline{s}-\lambda & -b\overline{x}e^{-\lambda\tau_1} & 0 \\ kb\overline{s}e^{-\lambda\tau_2} & -g-\overline{i}+kb\overline{x}-\lambda & -\overline{s} \\ 0 & \overline{i} & \overline{s}-h-\lambda \end{vmatrix} = 0. \tag{6}$$

For equilibrium $E_0 = (0,0,0)$, (6) reduces to

$$(\lambda - a)(\lambda + g)(\lambda + h) = 0. \tag{7}$$

Obviously, equation (7) has a root with positive real part. Hence , E_0 is a saddle point.

For equilibrium $E_1 = (1,0,0)$, (6) reduces to

$$(\lambda + a)(\lambda - kb + g)(\lambda + h) = 0.$$

If $k < g/b$, all roots of the equation have negative real parts. Thus, E_1 is locally asymptotically stable. Otherwise, it's unstable; meanwhile, E_2 exists.

For equilibrium $E_2 = (g/(kb), (a/b)(1-g/(kb)), 0)$, equation (6) reduces to

$$\left(\frac{a}{b}\left(1-\frac{g}{kb}\right)-h-\lambda\right)\left[\lambda\left(\lambda+\frac{ag}{kb}\right)+ag\left(1-\frac{g}{kb}\right)e^{-\lambda\tau}\right] = 0, \tag{8}$$

where $\tau = \tau_1 + \tau_2$. We change (8) into $(\lambda - \lambda^*)f(\lambda) = 0$, where $\lambda^* = (a/b)(1-g/(kb))-h$, $f(\lambda) = \lambda(\lambda+(ag)/(kb))+ag(1-g/(kb))e^{-\lambda\tau}$. For the existence of E_2, we have $f(0) = ag(1-g/(kb)) > 0$. So $f(\lambda) = 0$ must have two roots with negative real parts.

Then if $\lambda = \lambda^* < 0$, i.e., $k < (g/b)(a/(a-bh))$, E_2 is locally asymptotically stable. Otherwise, it is unstable; meanwhile, E_3 exists.

We summarize these results:

Theorem 3.1. *E_0 is an unstable saddle point. E_1 is locally asymptotically stable if $k < g/b$, and it is unstable if $k > g/b$.*

Then E_2 exists. Further, E_2 is locally asymptotically stable if $k < (g/b)(a/(a-bh))$. It is unstable if $k > (g/b)(a/(a-bh))$ and $a - bh > 0$; meanwhile, E_3 exists.

Theorem 3.2. *E_1 is globally asymptotically stable if $k \leq g/b$.*

Proof. Let $R^3_{+x} = \{(x, s, i) \in R^3_+ : x > 0, s \geq 0, i \geq 0\}$, and consider the scalar function $V : R^3_{+x} \to R^3_+$, $V(t) = x - 1 - \ln x + s$. From system (2) we have

$$\begin{aligned}\dot{V}(t) &= -(1-x)[1 - x - bs(t-\tau_1)] - [g - kbx(t-\tau_2)]s - si \\ &\leq -(1-x)[1 - x - bs(t-\tau_1)] - [g - kb]s.\end{aligned}$$

The first term on the righthand side of the above formula is always negative in $\operatorname{int} R^3_{+x}$; otherwise, $1 - x - bs(t-\tau_1) < 0$. It is inconsistent that E_0 is a saddle point.

If $k < g/b$, then $\dot{V}(t) < 0$ in $\operatorname{int} R^3_{+x}$ and $\dot{V}(t) = 0$ if and only if $(x, s) = (1, 0)$.

If $k = g/b$, then $\dot{V}(t) \leq -(1-x)[1 - x - bs(t-\tau_1)] \leq 0$ in $\operatorname{int} R^3_{+x}$. The largest positively invariant subset of the set where $\dot{V}(t) = 0$ is $(x, s) = (1, 0)$. Hence, for all solutions of system (2) starting in $\operatorname{int} R^3_{+x}$, we have $\lim_{t\to+\infty} x(t) = 1$, $\lim_{t\to+\infty} s(t) = 0$.

For any given $\varepsilon > 0$ sufficiently small and satisfying $\varepsilon > h$, there is a $T > 0$ such that for $t > T + \tau$, $di/dt \leq \varepsilon i - hi$. By the comparison theorem, we get $\limsup_{t\to+\infty} i(t) \leq 0$. Hence, $\lim_{t\to+\infty} i(t) = 0$. This completes the proof. □

3.2. Hopf bifurcation. The characteristic equation at E_3 is

$$\lambda^3 + (ax^*)\lambda^2 + s^*i^*\lambda + (ax^*)s^*i^* + b^2kx^*s^*e^{-\lambda\tau}\lambda = 0. \tag{9}$$

When $\tau = 0$, (9) yields

$$\lambda^3 + (ax^*)\lambda^2 + (s^*i^* + b^2kx^*s^*)\lambda + (ax^*)s^*i^* = 0. \tag{10}$$

According to the Routh-Hurwitz criterion we know that all roots of (10) have negative real parts, so E_3 is locally asymptotically stable when $\tau = 0$.

If $\lambda = i\omega$, $\omega > 0$, is a root of (10), separating real and imaginary parts, we have the following:

$$\begin{aligned} b^2kx^*s^*\cos\omega\tau &= \omega^2 - s^*i^* \\ b^2kx^*s^*\omega\cos\omega\tau &= ax^*\omega^2 - ax^*s^*i^*, \end{aligned} \tag{11}$$

which implies

$$z^3 + mz^2 + nz + r = 0, \tag{12}$$

where

$$\begin{aligned} \omega^2 &= z \\ m &= (ax^*)^2 - 2s^*i^* \\ n &= (s^*i^*)^2 - 2(ax^*)^2s^*i^* - (b^2kx^*s^*)^2 \\ r &= (ax^*)^2(s^*i^*)^2. \end{aligned}$$

We have $r > 0$ and $m^2 - 3n > 0$. Let

$$z_1^* = \frac{-m + \sqrt{m^2 - 3n}}{3}$$

and

$$f(z) = z^3 + mz^2 + nz + r.$$

We assume that $f(z_1^*) \leq 0$. Then, according to Lemma 2.2 of [**5**], equation (12)has a positive ω_0 satisfying (9), that is, the characteristic equation (9) has a pair of purely imaginary roots of the form $\pm i\omega_0$.

From (11) we know that τ_{0n} corresponding to ω_0 is

$$\tau_{0n} = \frac{1}{\omega_0} \arccos\frac{\omega^2 - s^*i^*}{b^2kx^*s^*} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2 \ldots .$$

For $\tau = 0$, E_3 is stable. Hence by Butler's lemma [**1**], E_3 remains stable for $\tau < \tau_0$, where $\tau_0 = \tau_{0n}$ as $n = 0$.

Next we will show that $d(\operatorname{Re}\lambda)/d\tau|_{x=x_0} > 0$. This will signify that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Then the conditions for Hopf bifurcation [**3**] are satisfied yielding the required periodic solution.

Now, differentiating (9) with respect τ, we get

$$[3\lambda^2 + 2(ax^*)\lambda + s^*i^* + b^2kx^*s^*e^{-\lambda\tau} - \tau b^2kx^*s^*e^{-\lambda\tau}\lambda]\frac{d\lambda}{d\tau} = \lambda^2b^2kx^*s^*e^{-\lambda\tau}.$$

Then

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{3\lambda^2 + 2(ax^*)\lambda + s^*i^* + (1-\tau\lambda)b^2kx^*s^*e^{-\lambda\tau}}{\lambda^2b^2kx^*s^*e^{-\lambda\tau}} \\ &= \frac{3\lambda^2 + 2(ax^*)\lambda + s^*i^*}{-\lambda(\lambda^3 + ax^*\lambda^2 + s^*i^*\lambda + ax^*s^*i^*)} + \frac{1}{\lambda^2} - \frac{\tau}{\lambda}.\end{aligned}$$

Thus,

$$\begin{aligned}&\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} \\ &= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} \\ &= \operatorname{sign}\left\{\operatorname{Re}\left[\frac{3\lambda^2 + 2(ax^*)\lambda + s^*i^*}{-\lambda(\lambda^3 + ax^*\lambda^2 + s^*i^*\lambda + ax^*s^*i^*)} + \frac{1}{\lambda^2}\right]\right\}_{\lambda=i\omega_0} \\ &= \operatorname{sign}\left(\frac{(3\omega_0^2 - s^*i^*)(\omega_0^4 - s^*i^*\omega_0^2) + 2(ax^*)\omega_0(ax^*\omega_0^3 - ax^*s^*i^*\omega_0)}{(\omega_0^4 - s^*i^*\omega_0^2)^2 + (ax^*\omega_0^3 - ax^*s^*i^*\omega_0)^2}\right. \\ &\qquad \left. - \frac{1}{\omega_0^2}\right) \\ &= \operatorname{sign}\left(\frac{(3\omega_0^2 - s^*i^*)(\omega_0^2 - s^*i^*) + 2(ax^*)^2(\omega_0^2 - s^*i^*)}{\omega_0^2(\omega_0^2 - s^*i^*)^2 + (ax^*)^2(\omega_0^2 - s^*i^*)^2} - \frac{1}{\omega_0^2}\right) \\ &= \operatorname{sign}\left(\frac{\omega_0^2[(3\omega_0^2 - s^*i^*) + 2(ax^*)^2] - (\omega_0^2 - s^*i^*)(\omega_0^2 + (ax^*)^2)}{\omega_0^2(\omega_0^2 - s^*i^*)(\omega_0^2 + (ax^*)^2)}\right) \\ &= \operatorname{sign}\left(\frac{2\omega_0^4 + (ax^*)^2\omega_0^2 + (ax^*)^2s^*i^*}{\omega_0^2(\omega_0^2 - s^*i^*)(\omega_0^2 + (ax^*)^2)}\right).\end{aligned}$$

For $z_1^* > s^*i^*$, $f(z_1^*) \leq 0$, we can get one of the positive roots which satisfies $\omega_0^2 > z_1^* > s^*i^*$. Then $(d(\operatorname{Re}\lambda))/d\tau|_{\tau=\tau_0} > 0$.

Hence, Hopf bifurcation occurs at $\omega = \omega_0$, $\tau = \tau_0$. Based on the analysis of Hopf bifurcation above, we can get Theorem 3.3.

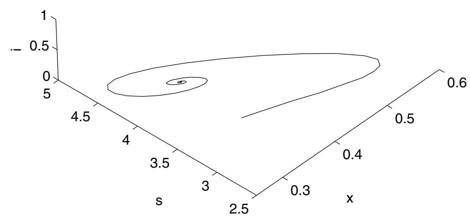

FIGURE 1. $\tau = 0.4$ and E_3 is asymptotically stable where $\tau_1 = 0.2$, $\tau_2 = 0.2$ and initial data $\{x_0, s_0, i_0\} = \{0.3, 3, 0.6\}$.

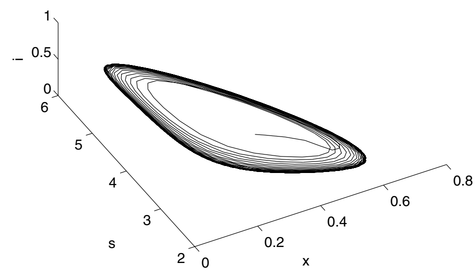

FIGURE 2. $\tau = 0.8$. Periodic solutions occur where $\tau_1 = 0.4$, $\tau_2 = 0.4$, and initial data $\{x_0, s_0, i_0\} = \{0.3, 3, 0.6\}$.

Theorem 3.3. *If $k > (g/b)(a/(a - bh))$ and $f((-m + \sqrt{m^2 - 3n})/3) \le 0$, then as $\tau = \tau_1 + \tau_2$ increases from zero, there is a value τ_0 such that the unique endemic equilibrium E_3 is locally asymptotically stable when $\tau < \tau_0$ and system* (2) *undergoes Hopf bifurcation at E_3 when $\tau = \tau_0$.*

4. Numerical example. In this section, we present some numerical results of system (2) at different values of τ. We consider the following system:

$$\begin{cases} \dfrac{dx}{dt} = 12x(1 - x) - 2xs(t - \tau_1) \\ \dfrac{ds}{dt} = -0.3s - si + 1.6sx(t - \tau_2) \\ \dfrac{di}{dt} = si - 4i. \end{cases}$$

By computing, we can get $\omega_0 = 1.5819$ and $\tau_0 = 0.7549$. Let $\tau_1 = 0.2$ and $\tau_2 = 0.2$. Then $\tau = 0.4 < \tau_0$ and E_3 is asymptotically stable, see

Figure 1. Let $\tau_1 = 0.4$ and $\tau_2 = 0.4$. Then $\tau = 0.8 > \tau_0$, bifurcating periodic solutions from E_3 occur, see Figure 2.

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