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XIA'S ANALYTIC MODEL OF A SUBNORMAL OPERATOR AND ITS APPLICATIONS

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ABSTRACT. In this paper, we provide an exposition and some applications of Xia's analytic model for pure subnormal operators and the associated mosaic. Using the model, we provide a complete set of examples of subnormal operators with rank one or two self-commutator.

1. Preface. One of the best ways to prove theorems about operators contained in a certain category is to show that all operators in the category can be expressed in a certain way. Examples of this include the model of multiplication by the independent variable on an $L^2(\mu)$ space for $*$-cyclic normal operators and the Toeplitz type model for hyponormal operators of Sz.-Nagy and Foiaş. In the late 1980's, Daoxing Xia created such a model for subnormal operators. In his papers, Xia proves that every pure subnormal operator can be expressed as multiplication by the independent variable on a certain type of vector-valued R^2 space.

The purpose of this paper is to give an exposition of this work. We offer new proofs of some of the results and provide many examples to help clarify the model. We also give several applications that have resulted from using the Xia model. It is our hope that this exposition encourages more work in the area. To that end we conclude each section with "Notes and open problems" which gives some additional background to the topics covered in the section as well as some of the related open problems. Furthermore, for the sake of brevity, only illustrative proofs are included. For the complete set of proofs and computations, the reader should contact the authors.

We have tried to be thorough and include all of the results related to the Xia model and to make correct references to theorems that have previously been published. For any omissions, we apologize.

Section 2 gives a thorough exposition of the works of Xia concerning his analytic model. We also describe some of the complete unitary

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invariants for pure subnormal operators that result from this model. One set of complete unitary invariants is the pair of operators $S^*|_{\mathcal{M}}$ and $[S^*, S]|_{\mathcal{M}}$ where $\mathcal{M}$ is the closure of the range of the self-commutator of S. We also explore the Xia mosaic which is another complete unitary invariant when the spectrum of the minimal normal extension has zero area measure.

Section 3 explores these unitary invariants when the self-commutator is of finite rank. In particular, matrices are constructed that classify all of the rationally cyclic, pure subnormal operators with rank 2 self-commutator. We also give a brief summary of the work of Dmitry Yakubovich which classifies all pure subnormal operators with finite rank self-commutator.

2. Xia's analytic model. One of the basic results in operator theory is the spectral theorem for normal operators. It states that if N is a normal operator on a Hilbert space, then there is a measure space (X, Ω, ν) and a function ϕ in $L^\infty(\nu)$ such that N is unitarily equivalent to M_ϕ acting on $L^2(\nu)$. It would be ideal to have a similar result for other classes of operators. One such result is the Foiaş, Pearcy and Sz.-Nagy theory of Hilbert space contractions (see [**22, 48**]). In this section we will show some of the work done towards this end for subnormal operators.

2.1. Notation and background. Throughout the paper all Hilbert spaces will be separable and the algebra of linear operators on a Hilbert space $\mathcal{H}$ will be denoted by $B(\mathcal{H})$. For $T \in B(\mathcal{H})$ the resolvent of T will be denoted by $\rho(T)$, and the spectrum of T will be denoted by $\sigma(T)$. Also, the set of eigenvalues of T will be denoted by $\sigma_p(T)$.

An operator S in $B(\mathcal{H})$ is called *subnormal* if there is a normal operator N on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $N|_{\mathcal{H}} = S$. Such an operator N is called a *normal extension* of S.

S is called *pure* if there does not exist any nontrivial invariant subspace of $\mathcal{H}$ on which S is normal.

S is said to be of *finite type* if it is pure and the rank of the self-commutator of S, $[S^*, S] = S^*S - SS^*$, is finite.

A normal extension N of S is called a *minimal normal extension* (mne) of S if there does not exist any reducing subspace $\mathcal{K}' \subset \mathcal{K}$ of

N which contains $\mathcal{H}$ and for which $N|_{\mathcal{K}'}$ is also a normal extension of S. Since any minimal normal extension of S is unitarily equivalent to any other minimal normal extension we can talk about "the" minimal normal extension of S. Thus, for each subnormal operator S we can define $\sigma_n(S)$ and $\rho_n(S)$ to be the spectrum and resolvent set, respectively, of the minimal normal extension of S. We will call these sets the normal spectrum and the normal resolvent set of S.

To provide some background we give the following common example of a subnormal operator.

Example 2.1. Let ν be a compactly supported measure on $\mathbf{C}$, and let $P^2(\nu)$ denote the closure of polynomials in $L^2(\nu)$. For any compact subset K of $\mathbf{C}$, we define $R^2(K,\nu)$ to be the closure in $L^2(\nu)$ of the rational functions with poles off of K. Finally, we define the Bergman space $L^2_a(K,\nu)$ to be the space of functions analytic in the interior of K. It is evident that if K contains the support of ν, then we have the following inclusion:

$$P^2(\nu) \subset R^2(K,\nu) \subset L^2_a(K,\nu) \subset L^2(\nu).$$

Note. There are examples of measures ν and compact spaces K for which each of the inclusions are strict.

Let N_ν be the operator on $L^2(\nu)$ defined by $N_\nu f = (\cdot)f(\cdot)$. Then the restriction of N_ν to each of these spaces is subnormal. In particular, we define S_ν to be the restriction of N_ν to $P^2(\nu)$.

Bram [**5**] and Singer [**51**] showed independently that every cyclic subnormal operator is unitarily equivalent to S_ν for some compactly supported measure ν on the plane. Then in [**31**] McCarthy and Yang proved that every rationally cyclic subnormal operator with finite rank self-commutator is unitarily equivalent to the restriction of N_ν to $R^2(K,\nu)$ for some compact set K which is a quadrature domain. In [**53**], Daoxing Xia showed that every subnormal operator can be written as a type of direct sum of these operators. He did this by showing that the subnormal operator is unitarily equivalent to multiplication by z on $R^2(K,e)$ for a certain type of compact space K and operator-valued measure e. This multiplication operator associated to a subnormal operator S will be called the *analytic model* of S.

2.2. The analytic model. Let $\mathcal{M}$ be a separable Hilbert space and $e(\cdot)$ a $B(\mathcal{M})$-valued positive measure on a compact support set $\gamma \subset \mathbf{C}$ such that $e(\gamma) = I_{\mathcal{M}}$. If in addition there are operators Λ and C in $B(\mathcal{M})$, with C positive, which satisfy the following properties:

$$\int_\gamma \frac{uI - \Lambda}{u - z} e(\mathrm{d}u) = 0 \tag{2.1}$$

for z in the unbounded component of $\mathbf{C} \setminus \gamma$, and

$$\int_F ((\bar{u}I - \Lambda^*)(uI - \Lambda) - C) e(\mathrm{d}u) = 0 \tag{2.2}$$

for every Borel set $F \subset \gamma$, then we will call $e(\cdot)$ a *compressed spectral measure.*

For each of these compressed spectral measures, we define $L^2(e)$ to be the Hilbert space of all measurable $\mathcal{M}$-valued functions f satisfying

$$||f||^2 = \int_\gamma (e(\mathrm{d}u) f(u), f(u)) < +\infty$$

where f and g are considered the same function if $||f - g|| = 0$.

Let D be the set of $z \in \mathbf{C} \setminus \gamma$ such that (2.1) holds, and let K be the set $\mathbf{C} \setminus D$. Then define the space $R^2(K, e)$ to be the closure in $L^2(e)$ of all linear combinations of the functions $f(\cdot) = (\lambda - (\cdot))^{-1}\alpha$ with $\lambda \in D$ and $\alpha \in \mathcal{M}$.

Then the operator S on $R^2(K, e)$ defined by

$$Sf = (\cdot)f(\cdot), \text{ for } f \in R^2(K, e)$$

is a pure subnormal operator with minimal normal extension N on $L^2(e)$ defined by

$$Nf = (\cdot)f(\cdot), \text{ for } f \in L^2(e).$$

We see from the following theorem that in fact every pure subnormal operator is of this type.

Theorem 2.2 [**53**, Theorem 1]. *Let S be a pure subnormal operator on a separable Hilbert space $\mathcal{H}$ with minimal normal extension N on* *a Hilbert space $\mathcal{K} \supset \mathcal{H}$. Let $\mathcal{M}$ be the defect space $\overline{[S^*, S]\mathcal{H}}$, and let $P_{\mathcal{M}} \in B(\mathcal{K})$ be the projection onto $\mathcal{M}$. If $E(\cdot)$ is the spectral measure of N, then the $B(\mathcal{M})$-valued positive measure $e(\cdot)$ on $\sigma_n(S)$ defined by $e(\cdot) = P_{\mathcal{M}}E(\cdot)P_{\mathcal{M}}^*$ satisfies the following:*

$$e(\sigma_n(S)) = I, \tag{2.3}$$

$$\int_{\sigma_n(S)} \frac{(uI - \Lambda)}{u - z}\, e(\mathrm{d}u) = 0 \tag{2.4}$$

for $z \in \rho(S)$. Taking $\Lambda^* = (S^*|_{\mathcal{M}})$ and $C = [S^*, S]|_{\mathcal{M}}$,

$$\int_F ((\bar{u}I - \Lambda^*)(uI - \Lambda) - C)e(\mathrm{d}u) = 0 \tag{2.5}$$

for every Borel set $F \subset \sigma_n(S)$.

Furthermore, the operator U defined by

$$Uh(N)\alpha = h(\cdot)\alpha,$$

for every bounded Borel function h and $\alpha \in \mathcal{M}$, extends to a unitary operator from $\mathcal{K}$ onto $L^2(e)$ satisfying

$$\begin{aligned} U\mathcal{H} &= R^2(\sigma(S), e), \\ USU^*f &= (\cdot)f(\cdot), \end{aligned}$$

and

$$US^*U^*f = (\bar{\cdot})(f(\cdot) - f(\Lambda)) + \Lambda^* f(\Lambda) \tag{2.6}$$

for $f \in R^2(\sigma(S), e)$, where

$$f(\Lambda) = \int_{\sigma_n(S)} e(\mathrm{d}u)f(u) = P_{\mathcal{M}}U^*f.$$

In the proof of Theorem 2.2, Xia makes use of a $B(\mathcal{M})$-valued analytic function

$$S(z, w) = P_{\mathcal{M}}(\bar{w}I - S^*)^{-1}(zI - S)^{-1}P_{\mathcal{M}}^* \tag{2.7}$$

for $z, w \in \rho(S)$ which he calls the *determining function*. He proves that this determining function can also be written as

$$S(z,w) = ((\bar{w}I - \Lambda^*)(zI - \Lambda) - C)^{-1}. \tag{2.8}$$

This determining function is one of the essential components in the classification of subnormal operators of finite type and so we will include the proof of the equation here.

Proof of equation (2.8). Let $N = \begin{pmatrix} S'^* & 0 \\ A & S \end{pmatrix}$ be the matrix decomposition of N with respect to $K = \mathcal{H}^\perp \oplus \mathcal{H}$. Since $[N^*, N] = 0$, we have that

$$[S^*, S] = AA^*, \qquad [S'^*, S'] = A^*A \text{ and } S^*A = AS'.$$

Thus, $\mathcal{M} = \overline{A\mathcal{H}^\perp}$ and $\mathcal{M}$ is an invariant subspace of S^*.

We now show that $\rho(S) \subset \rho(\Lambda)$ which makes the function

$$T(z,w) = (zI - \Lambda)^{-1}(\bar{w}I - \Lambda^*)^{-1}$$

well defined for $z, w \in \rho(S)$.

If $\lambda \in \rho(S^*)$, then $\lambda \in \rho(N^*)$ since $\sigma(S) \supset \sigma_n(S)$. Thus, for every $x \in \mathcal{H}^\perp$ there is a unique pair of vectors $x_1 \in \mathcal{H}^\perp$ and $x_2 \in \mathcal{H}$ such that

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = (\lambda I - N^*) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\lambda I - S')x_1 - A^*x_2 \\ (\lambda I - S^*)x_2 \end{pmatrix}.$$

Thus, $(\lambda I - S^*)x_2 = 0$ and, since $\lambda \in \rho(S^*)$, we conclude that $x_2 = 0$. Hence, $(\lambda I - S')x_1 = x$.

Multiplying both sides of the equation by A on the left and using the property that $S^*A = AS'$ we find that

$$Ax = A(\lambda I - S')x_1 = (\lambda I - S^*)Ax_1.$$

Thus, $(\lambda I - S^*)\mathcal{M}$ is dense in $\mathcal{M}$ and, since $(\lambda I - S^*)$ is invertible, we conclude that

$$(\lambda I - \Lambda^*)\mathcal{M} = (\lambda I - S^*)\mathcal{M} = \mathcal{M}$$

and

$$\ker(\lambda I - \Lambda^*) \subset \ker(\lambda I - S^*) = \{0\}\,.$$

Hence, $\lambda \in \rho(\Lambda^*)$ and we have that $\rho(S^*) \subset \rho(\Lambda^*)$ and $\sigma(\Lambda^*) \subset \sigma(S^*)$. Using the properties of the adjoint it follows that

$$\rho(S) \subset \rho(\Lambda).$$

Thus, the function $T(z,w) = (zI - \Lambda)^{-1}(\overline{w}I - \Lambda^*)^{-1}$ is well defined for $z, w \in \rho(S)$.

An equivalent definition of $T(\cdot,\cdot)$ is found using inner products. For $u, v \in \mathcal{M}$,

$$\begin{aligned}(T(z,w)u,v) &= ((zI-\Lambda)^{-1}(\overline{w}I-\Lambda^*)^{-1}u,v)\\ &= ((\overline{w}I-\Lambda^*)^{-1}u,(\bar{z}I-\Lambda^*)^{-1}v)\\ &= ((\overline{w}I-S^*)^{-1}u,(\bar{z}I-S^*)^{-1}v)\\ &= ((zI-S)^{-1}(\overline{w}I-S^*)^{-1}u,v).\end{aligned}$$

So $T(z,w) = P_{\mathcal{M}}(zI-S)^{-1}(\overline{w}I-S^*)^{-1}P_{\mathcal{M}}^*$ where $P_{\mathcal{M}}$ is the projection map onto $\mathcal{M}$.

Now the functions $S(\cdot,\cdot)$ and $T(\cdot,\cdot)$ satisfy the following identity.

$$\begin{aligned}&S(z,w) - T(z,w)\\ &= P_{\mathcal{M}}(\overline{w}I-S^*)^{-1}(zI-S)^{-1}P_{\mathcal{M}}^* - P_{\mathcal{M}}(zI-S)^{-1}(\overline{w}I-S^*)^{-1}P_{\mathcal{M}}^*\\ &= P_{\mathcal{M}}(\overline{w}I-S^*)^{-1}(zI-S)^{-1}P_{\mathcal{M}}^*\\ &\quad\times P_{\mathcal{M}}[S^*,S]P_{\mathcal{M}}^*P_{\mathcal{M}}(zI-S)^{-1}(\overline{w}I-S^*)^{-1}P_{\mathcal{M}}^*\\ &= S(z,w)CT(z,w).\end{aligned}$$

Equivalently, $S(z,w) - S(z,w)CT(z,w) = T(z,w)$. Since $T(z,w)$ is invertible this can be rewritten as

$$S(z,w)(T(z,w)^{-1} - C) = I$$

or

$$S(z,w)((\overline{w}I - \Lambda^*)(zI - \Lambda) - C = I.$$

Since $S(z,w) - T(z,w) = T(z,w)CS(z,w)$ as well, we have

$$((\overline{w}I - \Lambda^*)(zI - \Lambda) - C)S(z,w) = I$$

and hence

$$S(z,w) = ((\overline{w}I - \Lambda^*)(zI - \Lambda) - C)^{-1}. \qquad \square$$

In what follows we will always assume that any subnormal operator is in the form of the analytic model and we will omit the unitary operator U.

Before moving on, it might be useful to examine the meaning of $f(\Lambda)$. Note that when f is a real-valued function in $R^2(\sigma(S))$, $f(\Lambda)$ is an operator in $B(\mathcal{M})$ but when $f \in R^2(\sigma(S), e)$, $f(\Lambda) \in \mathcal{M}$. To illustrate, suppose that $f \in R^2(\sigma(S))$ and a is a fixed vector in $\mathcal{M}$. Define $\tilde{f} \in R^2(\sigma(S), e)$ by $\tilde{f}(z) = f(z)a$. Then

$$\begin{aligned} f(\Lambda) &= \int_{\sigma_n(S)} e(\mathrm{d}u) f(u) = \int_{\sigma_n(S)} P_{\mathcal{M}} e(\mathrm{d}u) P_{\mathcal{M}}^* f(u) \\ &= P_{\mathcal{M}} \int_{\sigma_n(S)} f(u) e(\mathrm{d}u) P_{\mathcal{M}}^* = P_{\mathcal{M}} f(N) P_{\mathcal{M}}^*. \end{aligned}$$

On the other hand,

$$\tilde{f}(\Lambda) = \int_{\sigma_n(S)} e(\mathrm{d}u) f(u) a = f(\Lambda) a.$$

Since the dual, S', of S is also a pure subnormal operator with minimal normal extension N^* it is useful to understand the model of S'. Therefore, we list some of the properties of S and S' in the following theorem.

Theorem 2.3. *Let S be the pure subnormal operator on $R^2(K, e)$ defined by $Sf = (\cdot)f(\cdot)$, and let N be its minimal normal extension defined on $L^2(e)$ by $Nf = (\cdot)f(\cdot)$. Let Λ and C be operators on $\mathcal{M}$ satisfying the conditions of the model. Let A and S' be the operators such that*

$$N = \begin{pmatrix} S'^* & 0 \\ A & S \end{pmatrix}$$

with respect to the decomposition $L^2(e) = R^2(K,e)^\perp \oplus R^2(K,e)$. *(Note that since* N *is normal we may assume that* $A : \overline{[S'^*, S']R^2(K,e)^\perp} \to \mathcal{M}$.*) Then for* $f \in R^2(K,e)$ *and* $g \in R^2(K,e)^\perp$ *we have the following:*

$$S^*f = (\bar{\cdot})(f(\cdot) - f(\Lambda)) + \Lambda^* f(\Lambda) \text{ where } f(\Lambda) = \int e(\mathrm{d}u) f(u), \tag{2.9}$$

$$A^* f = ((\bar{\cdot})I - \Lambda^*) f(\Lambda), \tag{2.10}$$

$$Ag = \int (uI - \Lambda) e(\mathrm{d}u) g(u), \tag{2.11}$$

$$S'g = (\bar{\cdot}) g(\cdot), \tag{2.12}$$

$$S'^* g = (\cdot) g(\cdot) - \int (vI - \Lambda) e(\mathrm{d}v) g(v), \tag{2.13}$$

$$C = [S^*, S]|_{\mathcal{M}} = AA^*, \qquad \Lambda^* = S^*|_{\mathcal{M}}, \tag{2.14}$$

and

$$\overline{[S^*, S]R^2(K,e)} = \mathcal{M}. \tag{2.15}$$

Furthermore, if we let

$$\mathcal{M}' = \overline{[S'^*, S']R^2(K,e)^\perp}, \text{ and } C' = [S'^*, S']|_{\mathcal{M}'}, \tag{2.16}$$

then

$$C'g = A^*Ag = ((\bar{\cdot})I - \Lambda^*) \int (vI - \Lambda) e(\mathrm{d}v) g(v) \tag{2.17}$$

and $\Omega = A^* C^{-1/2}|_{\mathcal{M}}$ *is a unitary mapping from* $\mathcal{M}$ *onto* $\mathcal{M}'$.

In order to gain some understanding of the analytic model we will include the proof of the theorem.

Proof. First note that equation (2.9) is the same as the description of S^* in equation (2.6) of Theorem 2.2.

Using the matrix representation of N and $[N^*, N] = 0$ we have the identities

$$[S^*, S] = AA^*, \quad [S'^*, S'] = A^*A, \quad S^*A = AS', \tag{2.18}$$

and, for $f \in R^2(K, e)$ and $g \in R^2(K, e)^\perp$, we have

$$N\begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} S'^*g \\ Sf + Ag \end{pmatrix} \quad \text{and} \quad N^*\begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} A^*f + S'g \\ S^*f \end{pmatrix}. \tag{2.19}$$

Viewing $f \in R^2(K, e)$ as an element of $L^2(e)$,

$$\begin{aligned} A^*f &= N^*f - S^*f \\ &= (\bar{\cdot})f(\cdot) - (\bar{\cdot})(f(\cdot) - f(\Lambda)) - \Lambda^*f(\Lambda) \\ &= ((\bar{\cdot})I - \Lambda^*)f(\Lambda) \end{aligned}$$

and we have (2.10). To prove (2.11) we take $f \in R^2(K, e)$ and $g \in R^2(K, e)^\perp$ and compute the inner product

$$\begin{aligned} (Ag, f)_{R^2} &= (g, A^*f)_{R^{2\perp}} \\ &= \int (e(\mathrm{d}u)g(u), (\bar{u}I - \Lambda^*)f(\Lambda))_{\mathcal{M}'} \\ &= \left(\int (uI - \Lambda)e(\mathrm{d}u)g(u), f(\Lambda)\right)_{\mathcal{M}} \\ &= \left(\int (uI - \Lambda)e(\mathrm{d}u)g(u), \int e(\mathrm{d}v)f(v)\right)_{\mathcal{M}} \\ &= \int \left(e(\mathrm{d}v)\int (uI - \Lambda)e(\mathrm{d}u)g(u), f(v)\right)_{\mathcal{M}} \\ &= \left(\int (uI - \Lambda)e(\mathrm{d}u)g(u), f\right)_{R^2}. \end{aligned}$$

Since $S'g = N^*g$ and $S'^*g = Ng - Ag$ for $g \in R^2(K, e)^\perp$ we have (2.12) and (2.13).

In order to prove (2.14), we look at $\mathcal{M}$ as the set of constant functions in $R^2(K,e)$. From equation (2.9) we see that $S^*a = (\bar{\cdot})(a-a) + \Lambda^* a = \Lambda^* a$ for $a \in \mathcal{M}$. Using (2.9) as well as equation (2.5) we get

$$\begin{aligned}[S^*, S]a &= S^*((\cdot)a) - S(\Lambda^* a)\\ &= (\bar{\cdot})((\cdot)I - \Lambda)a + \Lambda^*(\Lambda a) - (\cdot)\Lambda^* a\\ &= (\bar{\cdot})((\cdot)I - \Lambda)a - \Lambda^*((\cdot)I - \Lambda)a\\ &= ((\bar{\cdot})I - \Lambda^*)((\cdot)I - \Lambda)a\\ &= Ca.\end{aligned}$$

Equation (2.17) is proven in the same way. Since C is positive, we have (2.15). Finally, from (2.14), (2.15), (2.16), and (2.17) we see that $\mathcal{M} = \overline{AR^2(K,e)}$ and $\mathcal{M}' = \overline{A^*(R^2(K,e)^{\perp})}$ and that $\Omega = A^* C^{-1/2}$ is a unitary mapping of $\mathcal{M}$ onto $\mathcal{M}'$ such that $\Omega C \Omega^* = C'$. □

From this we see that S' can be thought of as the multiplication by $\bar{z}$ on the co-analytic functions with properties similar to that of S.

2.3. Unitary invariants. One of the goals of studying the analytic model is to be able to create a classification of all subnormal operators. Along these lines, we make use of two sets of unitary invariants.

The first set of unitary invariants is the pair of operators Λ and C.

Lemma 2.4. *The set of operators* $\{\Lambda, C\}$ *is a complete unitary invariant.*

The following proof of the Lemma is due to Putinar, [**39**]. It is distinct from the proof in [**53**] and is based on a matricial construction of the subnormal operator.

Proof. In [**39**], Putinar proves a decomposition theorem for hyponormal operators which we will use to prove that the set of operators $\{C, \Lambda\}$ discussed above form a complete set of unitary invariants for a pure subnormal operator. This decomposition resembles the Jacobi matrix decomposition of a self-adjoint matrix.

Much of the following comes directly from [**39**] and so we will state many of the results without proof.

If $S \in B(\mathcal{H})$ is a pure subnormal operator on a separable Hilbert space $\mathcal{H}$ with defect space $\mathcal{M}$, then for $n \geq 0$ the spaces

$$G_n = \bigvee_{k=0}^{n} S^k \mathcal{M},$$

form an increasing sequence of subspaces of $\mathcal{H}$. Furthermore, by letting $\mathcal{H}_n = G_n \ominus G_{n-1}$ for $n \geq 1$ and $\mathcal{H}_0 = \mathcal{M}$, we construct a canonical decomposition of the space $\mathcal{H}$ with respect to the operator S which satisfies the following relations:

(i) $\mathcal{H} = \oplus_{k \geq 0} \mathcal{H}_k$;

(ii) $\mathcal{H}_0$ is invariant under S^*;

(iii) $S^* \mathcal{H}_p \subset \mathcal{H}_{p-1} \oplus \mathcal{H}_p$, $p \geq 2$;

(iv) $S \mathcal{H}_p \subset \mathcal{H}_p \oplus \mathcal{H}_{p+1}$, $p \geq 1$;

(v) $\dim \mathcal{H}_{p+1} \leq \dim \mathcal{H}_p$, $p \geq 1$.

Using this decomposition and the fact that the image of the self-commutator of S is contained in $\mathcal{M}$ we have the following two-diagonal matrix representation of S:

$$S = \begin{pmatrix} D_0 & 0 & 0 & 0 & \cdots \\ C_1 & D_1 & 0 & 0 & \cdots \\ 0 & C_2 & D_2 & 0 & \cdots \\ 0 & 0 & C_3 & D_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Moreover, the operators $\{C_j, D_j\}_{j \geq 1}$ above can be constructed by setting

$$C_{n+1} = (C_n^2 - [D_n^*, D_n])^{1/2},$$

and

$$D_{n+1} = C_{n+1}^{-1} D_n C_{n+1}$$

for $n \geq 0$ with $C_0 = C^{1/2}$ and $D_0 = \Lambda$. Since the subnormal operator can be constructed from our pair of invariants, we conclude that the pair $\{C, \Lambda\}$ is a complete unitary invariant. □

This lemma raises the question, "What are the restrictions on operators Λ and C such that they represent a subnormal operator?" In the case when $\mathcal{M}$ is a finite-dimensional space, the question has been investigated by Yakubovich [**63, 64**] and will be discussed in Section 3. In the case that $\mathcal{M}$ is infinite dimensional, nothing is known in general.

Another invariant is what Xia refers to as the mosaic.

Given a subnormal operator S in analytic form, the *mosaic* of S is a function $\mu : \rho_n(S) \to B(\mathcal{M})$ defined by

$$\mu\left(z\right) = \int \frac{uI - \Lambda}{u - z} e(\mathrm{d}u).$$

It has the following property.

Theorem 2.5 [**53**, Theorem 6]. *Let S_1 and S_2 be subnormal operators on the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with minimal normal extensions N_1 and N_2 and mosaics μ_1 and μ_2. Suppose that $\gamma = \sigma(N_1) = \sigma(N_2)$ has zero area measure. If there is a unitary operator V from $\mathcal{M}_1 = \overline{[S_1^*, S_1]\mathcal{H}_1}$ onto $\mathcal{M}_2 = \overline{[S_2^*, S_2]\mathcal{H}_2}$ such that $\mu_2(z) = V\mu_1(z)V^{-1}$ for $z \in \mathbf{C}\backslash\gamma$, then there is a unitary operator $U : \mathcal{K}_1 \to \mathcal{K}_2$ satisfying*

$$S_2 = US_1U^{-1}|_{\mathcal{H}_2} \quad \textit{and} \quad N_2 = UN_1U^{-1}.$$

2.4. The mosaic. We will now develop some of the properties of the mosaic of a subnormal operator S. This will result in, among other things, the identification of $\sigma(S)$, the spectrum of S, in terms of the mosaic. Recall that for a subnormal operator S in analytic form, i.e., S is multiplication by this independent variable on $R^2(K, e)$, the *mosaic* of S is the function $\mu : \rho(S) \to \mathcal{B}(\mathcal{M})$ defined by

$$\mu\left(z\right) = \int \left(u - z\right)^{-1}\left(uI - \Lambda\right) e\left(du\right).$$

We begin by extending $f \in R^2(K, e)$ to be defined on all of $\rho_n(S) = \rho(N)$, the resolvent set of the minimal normal extension of S.

Lemma 2.6 [**53**, Lemma 3]. *If $z \in \rho_n(S)$, then for every $f \in R^2(K, e)$, there is a unique vector in $\mathcal{M}$, denoted by $f[z]$, such that*

$$\frac{f - f\left[z\right]}{\left(\cdot\right) - z} \in R^2\left(K, e\right) \quad \textit{and} \quad \frac{f\left[z\right]}{\left(\cdot\right) - z} \in R^2\left(K, e\right)^{\perp}.$$

$f[z]$ is said to be the value of f at z and satisfies the following properties for any $f \in R^2(K, e)$.

(i) $(\alpha f + \beta g)[z] = \alpha f[z] + \beta g[z]$, *for* $f, g \in R^2(K, e)$, $\alpha, \beta \in \mathbf{C}$ *and* $z \in \rho_n(S)$.

(ii) $((\lambda I - S)^{-1} f)[z] = (\lambda - z)^{-1} f[z]$ *for* $\lambda \in \rho(S)$.

(iii) *If* $\{f_n\} \subset R^2(K, e)$ *and* $\|f_n - f\| \to 0$, *then* $\|f_n[z] - f[z]\| \to 0$.

(iv) $f[z] = \mu(z) f[z]$.

(v) *$f[z]$ is an analytic function of $z \in \rho_n(S)$ which can be computed as*

$$f\,[z] = \int (u - z)^{-1}\,(uI - \Lambda)\, e\,(\mathrm{d}u)\, f\,(u)\,.$$

In his paper, Xia uses the notation $f(z)$ for the value of f at z. We have chosen the alternate notation $f[z]$ since Xia's sense of the value of f differs from the usual sense. For example, if we view $a \in \mathcal{M}$ as a constant function in $R^2(K, e)$, then the usual notion of value would have $a(z) = a$. As we shall see, $a[z]$ need not be a. Moreover, in some of the examples to follow, we will identify $\mathcal{M}$ as a set of complex-valued functions. In this case for a fixed $z_0 \in \rho_n(S)$, $f(z_0)$ is a complex number, while $f[z_0]$ is a function.

The proof of Lemma 2.6 is based on the decomposition of $g = (N - zI)^{-1} f$ as $g_1 + g_2 \in R^2(K, e) \oplus R^2(K, e)^{\perp}$ where N is the minimal normal extension of S. In order to better understand this decomposition we will include the proof.

Proof of Lemma 2.6 (*Existence*). Let N be the minimal normal extension of S and decompose $g = (N - zI)^{-1} f$ as $g_1 + g_2$ from $R^2(K, e) \oplus R^2(K, e)^{\perp}$. Using the matrix decomposition of Theorem 2.3 to re-express the action of N on the two subspaces, we find that

$$\begin{aligned} f &= (N - zI)\, g_1 + (N - zI)\, g_2 \\ &= ((S - zI)\, g_1 + ((S'^{*} + A) - zI)\, g_2 \\ &= ((S - zI)\, g_1 + A g_2) + (S'^{*} - zI) g_2. \end{aligned}$$

Since $f \in R^2(K, e)$, the component from $R^2(K, e)^{\perp}$ must be 0. Applying the analytic definitions of S, S'^{*} and A from (2.11) and (2.13) of

Theorem 2.3, we see that

$$f = (S - zI)\, g_1 + Ag_2 = ((\cdot) - z)\, g_1 + \int (uI - \Lambda)\, e\,(du)\, g_2\,(u)$$

and

$$0 = (S'^{*} - zI)g_2 = ((\cdot) - z)\, g_2 - \int (uI - \Lambda)\, e\,(du)\, g_2\,(u)\,.$$

If we set $f[z] = \int (uI - \Lambda) e(\mathbf{d}u) g_2(u)$, then

$$(2.20)\quad \frac{f - f\,[z]}{(\cdot) - z} = g_1 \in R^2\,(K, e) \quad \text{and} \quad \frac{f\,[z]}{(\cdot) - z} = g_2 \in R^2\,(K, e)^{\perp}\,.$$

(*Uniqueness*). If $a \in \mathcal{M}$ is a vector satisfying

$$(2.21)\qquad \frac{f - a}{(\cdot) - z} \in R^2\,(K, e) \quad \text{and} \quad \frac{a}{(\cdot) - z} \in R^2\,(K, e)^{\perp}\,,$$

then subtraction of each of the corresponding components in (2.20) and (2.21) produces

$$\frac{f\,[z] - a}{(\cdot) - z} \in R^2\,(K, e) \cap R^2\,(K, e)^{\perp} = \{0\}\,.$$

Hence, $f\,[z] = a$.

(iv) Substituting $g_2 = ((\cdot) - z)^{-1} f[z]$ from (2.20) in the definition of $f[z]$ and moving the scalar, we see that

$$\begin{aligned} f\,[z] &= \int (uI - \Lambda)\, e\,(\mathbf{d}u)\, g_2\,(u) \\ &= \int (u - z)^{-1}\,(uI - \Lambda)\, e\,(du)\, f[z] = \mu\,(z)\, f\,[z]\,. \end{aligned}$$

(v) Suppose that $b \in \mathcal{M}$. Then $((\bar{\cdot})I - \Lambda^{*})b \in R^2(K, e)^{\perp}$. Hence,

$$0 = \left(\frac{f - f\,[z]}{(\cdot) - z}, ((\bar{\cdot})\, I - \Lambda^{*})\, b\right) = \left(\int \frac{uI - \Lambda}{u - z} e\,(du)\,(f\,(u) - f\,[z])\,, b\right).$$

Thus,

$$0 = \int \frac{uI - \Lambda}{u - z} e\,(du)\,(f\,(u) - f\,[z])$$

or

$$\int (u-z)^{-1}\,(uI - \Lambda)\,e\,(du)\,f\,(u)$$
$$= \int (u-z)^{-1}\,(uI - \Lambda)\,e\,(du)\,f\,[z]$$
$$= \mu\,(z)\,f\,[z] = f\,[z]\,.$$

(i) and (iii) are immediate consequences of (v).

(ii) Since $[(\lambda - z)^{-1} f[z]]/[(\cdot) - z]$ is a scalar multiple of $f[z]/[(\cdot) - z] \in R^2(K, e)^{\perp}$, we see that $(\lambda - z)^{-1} f(z)$ satisfies the second condition to be the value of $(\lambda I - S)^{-1} f$ at z. To verify the first condition, note that

$$\frac{(\lambda I - S)^{-1}\,f - (\lambda - z)^{-1}\,f\,[z]}{(\cdot) - z}$$
$$= \frac{(\lambda - (\cdot))^{-1}\,f - (\lambda - z)^{-1}\,f\,[z]}{(\cdot) - z}$$
$$= \frac{(\lambda - (\cdot))^{-1}\,(f - f\,[z]) + \Big((\lambda - (\cdot))^{-1} - (\lambda - z)^{-1}\Big)\,f\,[z]}{(\cdot) - z}$$
$$= (\lambda - (\cdot))^{-1}\,\frac{f - f\,[z]}{(\cdot) - z} + (\lambda - (\cdot))^{-1}\,(\lambda - z)^{-1}\,f\,[z]$$
$$= (\lambda I - (\cdot))^{-1} \left(\frac{f - f\,[z]}{(\cdot) - z} + (\lambda - z)^{-1}\,f\,[z]\right).$$

This last expression is in $R^2(K, e)$ since both terms in the right factor are in $R^2(K, e)$ and $(\lambda I - (\cdot))^{-1} = (\lambda I - S)^{-1}$. □

Since $f \in R^2(K, e)$ and $K \cup \rho_n(S) = \mathbf{C}$, it is tempting to conclude from (v) that $f[\cdot]$ is analytic on all of $\mathbf{C}$. This, however, is certainly not the case as (iv) insures that $f[\cdot]$ is zero on the unbounded component of the complement of K.

That $\mu(z) = 0$ on $\mathbf{C}\backslash K$ together with (iv) of Lemma 2.6 suggests that $\mu(z)$ exhibits projection-like behavior. Indeed $\mu(z)$ and $I - \mu(z)$ act as parallel projections of $\mathcal{M}$ onto the spaces

$$\mathcal{M}_z = \left\{ b \in \mathcal{M} : ((\cdot) - z)^{-1} b \in R^2 (K, e) \right\}$$

and

$$\mathcal{M}'_z = \left\{ b \in \mathcal{M} : ((\cdot) - z)^{-1} b \in R^2 (K, e)^{\perp} \right\}.$$

By viewing $a \in \mathcal{M}$ as a constant function in $R^2(K,e)$, we see that $a[z]$, the value of a at z, is that unique vector in $\mathcal{M}$ satisfying

$$a - a[z] \in \mathcal{M}_z \quad \text{and} \quad a[z] \in \mathcal{M}'_z.$$

Moreover, by factoring the constant function a out of the integral in (v) of Lemma 2.6 we see that $a[z] = \mu(z)a$. We conclude that $\mathcal{M} = \mathcal{M}_z + \mathcal{M}'_z$, $\mathcal{M}_z \cap \mathcal{M}'_z = \{0\}$, and that $\mu(z)$ and $I - \mu(z)$ serve as parallel projections from $\mathcal{M}$ onto $\mathcal{M}'_z$ and $\mathcal{M}_z$, respectively. We can now identify $\mathcal{M}'_z$ as the space of values of the constant functions in $R^2(K,e)$. By (iv) of Lemma 2.6 we see that $\mathcal{M}'_z$ also can be identified as the space of values for all functions in $R^2(K,e)$.

These facts are recorded in the following theorem.

Theorem 2.7 [**53**, Theorem 3]. *If μ is the mosaic for the pure subnormal S in analytic form and if $z \in \rho_n(S)$, then*

(i) $\mu(z) = \mu(z)^2$,

(ii) $\mu(z)\mathcal{M} = \mathcal{M}'_z$,

(iii) $(I - \mu(z))\mathcal{M} = \mathcal{M}_z$,

(iv) *when $a \in \mathcal{M}$ is viewed as a constant function in $R^2(K,e)$, $\mu(z)a$ is $a[z]$, the value of a at z, and*

(v) $\mathcal{M}'_z = \{f[z] : f \in R^2(K,e)\}$. *In other words, $\mathcal{M}'_z$ is the space of all values at z.*

If one begins instead with the decomposition of $g = (N^* - \bar{z}I)^{-1}f$ as $g_1 + g_2$ from $R^2(K,e) \oplus R^2(K,e)^{\perp}$, then Lemma 2.6 and Theorem 2.7

become Lemmas 2.8 and 2.9. The proofs of Lemmas 2.8 and 2.9 follow the same lines as the proofs of Lemma 2.6 and Theorem 2.7 and are omitted.

Lemma 2.8 [**53**, Lemma 4]. *If $z \in \rho_n(S)$, then for every $f \in R^2(K,e)^\perp$ there is a unique vector $f^*[z] \in \mathcal{M}$ such that*

$$\frac{\left(\left(\bar{\cdot}\right) - \Lambda^*\right)\left(f - f^*\left[z\right]\right)}{\left(\bar{\cdot}\right) - \bar{z}} \in R^2\left(K,e\right)^\perp$$

and

$$\frac{\left(\left(\bar{\cdot}\right) - \Lambda^*\right) f^*\left[z\right]}{\left(\bar{\cdot}\right) - \bar{z}} \in R^2\left(K,e\right).$$

Lemma 2.9 [**53**, Lemma 5]. *If $z \in \rho_n(S)$, then the operators $\mu(z)^*$ and $I - \mu(z)^*$ are projections from $\mathcal{M}$ onto*

$$\mathcal{M}_z^* = \left\{b \in \mathcal{M} : \left(\left(\cdot\right) - z\right)^{-1}\left(\left(\bar{\cdot}\right) - \Lambda^*\right) b \in R^2\left(K,e\right)\right\}$$

and

$$\mathcal{M}_z'^* = \left\{b \in \mathcal{M} : \left(\left(\cdot\right) - z\right)^{-1}\left(\left(\bar{\cdot}\right) - \Lambda^*\right) b \in R^2\left(K,e\right)^\perp\right\},$$

respectively.

Note that the use of N^* instead of N results in the introduction of the factor $((\bar{\cdot}) - \Lambda^*)$ as well as the reversals of the roles of $R^2(K,e)$ and $R^2(K,e)^\perp$ and of the roles of $\mu(z)$ and $I - \mu(z)$.

The mosaic can be used to identify several special subsets of $\sigma(S)$ and $\rho(S)$. The mosaic tags the resolvent set of N in a manner similar to that of the Fredholm index. Of course, a pure subnormal operator has no point spectrum, but S^* and S'^* may have point spectrum. Additionally, they can be associated with other sets of points with special properties. Let

$$\tau_p\left(S'\right) = \left\{z \in \mathbf{C} : \left(S' - zI\right) R^2\left(K,e\right)^\perp \cap \mathcal{M}' \neq 0\right\}$$

and

$$\nu_p(S) = \left\{ z \in \mathbf{C} : (S - zI)\, R^2(K, e) \cap \mathcal{M} \neq 0 \right\}.$$

It is easy to see that both $\tau_p(S')$ and $\nu_p(S)$ contain $\rho(S)$. The next series of theorems shows how the mosaic can be used to identify these sets as well as the eigenspaces of S^* and S'^*.

Theorem 2.10 [**53**, Theorem 4]. *If S is a pure subnormal operator in analytic form with mosaic μ, then*

$$\sigma_p(S^*) \backslash \sigma_n(S^*) = \{\bar{z} \in \rho_n(S^*) : \mu(z) \neq 0\}$$

and

$$\tau_p(S') \backslash \sigma_n(S^*) = \{\bar{z} \in \rho_n(S^*) : \mu(z) \neq I\}.$$

Moreover, if we define

$$R_z^* = ((\cdot) - \bar{z})^{-1} ((\cdot)\, I - \Lambda^*)\, \mu(z)^* \mathcal{M}$$

and

$$R_z'^* = ((\cdot) - \bar{z})^{-1} ((\cdot)\, I - \Lambda^*) \left(I - \mu(z)^*\right) \mathcal{M},$$

then R_z^* is the eigenspace of S^* corresponding to $\bar{z} \in \sigma_p(S^*) \backslash \sigma(N^*)$,

$$R_z'^* = (S' - \bar{z}I)^{-1} \mathcal{M}' \tag{2.22}$$

and

$$R_z^* \oplus R_z'^* = (N^* - \bar{z}I)^{-1} \mathcal{M}'.$$

The corresponding facts for S'^* are recorded in Theorem 2.11. The statement of the theorem involves the notation K^* which denotes the set $K^* = \{\bar{z} \in \mathbf{C} : z \in K\}$ where K is a subset of $\mathbf{C}$.

Theorem 2.11 [**53**, Theorem 5]. *With the same assumptions as Theorem* 2.10 *if* S' *is the dual of* S, *then*

$$\begin{aligned}\sigma_p(S'^*) \setminus \sigma_n(S) &= \{z \in \boldsymbol{\rho}_n(S) : \mu(z) \neq 0\} \\ &= \sigma_p(S^*)^* \setminus \sigma_n(S)\end{aligned}$$

and

$$\begin{aligned}\nu_p(S) \setminus \sigma_n(S) &= \{z \in \boldsymbol{\rho}(N) : \mu(z) \neq I\} \\ &= \tau_p(S')^* \setminus \sigma_n(S).\end{aligned}$$

If

$$R_z = ((\cdot) - z)^{-1}(I - \mu(z))\mathcal{M}$$

and

$$R_z' = ((\cdot) - z)^{-1}\mu(z)\mathcal{M},$$

then R_z' *is the eigenspace of* S'^* *corresponding to* z,

$$R_z = (S - zI)^{-1}\mathcal{M},$$

and

$$R_z \oplus R_z' = (N - zI)^{-1}\mathcal{M}.$$

Corollary 2.12 [**53**, Corollary 5]. *Under the assumptions of Theorem* 2.10,

$$\dim\ker(S^* - \bar{z}) = \operatorname{rank}\left(\mu(z)^*\right),$$

and

$$\dim\ker(S'^* - z) = \operatorname{rank}(\mu(z)).$$

These results allow a clear characterization of the resolvent and spectrum of S.

Corollary 2.13 [**53**, Corollary 6]. *Under the assumptions of Theorem* 2.10*,*

$$\rho(S) = \{z \in \rho_n(S) : \mu(z) = 0\}, \tag{2.23}$$

$$\sigma(S) = \sigma_n(S) \cup \sigma_p(S^*)^* \tag{2.24}$$

and

$$\sigma(S'^*) = \sigma(S). \tag{2.25}$$

Corollary 2.14 [**53**, Corollary 7]. *Under the assumptions of Theorem* 2.10*,* $\sigma_n(S)$ *contains the boundary of* $\sigma(S)$*.*

As we have seen, the mosaic captures significant information about a subnormal operator. Moreover, the mosaic is a unitary invariant. If S_1 and S_2 are two subnormal operators on the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with minimal normal extensions N_1 and N_2 on $\mathcal{K}_1$ and $\mathcal{K}_2$ and there is a unitary operator $U : \mathcal{K}_1 \to \mathcal{K}_2$ satisfying

$$S_2 = US_1U^{-1}|_{\mathcal{H}_2} \quad \text{and} \quad N_2 = UN_1U^{-1},$$

then the operator $V = U|_{\mathcal{M}_1}$ is a unitary mapping from $\mathcal{M}_1 = \overline{[S_1^*, S_1]\mathcal{H}_1}$ onto $\mathcal{M}_2 = \overline{[S_2^*, S_2]\mathcal{H}_2}$. Moreover,

$$\mu_2(z) = V\mu_1(z)V^{-1} \quad \text{for } z \in \rho(N_1).$$

This fact has a partial converse which gives sufficient conditions under which the mosaic forms a complete unitary invariant.

Theorem 2.15 [**53**, Theorem 6]. *Let* S_1 *and* S_2 *be subnormal operators on the spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *with minimal normal extensions* N_1 *and* N_2 *on* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *and mosaics* μ_1 *and* μ_2*. Suppose that* $\gamma = \sigma(N_1) = \sigma(N_2)$ *has zero area measure. If there is a unitary operator* V *from* $\mathcal{M}_1 = \overline{[S_1^*, S_1]\mathcal{H}_1}$ *onto* $\mathcal{M}_2 = \overline{[S_2^*, S_2]\mathcal{H}_2}$ *such that* $\mu_2(z) = V\mu_1(z)V^{-1}$ *for* $z \in \mathbf{C}\backslash\gamma$*, then there is a unitary operator* $U : \mathcal{K}_1 \to \mathcal{K}_2$ *satisfying*

$$S_2 = US_1U^{-1}|_{\mathcal{H}_2} \quad \textit{and} \quad N_2 = UN_1U^{-1}.$$

The key to the proof of the theorem is that, when γ has zero area measure, the continuous functions on γ are uniformly approximated by rational functions with poles off γ. This allows the unitary equivalence on $\mu(z) = \int (u-z)^{-1}(uI - \Lambda)e(\mathrm{d}u)$ to be extended to any operator of the form

$$\int f(u)\,(uI - \Lambda)\,e\,(\mathrm{d}u)$$

where f is continuous on K. The unitary equivalence is then extended to S by means of the determining function $S(\cdot,\cdot)$ of (2.7).

The requirement that γ have zero area measure is not as onerous as it might first appear. The reason for this will become more clear as we consider the general analytic model for a subnormal of finite type which we will study in Section 3.

2.5. Notes and open problems. In [**56**] Daoxing Xia also created an analytic model for a subnormal tuple of operators. It would be useful to have an exposition of this work similar to the present exposition of the single variable case. Other papers dealing with the analytic model of subnormal tuples include [**24, 34, 58, 59, 61**]. In this extended analytic model the mosaic is a matrix of operators and, instead of a single determining function, we have a set of determining functions.

3. Subnormal operators of finite type. A pure subnormal operator, S, is said to be of *finite type* if the self-commutator, $[S^*, S]$, has finite rank. The most common example of an operator in this class is the unilateral shift which has a rank 1 self-commutator. One of the reasons that we restrict ourselves to this class of subnormal operators is that, for subnormal operators of finite type, the normal spectrum is thin, consisting of an analytic curve along with a finite number of isolated points. This enables us to relate the subnormal operators to certain regions in the complex plane.

One of the best results to date along these lines is the work of McCarthy and Yang, [**30**]. This work creates a classification of rationally cyclic subnormal operators of finite type by classifying them according to their associated domains called quadrature domains.

An alternate strategy would be to classify the subnormal operators of finite type according to the possibilities for the Xia invariants Λ and

C. In the following we exhibit all of the possibilities for Λ and C for rationally cyclic subnormal operators whose self-commutator is rank 1 or 2.

Following this complete discussion of the special cases when the self-commutator has rank one or two, we discuss the work of Yakubovich which moves toward a general classification. Finally, we discuss a complete classification of the subnormal operators in terms of simple operators and isolated points of the spectrum by Yakubovich as found in [**63**].

3.1. Quadrature domains and subnormal operators. A domain Ω in $\mathbf{C}$ is called a *quadrature domain* if there exists a distribution u with finite support in Ω such that

$$\int_\Omega f \mathrm{d}A = u(f)$$

for every integrable analytic function f in Ω where $\mathrm{d}A$ is the area measure in $\mathbf{C}$. The standard first example of a quadrature domain is the unit disk. By Cauchy's theorem,

$$\begin{aligned} \int_{\mathbf{D}} f \mathrm{d}A &= \int_0^1 \int_0^{2\pi} f\left(re^{i\theta}\right) r \,\mathrm{d}\theta \,\mathrm{d}r \\ &= \int_0^1 \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z} \,\mathrm{d}z \,\mathrm{d}r \\ &= \int_0^1 f(0) \,\mathrm{d}r = f(0). \end{aligned}$$

The connection between quadrature domains and subnormal operators was studied by McCarthy and Yang in [**30**] where they prove the following theorem.

Theorem 3.1 [**30**, Theorem 1.12]. *Let S be a rationally cyclic subnormal operator with spectrum K. Since any rationally cyclic subnormal operator is unitarily equivalent to some S_ν we can assume S is multiplication by the coordinate function of some $R^2(K,\nu)$. Let Ω be the interior of K. Then S_ν is irreducible and has finite rank self-commutator if and only if the following conditions are satisfied*:

(i) $K = \overline{\Omega}$.

(ii) Ω *is a quadrature domain.*

(iii) $\nu|_{\partial\Omega}$ *is absolutely continuous with respect to harmonic measure for* Ω*, which we will denote by* ω*, and*

$$\int_{\partial\Omega} \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\omega}\right) \mathrm{d}\omega > -\infty.$$

(iv) $\nu|_{\Omega}$ *is either zero or a finite sum of point masses.*

It would be useful to know more about quadrature domains and their basic properties. A few of the most important and best known properties are identified in the following lemmas.

Lemma 3.2. *Let* Ω *be a bounded open set in* $\mathbf{C}$*.* Ω *is a quadrature domain if and only if there is a function* R *meromorphic in* Ω *and continuously extendable to each point of* $\partial\Omega$ *so that* $R(z) = \bar{z}$ *on the boundary of* Ω*. This function is called the Schwarz function of* Ω*.*

For two distinct proofs of this theorem, the reader is referred to [**1**, Lemma 2.3] and [**14**, page 154].

Lemma 3.3 [**1**, Theorem 3]. *The boundary of* Ω *is an irreducible algebraic curve, except for possibly finitely many points.*

Lemma 3.4 [**25**]. *The Schwarz function* $R(z) = \bar{z}$ *has at most finitely many solutions inside* Ω*.*

Lemma 3.5 [**1**, Theorem 1]. *A bounded simply connected domain is a quadrature domain if and only if it is the conformal image of the unit disk under a rational function.*

For more information on quadrature domains and the Schwarz function the reader is referred to [**1, 14**].

3.2. A study of the one-dimensional case. In order to gain some basic intuition into the analytic model and the mosaic, we will investigate the model when $\mathcal{M} = \mathbf{C}$. In this case, the operators C and Λ are scalars (which we will denote by c and λ) and $e(\cdot)$ is a positive, scalar-valued measure. Note that $c > 0$. The condition that $e(\gamma) = I_{\mathcal{M}} = 1$ implies that $e(\cdot)$ is a probability measure. Now condition (2.2) requires that, for every Borel set F contained in γ,

$$\begin{aligned} 0 &= \int_F \left(\left(\bar{u} - \bar{\lambda}\right)(u - \lambda) - c\right) e\,(\mathrm{d}u) \\ &= \int_F \left(|u - \lambda|^2 - c\right) e\,(\mathrm{d}u)\,. \end{aligned}$$

Hence $|u - \lambda| = \sqrt{c}$ almost everywhere $e(\cdot)$. This implies that γ, the support of $e(\cdot)$, is contained in the circle of radius $\sqrt{c}$ centered at λ. In fact, γ must be the circle of radius $\sqrt{c}$ centered at λ. If it were not, then $\mathbf{C}\backslash\gamma$ would consist of a single unbounded component on which $\mu(z) = 0$. Then Corollary 2.13 would imply that $K = \sigma(S) = \gamma$. But then we would have $R^2(K, e) = L^2(e)$ and S would be normal.

Given this information, we can determine the mosaic.

Any $z \in \mathbf{C}$ satisfying $|z - \lambda| > \sqrt{c}$ is in the unbounded component of $\mathbf{C}\backslash\gamma$ and so

$$\mu\,(z) = \int_\gamma \frac{u - \lambda}{u - z} e\,(du) = 0.$$

For $|z - \lambda| < \sqrt{c}$, we cannot have $\mu(z)$ identically 0 or else we would have $K = \gamma$ and S would be normal. Since $\mu(z)^2 = \mu(z)$ by Theorem 2.7, we can conclude that

$$\mu\,(z) = \int_\gamma \frac{u - \lambda}{u - z} e\,(du) = 1 \text{ for } |z - \lambda| < \sqrt{c}.$$

It is instructive to also determine $\mu(z)$ by considering its properties as a projection onto $\mathcal{M}'_z = \{b \in \mathcal{M} : ((\cdot) - z)^{-1} b \in R^2(K, e)^\perp\}$. Since $a[z] = \mu(z)a$ for $a \in \mathbf{C}$ and $\mu(z)$ must be either 0 or 1, we know that $a[z] = a$ or $a[z] = 0$. Hence, one of

$$\frac{a - a\,[z]}{(\cdot) - z} \quad \text{and} \quad \frac{a\,[z]}{(\cdot) - z}$$

will be 0 and the other will be $a/((\cdot)-z)$. Moreover, this cannot change on a component of $\mathbf{C}\backslash\gamma$ since $a[z]$ is analytic on $\mathbf{C}\backslash\gamma$.

When z is in the unbounded component of $\mathbf{C}\backslash\gamma$,

$$\frac{a}{(\cdot)-z} \in R^2(K,e)$$

so that $a[z]=0$ and $\mu(z)=0$. When z is inside the circle $|u-\lambda|=\sqrt{c}$, we must have

$$\frac{a}{(\cdot)-z} \in R^2(K,e)^{\perp}$$

or else $R^2(K,e)$ would contain all of $C(\gamma)$ and S would be normal. Hence for z inside the circle, $a[z]=a$ and $\mu(z)=1$.

Finally, note that when $\lambda=0$ and $c=1$, we have $R^2(\mathbf{D},e)=P^2(\mathbf{D},e)$ which produces the analytic model of the standard unilateral shift U. All the other possible analytic models when $\mathcal{M}=\mathbf{C}$ are obtained by scaling and translating the model on $\mathbf{D}$. Thus, we have established Morrel's theorem [**32**] that any pure subnormal operator with rank one self-commutator $[S^*,S]$ is unitarily equivalent to $\alpha U+\beta I$ where $\alpha\in\mathbf{R}^+$ and $\beta\in\mathbf{C}$.

3.3. A study of the two-dimensional case. We now turn our attention to the case that $[S^*,S]$ is rank 2.

We begin by considering the case when the subnormal operator is reducible. In this case, the subnormal operator with rank 2 self-commutator is the direct sum of two subnormal operators with rank 1 self-commutators. From our study of the one-dimensional case above, $S=S_1\oplus S_2$ on $\mathcal{H}_1\oplus\mathcal{H}_2$ where $S_j=\alpha_j U+\beta_j$, U is the unilateral shift, $\beta_j\in\mathbf{C}$, and $\alpha_j>0$ for $j=1,2$. Thus, we can write Λ and C as

$$\Lambda=\begin{pmatrix}\beta_1 & 0\\ 0 & \beta_2\end{pmatrix} \quad\text{and}\quad C=\begin{pmatrix}\alpha_1^2 & 0\\ 0 & \alpha_2^2\end{pmatrix}$$

and $\sigma(S)$ is the union of the two circles with radius α_j and center β_j. We also see from the one-dimensional case that

$$\mu(z_0)=\begin{pmatrix}\chi_{\{|z-\beta_1|<\sqrt{\alpha_1}\}}(z_0) & 0\\ 0 & \chi_{\{|z-\beta_2|<\sqrt{\alpha_2}\}}(z_0)\end{pmatrix}$$

where χ_A is the characteristic function of the set A.

From now on we will assume that S is an irreducible, rationally cyclic, pure subnormal operator. From the following well-known lemma we know that the spectrum of S is connected.

Lemma 3.6. *If S is a subnormal operator and $\sigma(S)$ is disconnected, then S is reducible.*

Letting $\mathcal{M} = [S^*, S]\mathcal{H}$, we can write Λ and C as 2×2 matrices. With the appropriate shift we may assume that 0 is in the spectrum of Λ so that

$$\Lambda = \begin{pmatrix} 0 & 0 \\ \lambda & \eta \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} c_{11} & c_{12} \\ \bar{c}_{12} & c_{22} \end{pmatrix}$$

for some λ, η, and c_{12} in $\mathbf{C}$ and some c_{11} and c_{22} positive real numbers. Moreover, $\{0, \eta\} \subset \sigma(S)$ since $\sigma(\Lambda) \subset \sigma(S)$, as was established in the proof of (2.8).

Note that equation (2.5) is equivalent to

$$\det\left(C - (\overline{u}I - \Lambda^*)(uI - \Lambda)\right) = 0 \tag{3.1}$$

for all $u \in \sigma_n(S)$. When u is not in $\sigma(\Lambda)$, equation (3.1) is equivalent to

$$\det\left(C(uI - \Lambda)^{-1} + \Lambda^* - \overline{u}I\right) = 0 \tag{3.2}$$

which, after taking the determinant, becomes

$$\left(\frac{c_{22}}{u-\eta} + \bar{\eta} - \overline{u}\right)\left(\frac{c_{11}}{u} + \frac{\lambda c_{12}}{u(u-\eta)} - \overline{u}\right) = \left(\frac{c_{12}}{u-\eta} + \bar{\lambda}\right)\left(\frac{\overline{c_{12}}}{u} + \frac{\lambda c_{22}}{u(u-\eta)}\right). \tag{3.3}$$

If S is rationally cyclic then we know from [**31**] that $\sigma(S)$ is a quadrature domain. Therefore, there is a Schwarz function, ϕ, associated to $\sigma(S)$, that is meromorphic on $\operatorname{int}(\sigma(S))$, extends continuously to all of $\sigma(S)$ and satisfies $\phi(z) = \bar{z}$ on the boundary of $\sigma(S)$. Since S is rationally cyclic and the corresponding quadrature domain is simply connected, $\sigma(S)$ is the image of the unit disk under a rational function ψ with poles outside of $\mathbf{D}$. By pre-composing ψ with an appropriate linear fractional transformation, we may assume that ψ sends 0 to a point of our choosing from $\operatorname{int}(\sigma(S))$. If we define the function $\widetilde{\psi}$ on the complement of the unit disk by

$$\widetilde{\psi}(z) = \overline{\phi\left(\psi\left(\frac{1}{\bar{z}}\right)\right)}, \tag{3.4}$$

then $\widetilde{\psi}$ is analytic on the complement of the unit disk sending $|z| > 1$ to $\rho(S)$. Note that $\psi(z) = \widetilde{\psi}(z)$ on $\partial\mathbf{D}$ since ψ maps $\partial\mathbf{D}$ to the boundary of $\sigma(S)$ on which $\phi(z) = \bar{z}$. Hence, $\widetilde{\psi}$ is the analytic extension of ψ. We will use the analytic model to determine the possibilities for ϕ and hence for ψ.

Since the normal spectrum contains the boundary of $\sigma(S)$ we can substitute $\phi(u)$ for $\overline{u}$ (3.3) and find that

$$\left(\frac{c_{22}}{u-\eta} + \bar{\eta} - \phi(u)\right)\left(\frac{c_{11}}{u} + \frac{\lambda c_{12}}{u(u-\eta)} - \phi(u)\right) = \left(\frac{c_{12}}{u-\eta} + \bar{\lambda}\right)\left(\frac{\overline{c_{12}}}{u} + \frac{\lambda c_{22}}{u(u-\eta)}\right)$$

on the boundary of $\sigma(S)$, a set large enough to determine the essential character of ϕ. Comparing the poles on either side of the equality, we find that ϕ satisfies one of the following descriptions:

(i) ϕ is an entire function,

(ii) ϕ only has a simple pole at 0,

(iii) ϕ only has a simple pole at η,

(iv) ϕ has a simple pole at 0 and a simple pole at η, or

(v) $\eta = 0$ and ϕ has a double pole at 0.

If ϕ is an entire analytic function then it is constant and $\sigma(S) = \sigma_n(S)$ is a point. In this case, S is normal which contradicts our assumption that S was pure.

For each of the remaining options we will determine the character of the corresponding ψ and provide an example of a pure subnormal operator with rank 2 self-commutator for which we will determine $\mathcal{M}$, Λ, C and $\mu(z)$.

If ϕ has only one simple pole in $\operatorname{int}(\sigma(S))$ (cases (ii) and (iii)), we select ψ to send 0 to that simple pole. When ψ is analytically extended to $\mathbf{C}$ by (3.4) it has one simple pole located at ∞. Therefore, $\psi(z) = bz + c$ for some $b \in \mathbf{C} \setminus \{0\}$ and $c \in \mathbf{C}$. Hence, $\sigma(S)$ is a disk centered at c with radius $|b|$. By scaling and translating, we may assume that $\sigma(S) = \mathbf{D}$.

From the discussion [**31**, page 68] we know that the degree of the rational function plus the number of point masses is equal to the rank of the self-commutator. Since the circle is given by an algebraic function of degree 1, we know that there must be a single point mass in the interior of the disk in order for the self-commutator to have rank 2. Therefore, we know that S is unitarily equivalent to multiplication by z on $P^2(\lambda)$ where λ is the harmonic measure on the boundary of the disk with a point mass in the interior of the disk.

This brings us to the following example.

Example 3.7. Let $\mathbf{T}_a = \partial\mathbf{D} \cup \{a\}$ where $|a| < 1$. Let λ be a measure on $\mathbf{T}_a$ such that

$$d\lambda\left(e^{i\theta}\right) = \frac{d\theta}{2\pi}$$

and $\lambda(\{a\}) = \nu$. Let $S = U_\lambda$ where U_λ is multiplication by z on $P^2(\boldsymbol{\lambda})$, the closure of the polynomials under the inner product $(f,g) = \int_{\partial\mathbf{D}} f(z)\overline{g(z)}\, d\lambda(z) = 1/(2\pi i) \int_{\partial\mathbf{D}} f(z)\overline{g(z)}(dz/z) + \nu f(a)\overline{g(a)}$.

If $a = 0$, then $\{(1/\sqrt{1+\nu}), z\}$ is an orthonormal basis for $\mathcal{M}$ relative to which

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 1/\sqrt{1+\nu} & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 1/(1+\nu) & 0 \\ 0 & \nu/(1+\nu) \end{pmatrix},$$

and

$$\mu(z_0) = \begin{pmatrix} 1/(1+\nu) & z_0/\sqrt{1+\nu} \\ \nu/(z_0(1+\nu)^{3/2}) & \nu/(1+\nu) \end{pmatrix},$$

for $z_0 \in \sigma(S) \setminus \sigma_n(S)$. It is easy to verify that $\operatorname{tr}(\mu(z_0)) = 1$ and that $\mu(z_0)^2 = \mu(z_0)$ for $z_0 \in \sigma(S) \setminus \sigma_n(S)$.

If $a \neq 0$, then $\mathcal{M}$ is spanned by the orthogonal basis $\{f_1, f_2\}$ where $f_2(z) = \bar{a}z/(1-\bar{a}z)$ and $f_1 = (1+\nu\rho^2) - \nu f_2$ with $\rho^2 = |a|^2/(1-|a|^2)$. Relative to the normalization of this basis,

$$\Lambda = \begin{pmatrix} 0 & 0 \\ (a\left(1-|a|^2\right))/(|a|\sqrt{1+\nu-|a|^2}) & a \end{pmatrix},$$

$$C = \begin{pmatrix} 1-\nu\left(1-|a|^2\right)^3\left(\frac{1+\nu|a|^2-|a|^2}{1+\nu-|a|^2}\right) & -\frac{\nu|a|\left(1-|a|^2\right)}{\left(1+\nu|a|^2-|a|^2\right)\sqrt{1+\nu-|a|^2}} \\ -\frac{\nu|a|\left(1-|a|^2\right)}{\left(1+\nu|a|^2-|a|^2\right)\sqrt{1+\nu-|a|^2}} & \nu\left(1-|a|^2\right)^3\left(\frac{1+\nu|a|^2-|a|^2}{1+\nu-|a|^2}\right) \end{pmatrix},$$

and

$$\mu(z_0) = \frac{1}{(1+\nu\rho^2)(1-\bar{a}z_0)(a-z_0)}$$
$$\begin{pmatrix} \left(1+\nu\rho^2\right)(1-\bar{a}z_0)(a-z_0) + \frac{\nu z_0\left(1-|a|^2\right)}{1+\nu\rho^2+\nu} & -\frac{z_0\nu|a|^2+z_0\bar{a}\left(1+\nu\rho^2\right)(a-z_0)}{\sqrt{\rho^2(1+\nu\rho^2+\nu)}} \\ \frac{\nu a\left(1-|a|^2\right)+\nu|a|^2\left(1+\nu\rho^2+\nu\right)(a-z_0)}{\sqrt{\rho^2(1+\nu\rho^2+\nu)^3}} & -\frac{\nu z_0\left(1-|a|^2\right)}{1+\nu\rho^2+\nu} \end{pmatrix}$$

for $z_0 \in \mathbf{D}$. While verifying that $\mu(z_0)^2 = \mu(z_0)$ is quite tedious, it is easy to see that for $z_0 \in \mathbf{D}$, $\operatorname{tr}(\mu(z_0)) = 1$.

Next consider case (v) when ϕ has a double pole at 0. Let ψ be a conformal map from the unit disk onto the spectrum of the subnormal operator so that $\psi(0) = 0$. When we analytically extend ψ to include $|z| \geq 1$ via (3.4) we find that ψ has a double pole at ∞. Therefore, $\psi(z) = bz(z+\alpha) + c$ for some $b, \alpha \in \mathbf{C} \setminus \{0\}$, $c \in \mathbf{C}$. Note that $\alpha \neq 0$ since we are assuming the subnormal operator is irreducible. Since ψ is a degree 2 polynomial we know that there are no point masses in the interior of the spectrum. Hence S is a multiple and shift of the following operator.

Example 3.8. If $S = U(U+\alpha)$ where U is the unilateral shift, then $\mathcal{M}$ has $\{1, z\}$ as an orthonormal basis. Relative to this basis,

$$C = \begin{pmatrix} 1+|\alpha|^2 & \alpha \\ \bar{\alpha} & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix},$$

and

$$\mu(z_0) = \begin{cases} 0_{\mathcal{M}} & 1 < |\gamma_{z_0}|, \\ \begin{pmatrix} \frac{z_0}{\gamma_{z_0}\left(\gamma_{z_0} - \gamma'_{z_0}\right)} & \frac{z_0}{\gamma_{z_0} - \gamma'_{z_0}} \\ \frac{z_0 - \alpha\gamma_{z_0}}{\gamma_{z_0}^2\left(\gamma_{z_0} - \gamma'_{z_0}\right)} & \frac{z_0 - \alpha\gamma_{z_0}}{\gamma_{z_0}\left(\gamma_{z_0} - \gamma'_{z_0}\right)} \end{pmatrix} & |\gamma_{z_0}| < 1 < \left|\gamma'_{z_0}\right|, \\ I_{\mathcal{M}} & \left|\gamma'_{z_0}\right| < 1, \end{cases}$$

where γ_{z_0} and γ'_{z_0} are the two roots of $z(z+\alpha) - z_0$ labeled so that $|\gamma_{z_0}| \le |\gamma'_{z_0}|$. One can verify that $2z_0 - \alpha\gamma_{z_0} = \gamma_{z_0}(\gamma_{z_0} - \gamma'_{z_0})$ from which it will follow, in the middle case, that $\operatorname{tr}(\mu(z_0)) = 1$ and that $\mu(z_0)$ is a projection. It is clear from this formulation that $\operatorname{tr}(\mu(z_0))$ is measuring the multiplicity of preimages of z_0 that lie inside the unit disk.

We now turn to the case that the Schwarz function, ϕ, has a simple pole at η and a simple pole at 0. As before we let ψ be the conformal map from the unit disk onto the subnormal spectrum that takes 0 to 0. Using the same analysis as above, we see that ψ has a simple pole at ∞ and a simple pole at $1/\delta$ for some δ in the unit disk. Since ψ has rational degree 2 we conclude that the subnormal is a dilation and translation of the following operator.

Example 3.9. Consider $S = \alpha U + \delta U(I - \delta U)^{-1}$ where $\alpha \in \mathbf{C} \setminus \{0\}$, $0 < |\delta| < 1$, and U is the unilateral shift.

Then $\{1, (1/\rho)(\delta z/1 - \delta z)\}$ is an orthonormal basis for $\mathcal{M}$ where $\rho = \sqrt{|\delta|^2/(1-|\delta|^2)}$. Relative to this basis,

$$\Lambda = \begin{pmatrix} 0 & 0 \\ \left(\alpha\bar{\delta} + \rho^2\right)\rho & \alpha\bar{\delta} + \rho^2 \end{pmatrix},$$

$$C = \begin{pmatrix} |\alpha|^2 + \alpha\bar{\delta} + \bar{\alpha}\delta + \rho^2 & \alpha\bar{\delta}\rho + \rho^3 \\ \overline{\alpha}\delta\rho + \rho^3 & \rho^4 \end{pmatrix},$$

and, letting γ_{z_0} and γ'_{z_0} be the two roots of $\alpha\delta z^2 - (\alpha + \delta + z_0\delta)z + z_0$ labeled so that $|\gamma_{z_0}| \le |\gamma'_{z_0}|$, we have $\mu(z_0) = 0$ if $1 < |\gamma_{z_0}|$,

$$\mu(z_0) = \begin{pmatrix} \frac{-z_0\left(1 - \delta\gamma_{z_0}\right)}{\alpha\delta\gamma_{z_0}\left(\gamma_{z_0} - \gamma'_{z_0}\right)} & \frac{-z_0}{\alpha\rho\left(\gamma_{z_0} - \gamma'_{z_0}\right)} \\ \frac{\left(\alpha|\delta|^2 + \delta\rho^2\right)\gamma'_{z_0} + \delta\rho^2 z_0\gamma'_{z_0} - \rho^2 z_0}{\rho\alpha^2\delta\gamma'_{z_0}\left(\gamma_{z_0} - \gamma'_{z_0}\right)\left(\bar{\delta} - \gamma'_{z_0}\right)} & \frac{\alpha\delta\rho^2\left(\gamma_{z_0} + \left(\gamma'_{z_0}\right)^2\right) - (\alpha+1)\rho^2\gamma'_{z_0} - |\delta|^2\rho^2 z_0}{\rho^2\alpha\delta\gamma'_{z_0}\left(\gamma_{z_0} - \gamma'_{z_0}\right)\left(\bar{\delta} - \gamma'_{z_0}\right)} \end{pmatrix}$$

if $|\gamma_{z_0}| < 1 < |\gamma'_{z_0}|$, and

$$\mu(z_0) = I_{\mathcal{M}} \text{ if } \left|\gamma'_{z_0}\right| < 1.$$

We have proven the following theorem.

Theorem 3.10. *If S is a pure rationally cyclic subnormal operator with rank 2 self-commutator $[S^*, S]$, then S is unitarily equivalent to a dilation and shift of one of the following operators*:

(i) $S = S_1 \oplus S_2$ *on* $\mathcal{H}_1 \oplus \mathcal{H}_2$ *where* $S_j = \alpha_j U + \beta_j$ *with* $\beta_j \in \mathbf{C}$ *and* $\alpha_j > 0$ *for* $j = 1, 2$,

(ii) $S = U_\lambda$,

(iii) $S = U(U + \alpha)$ *for some* $\alpha \in \mathbf{C} \setminus \{0\}$,

(iv) $S = \alpha U + \delta U(I - \delta U)^{-1}$ *where* $\alpha \in \mathbf{C} \setminus \{0\}$ *and* $0 < |\delta| < 1$.

where U_λ *is the operator from Example* 3.7 *above and* U *is the unilateral shift.*

To further understand this characterization of S in terms of Λ and C, we will examine the operator $S = (1 - \delta U)^{-2}$ in the example below. This example is motivated by the fact that the image of the unit disk under $\phi(z) = (1 - \delta z)^{-2}$ produces a quadrature domain whose associated distribution involves point evaluation of the derivative.

Example 3.11. If $S = (1 - \delta U)^{-2}$ with $0 < |\delta| < 1$, then it is the rational image of degree 2 of the unilateral shift and so has a rank 2 self-commutator, but it is not immediately apparent that S is included in the list of operators in Theorem 3.10. The corresponding computations reveal that $\mathcal{M} = \vee\{(1/1 - \delta z), (1/(1 - \delta z)^2)\}$ and that relative to the normalization of this orthogonal basis

$$\Lambda = \frac{|\delta|}{\left(1 - |\delta|^2\right)^2} \begin{pmatrix} 1/|\delta| & 0 \\ 2 & 1/|\delta| \end{pmatrix}$$

and

$$C = \frac{|\delta|^2}{\left(1 - |\delta|^2\right)^4} \begin{pmatrix} 4 + |\delta|^2 & 2\,|\delta| \\ 2\,|\delta| & |\delta|^2 \end{pmatrix}.$$

If we let $T = (1-|\delta|^2)^2/|\delta|^2 S - 1/|\delta|^2 I$, then

$$\Lambda_T = \begin{pmatrix} 0 & 0 \\ \frac{2}{|\delta|} & 0 \end{pmatrix} \text{ and } C_T = \begin{pmatrix} 1+(4/|\delta|^2) & 2/|\delta| \\ 2/|\delta| & 1 \end{pmatrix}.$$

Thus, we see that S is unitarily equivalent to a dilation and shift of the operator $U(U+(2/|\delta|)I)$ from Example 3.8.

3.4. Restrictions on Λ and C. We know that Λ and C form a complete unitary invariant for a subnormal operator S. We also know that, given Λ and C corresponding to a subnormal operator S, we can reconstruct S. Consequently, it would be useful to have a complete description of the possible operators, Λ and C, that corresponds to a subnormal operator. In the case that Λ and C operate on a finite-dimensional Hilbert space, Yakubovich has discovered such a description based on a topological property of an algebraic curve. We will give a brief exposition of this description in the following. For the proofs of the results the reader is referred to the paper, [**64**].

Let $C > 0$ and Λ be operators on a finite-dimensional Hilbert space $\mathcal{M}$. We associate to C and Λ a polynomial

$$\tau(z,w) = \det\left(C - (w-\Lambda^*)(z-\Lambda)\right)$$

and the algebraic curve

$$\Delta = \left\{(z,w) \in \mathbf{C}^2 : \tau(z,w) = 0\right\},$$

which is called the *discriminant curve* of S. Each point of the discriminant curve will be denoted by $\delta = (z,w)$ with $\mathrm{z}(\delta)$ being the first coordinate of δ and $\mathrm{w}(\delta)$ being the second.

If we decompose τ into irreducible factors τ_j so that

$$\tau(z,w) = \prod_{j=1}^{T} \tau_j(z,w)^{\alpha_j},$$

then we are also able to decompose Δ as

$$\Delta = \bigcup_{j=1}^{T} \Delta_j$$

where $\Delta_j = \{(z, w) : \tau_j(z, w) = 0\}$. A point $\delta \in \Delta$ will be called *regular* if it belongs to only one Δ_j and either

$$\frac{\partial \tau_j}{\partial z}(\delta) \neq 0 \text{ or } \frac{\partial \tau_j}{\partial w}(\delta) \neq 0.$$

δ is called *singular* if it is not regular. The set of all singular points of Δ will be denoted by Δ_s.

Since there are only finitely many points in Δ_s we remove these points from Δ to create $\Delta_0 = \Delta \backslash \Delta_s$. From Δ_0 we can create a unique abstract compact Riemann surface, $\widehat{\Delta}$, that consists of exactly T compact, connected components, $\widehat{\Delta}_j$, where each component is obtained by adding a finite number of points to $\Delta_j \cap \Delta_0$.

A component, $\widehat{\Delta}_j$, will be called *degenerate* if either z or w is constant on the component and *nondegenerate* if it is not degenerate. Let $\widehat{\Delta}_{\mathrm{ndeg}}$ be the union of all of the nondegenerate components of $\widehat{\Delta}$.

The functions $\delta \mapsto \mathrm{z}(\delta)$ and $\delta \mapsto \mathrm{w}(\delta)$ extend to meromorphic functions on $\widehat{\Delta}$, and the function

$$\eta = -\frac{\mathrm{dz}}{\mathrm{dw}},$$

initially defined on regular points of $\widehat{\Delta}_{\mathrm{ndeg}}$, can be extended to a meromorphic function on all of $\widehat{\Delta}$.

Let

$$\widehat{\Delta}_+ = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} : |\eta(\delta)| < 1\} \text{ and } \widehat{\Delta}_- = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} : |\eta(\delta)| > 1\}.$$

Furthermore, if

$$\widehat{\Delta}_{\mathbf{R}} = \{\delta \in \widehat{\Delta}_{\mathrm{ndeg}} : \delta = \delta^*\}$$

is the set of real points of $\widehat{\Delta}$, then the algebraic curve Δ is called *separated* if for any nondegenerate component of $\widehat{\Delta}_k$ of $\widehat{\Delta}$, the set $\widehat{\Delta}_{\mathbf{R}} \cap \widehat{\Delta}_k$ separates $\widehat{\Delta}_k$ into at least two connected components.

For every square matrix, A, and every $\lambda \in \sigma(A)$, define

$$\Pi_\lambda(A) = \chi_\lambda(A),$$

by means of the Reisz-Dunford functional calculus for an analytic function χ_λ where $\chi_\lambda(\lambda) = 1$ and $\chi_\lambda \equiv 0$ on $\sigma(A) \setminus \{\lambda\}$, a finite set of isolated points. For z not in $\sigma(\Lambda)$, a point (z, w) is in Δ if and only if w belongs to $\sigma(C(z-\Lambda)^{-1} + \Lambda^*)$. Recalling that $\mathrm{z}(\cdot)$ is the coordinate function, for $(z,w) \in \Delta_0 \backslash \mathrm{z}^{-1}(\sigma(\Lambda))$ we can define

$$P(z,w) := \Pi_w \left(C(zI - \Lambda)^{-1} + \Lambda^* \right).$$

Then P is a nonzero parallel projection in $\mathcal{M}$.

Theorem 3.12 [**64**, Theorem 1]. *Let $\mathcal{M}$ be a finite-dimensional Hilbert space, let $C > 0$ and Λ be operators on $\mathcal{M}$, and let*

$$\gamma = \{u \in \mathbf{C} | \det\left(C - (\overline{u}I - \Lambda^*)(uI - \Lambda)\right) = 0\}.$$

Define Δ, $\widehat{\Delta}_\pm$, and P as above. Let

$$\mu(z) = \sum_{w:(z,w)\in\widehat{\Delta}_+} P(z,w), \quad z \in \mathbf{C} \setminus (\sigma(\Lambda) \cup \gamma \cup \mathrm{z}(\Delta_s)).$$

There exists a subnormal operator S associated to Λ and C if and only if the following conditions hold:

(i) *Δ is separated.*

(ii) *There exists a positive $B(\mathcal{M})$-valued measure $e(\cdot)$ such that*

$$(3.5)\quad (\Lambda - zI)^{-1}(I - \mu(z)) = \int \frac{e(\mathrm{d}u)}{u - z}, \quad z \in \mathbf{C} \setminus (\sigma(\Lambda) \cup \gamma \cup \mathrm{z}(\Delta_s))$$

and

$$(3.6)\quad (C - (\bar{u}I - \Lambda^*)(uI - \Lambda))\, e(\mathrm{d}u) \equiv 0.$$

If both conditions (3.5) *and* (3.6) *hold, then the measure, $e(\cdot)$, is the compressed spectral measure of S and μ is the Xia mosaic.*

3.5. Classification of subnormal operators of finite type. Another path of research is to classify all of the subnormal operators of finite type. An example of this type of classification is that every rationally cyclic subnormal operator is unitarily equivalent to multiplication by the independent variable on $R^2(K, \nu)$ for some measure ν and compact set K whose interior is a quadrature domain. In the paper, [**63**], Yakubovich gives a classification of all subnormal operators of finite type as multiplication by the independent variable on a vector-valued R^2 space over a quadrature Riemann surface with some possible additional point masses.

As before, we let $\widehat{\Delta}$ be the discriminant surface of a subnormal S of finite type on a Hilbert space $\mathcal{H}$ such that the spectral measure of the minimal normal extension N has no point masses. In this case we can define an $H^\infty(\widehat{\Delta}_+)$ functional calculus for N. The operator S is called *simple* if it admits the $H^\infty(\widehat{\Delta}_+)$ functional calculus, that is, $f(N)\mathcal{H} \subset \mathcal{H}$ for all $f \in H^\infty(\widehat{\Delta}_+)$.

For each pure subnormal operator of finite type, we are able to construct a corresponding simple subnormal operator of finite type in the following canonical way. Since the minimal normal extension, $N \in B(\mathcal{K})$, of S is unitarily equivalent to the operator of multiplication by z on $L^2(e)$, S has no point masses if and only if $e(\cdot)$ is absolutely continuous with respect to arc length measure. Decompose $e(\cdot)$ into $e_a(\cdot)+e_s(\cdot)$ where e_a is absolutely continuous with respect to arc length measure and e_s is a finite sum of point masses. Similarly, decompose $\mathcal{K}$ and N as

$$\mathcal{K} = \mathcal{K}_a \bigoplus \mathcal{K}_s, \text{ and } N = N_a \bigoplus N_s.$$

Now let $\mathcal{H}_0$ be the intersection of $\mathcal{K}_a$ with $\mathcal{H}$, and let L be the projection onto $\mathcal{H}_0$. Then the operator $S_0 := LSL^*$ is a pure subnormal operator without point masses.

Define $\widetilde{\mathcal{H}}_0$ to be the linear manifold

$$\widetilde{\mathcal{H}}_0 := \operatorname{span}\left\{f(N)\mathcal{H}_0 : f \in H^\infty(\widehat{\Delta}_+)\right\}.$$

Then $\widetilde{\mathcal{H}}_0$ is a closed invariant subspace of N containing $\mathcal{H}_0$, $\dim(\widetilde{\mathcal{H}}_0 \ominus \mathcal{H}_0)$ is finite, and $\widetilde{S}_0 := N|_{\widetilde{\mathcal{H}}_0}$ is a simple subnormal operator. We call $\widetilde{S}_0$ the canonical simple subnormal operator that corresponds to S. With these definitions we are able to state the main theorem of [**63**].

Theorem 3.13 [**63**, Theorem 12.3]. *Let S be a subnormal operator of finite type. If S_0 is the operator obtained from S by eliminating* *point masses, and if* $\widetilde{S}_0$ *acting on* $\widetilde{\mathcal{H}}_0$ *is the canonical simple operator corresponding to* S_0*, then*

(i) *There exist eigenvalues* $\bar{\lambda}_k$, $1 \leq k \leq r$ *of* $\widetilde{S}_0^*$ *and corresponding Jordan chains* $\{\psi_{\lambda_k}^j\}_{j=0}^{m_k}$ *of generalized eigenvectors:*

$$\left(\widetilde{S}_0^* - \bar{\lambda}_k I\right)\psi_{\lambda_k}^0 = 0, \quad \left(\widetilde{S}_0^* - \bar{\lambda}_k I\right)\psi_{\lambda_k}^j = \psi_{\lambda_k}^{j-1}, \quad j = 1, \dots, m_k,$$

such that $S_0 = \widetilde{S}_0|_{\mathcal{H}_0}$*, where*

$$\mathcal{H}_0 = \left\{x \in \widetilde{\mathcal{H}}_0 : (x, \psi_{\lambda_k}^j) = 0,\ 1 \leq k \leq r,\ 0 \leq j \leq m_k\right\}$$

(*the* λ_k*'s are not necessarily distinct*).

(ii) *There is a finite set* $\{\mu_j\}$ *and operators* $L_j : \mathcal{H}_0 \to \mathbf{C}^{t_j}$, $t_j \in \mathbf{N}$, $1 \leq j \leq m$, *with* $(S_0^* - \bar{\mu}_j)L_j^* = 0$ *such that the operator* S *coincides with* S_0*, acting on the renormed space* $(\mathcal{H}_0, \|\cdot\|_1)$*, where*

$$\|x\|_1^2 := \|x\|^2 + \sum_{j=1}^{m} \|L_j x\|^2.$$

Conversely, if $\widetilde{S}_0$ *is any simple subnormal operator of finite type and if* S *is obtained from* $\widetilde{S}_0$ *by applying the above procedure, where* $\{\psi_{\lambda_j}^j\}$ *and* $\{L_j\}$ *are arbitrary finite families with the above properties, then* S *is a pure subnormal of finite type.*

3.6. Notes and open problems. With the results of Yakubovich we see that it may be possible to find a better description of which operators Λ and C can be associated with a subnormal operator. However, from the examples of rank 2 self-commutators, we see that this is not going to be simple.

A modest step in this direction can be found in Theorems 1 and 2 of [46]. These theorems address the special case when $\sigma(S) = \overline{\mathbf{D}}$. In this case,

$$\sigma_n(S) = \partial\mathbf{D} \cup \{a_1, a_2, \dots, a_n\}$$

with $\{a_1, a_2, \dots, a_n\} \subset \mathbf{D}$. If we define $Q = C - [\Lambda^*, \Lambda]$, then

(i) Q is a nonzero orthogonal projection,

(ii) $Qe(\{a_i\}) = 0$, $i = 1, 2, \dots, n$, so Q is supported on $\operatorname{ran}(e(\partial\mathbf{D}))$,

(iii) $\Lambda Q = 0$,

(iv) $\Lambda^* e(\{a_i\}) = \bar{a}_i e(\{a_i\})$, and

(v) $C = I - \Lambda\Lambda^* - \sum_{i=1}^n (1 - |a_i|^2) e(\{a_i\})$.

Moreover, these conditions are sufficient for Λ and C to define a subnormal operator.

Note that, in general, Q^2 is the operator C_1 from Putinar's model of a subnormal operator found in the proof of Corollary 2.4. When $\sigma_n(S)$ is the boundary of the disk together with a finite number of points inside the disk, the Stewart-Xia theorems imply that $C_1 = Q$ and that Λ partially decomposes relative to $Q\mathcal{M}$. This does not happen in general. In particular, $Q = C - [\Lambda^*, \Lambda]$ need not be an orthogonal projection as can be seen by considering example Example 3.8 above.

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