ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 1, 2008

WHEN IS BETWEENNESS PRESERVED?

XIANG-DONG HOU AND GREGORY MCCOLM

ABSTRACT. Let V and W be vector spaces over an ordered field $\mathbf{F}$. A map $f : V \to W$ is said to preserve betweenness if, for every $v_1, v_2, v_3 \in V$, if v_2 is between v_1 and v_3 on a line, then $f(v_2)$ is between $f(v_1)$ and $f(v_3)$ on a line. Wetzel asks whether every injection $f : \mathbf{R}^2 \to \mathbf{R}^2$ which preserves betweenness must be affine. We give an affirmative answer after treating the problem in a more general setting.

1. Introduction. Let V and W be vector spaces over an ordered field $\mathbf{F}$ such that $\dim V \geq 2$. It is immediate that every affine injection f from V to W "preserves betweenness" in the sense that for any $v_1, v_2, v_3 \in V$, if v_2 is between v_1 and v_3 on a line, then $f(v_2)$ is between $f(v_1)$ and $f(v_3)$ on a line. But to what extent is the converse true? For this question, we will see that it suffices to assume $V = W = \mathbf{F}^2$, cf. Remark 1.2 below.

It is known that if $\mathbf{F} = \mathbf{R}$, and if $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a bijection that preserves betweenness, then f is affine (see e.g., [**1**, IV.10.4]). This uses the fact [**1**, IV.10.1] that any bijection of $\mathbf{R}^2$ which maps lines to lines is affine. Wetzel [**5**] raises a natural question:

Problem 1.1. *Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be an injection which preserves betweenness. Must f be affine*?

We will answer this question affirmatively (Corollary 4.6). The property of preserving nonbetweenness turns out to be critical. We will show that for any subfield $\mathbf{F} \subseteq \mathbf{R}$, if a map $f : \mathbf{F}^2 \to \mathbf{F}^2$ preserves betweenness and nonbetweenness, then f is affine.

We now define our terms.

Let V be a vector space over an ordered field $\mathbf{F}$. For $a, b \in V$, the closed segment and open segment between a and b, denoted by seg $[a, b]$

2000 AMS *Mathematics subject classification.* Primary 26B99, 51N10.
Keywords and phrases. Archimedean field, affine transformation, betweenness, ordered field, real analysis, injections on vector spaces.
Received by the editors on October 4, 2005.

DOI:10.1216/RMJ-2008-38-1-123

and seg (a,b), respectively, are defined by

$$\operatorname{seg}[a,b] = \{\tau a + (1-\tau)\, b : \tau \in \mathbf{F},\ 0 \le \tau \le 1\},$$
$$\operatorname{seg}(a,b) = \operatorname{seg}[a,b] \setminus \{a,b\}.$$

Points in seg $[a,b]$ are said to be *between* a and b. If $a \neq b$, the unique line in V through a and b is denoted by $\overline{ab}$ and consists of the points $\{\tau a + (1-\tau)b : \tau \in \mathbf{F}\}$.

Let W be another vector space over $\mathbf{F}$, and let $f : V \to W$ be a map. We say that f *preserves betweenness* if for any $a, b \in V$, $f(\operatorname{seg}[a,b]) \subseteq \operatorname{seg}[f(a), f(b)]$. We say that f *preserves nonbetweenness* if for any $a, b, c \in V$ with $b \notin \operatorname{seg}[a,c]$, $f(b) \notin \operatorname{seg}[f(a), f(c)]$. Recall that, for any k, a k-flat in V is a k-dimensional affine subspace of V. Let $|A|$ be the cardinality of a set A.

Let "$\cong$" refer to isomorphism between affine spaces and, given a map f and a subset A of its domain, let $f|_A$ be the restriction of f to A.

Let $\mathbf{F}$ be an ordered field and let $\mathcal{I}(\mathbf{F})$ be the set of all nonempty open intervals in $\mathbf{F}$. (An open interval in $\mathbf{F}$ is of the form $(\alpha, \beta) := \{\gamma \in \mathbf{F} : \alpha < \gamma < \beta\}$ for some $\alpha, \beta \in \mathbf{F}$.) A map $f : \mathbf{F} \to \mathcal{I}(\mathbf{F})$ is called *order preserving* if, for $\gamma_1, \gamma_2 \in \mathbf{F}$ with $\gamma_1 < \gamma_2$, $f(\gamma_i) = (\alpha_i, \beta_i)$, $i = 1, 2$, where $\beta_1 \le \alpha_2$. Call $\mathbf{F}$ *compressible* if there exists an order preserving map $f : \mathbf{F} \to \mathcal{I}(\mathbf{F})$. We will investigate the relations among the following properties of ordered fields $\mathbf{F}$:

P1. Every injection $f : \mathbf{F}^2 \to \mathbf{F}^2$ which preserves betweenness is affine.

P2. Every injection $f : \mathbf{F}^2 \to \mathbf{F}^2$ which preserves betweenness must also preserve nonbetweenness.

P3. There is no injection $f : \mathbf{F}^2 \to \mathbf{F}$ which preserves betweenness.

P4. The field $\mathbf{F}$ is incompressible.

Remark 1.2. We should observe that P1 and P2 are equivalent to the following statements P1$'$ and P2$'$ respectively.

P1$'$. For any vector spaces V and W over $\mathbf{F}$ with $\dim V \ge 2$, every injection $f : V \to W$ which preserves betweenness is affine.

P2$'$. For any vector spaces V and W over $\mathbf{F}$ with $\dim V \ge 2$, every injection $f : V \to W$ which preserves betweenness must also preserve nonbetweenness.

To see that P1 $\Rightarrow$ P1$'$, let A be any 2-flat in V. Since f preserves betweenness, $f(A)$ is contained in a 2-flat B in W. Then the map

$$\mathbf{F}^2 \cong A \xrightarrow{f|_A} B \cong \mathbf{F}^2$$

is an injection which preserves betweenness. Thus $f|_A$ is affine. We claim that since this is true for all 2-flats A in V, f is affine. In fact, for any $x, y \in V$ and $\alpha \in \mathbf{F}$, let $A \subseteq V$ be any 2-flat containing x and y. Then $f(\alpha x+(1-\alpha)y) = f|_A(\alpha x+(1-\alpha)y) = \alpha f|_A(x)+(1-\alpha)f|_A(y) = \alpha f(x) + (1-\alpha)f(y)$. The proof of P2 $\Rightarrow$ P2$'$ is similar to the proof of P1 $\Rightarrow$ P1$'$.

In addition, note that if $\dim(V) \geq 2$ and if $f : V \to W$ preserves betweenness and nonbetweenness, then f must be injective: if $a, b \in V$, $a \neq b$, then choose $x, y \in V$ so that a but not b is between x and y, so $f(a) \in \operatorname{seg}[f(x), f(y)]$ while $f(b) \notin \operatorname{seg}[f(x), f(y)]$, forcing $f(a) \neq f(b)$. P1$'$ implies that every map $f : V \to W$ that preserves betweenness and nonbetweenness is affine.

We will prove the following results.

Theorem 1.3. P1 $\Rightarrow$ P2 $\Leftrightarrow$ P3 $\Rightarrow$ P4.

Theorem 1.4. P3 *holds if* $\mathbf{F} = \mathbf{R}$.

Theorem 1.5. *If* $\mathbf{F} \subseteq \mathbf{R}$*,* P2 $\Rightarrow$ P1*.*

Theorem 1.6. P4 *fails for countable* $\mathbf{F}$*.*

Thus P1 fails for countable ordered fields, but holds for $\mathbf{R}$. This leaves open the situation for uncountable proper subfields of $\mathbf{R}$ (i.e., for uncountable, incomplete Archimedean fields, see [**4**, Section 67, Exercise 3]) and for uncountable non-Archimedean fields. It is also open for which uncountable fields do we have P4 $\Rightarrow$ P3.

In Section 2, we observe certain facts about betweenness and prove Theorems 1.3 and 1.6. The proofs of Theorems 1.4 and 1.5 are more involved and are given in Sections 3 and 4.

2. Betweenness, nonbetweenness, and incompressible fields. We first observe some simple facts about betweenness and nonbetweenness. Again, let V and W be vector spaces over an ordered field $\mathbf{F}$, and let $f : V \to W$ be a map which preserves betweenness. Then, for every line L in V, $f(L)$ is contained in a line in W. If f also preserves nonbetweenness, it is one-to-one, so if L and L' are distinct lines in V intersecting at P, then $f(L) \subseteq M$, $f(L') \subseteq M'$, where M and M' are distinct lines in W intersecting at $f(P)$.

First, a word of warning. A map $f : V \to W$ is said to *preserve noncolinearity* if, for any $a, b, c \in V$ not colinear, $f(a), f(b), f(c) \in W$ are not colinear. While maps preserving nonbetweenness preserve noncolinearity, the converse is not true, which we prove as follows.

Lemma 2.1. *Let $\mathbf{F}$ be any infinite field, and let $a_1, a_2, a_3 \in \mathbf{F}^2$ be distinct but colinear. Then there exists $X = X(a_1, a_2, a_3; \mathbf{F}) \subseteq \mathbf{F}^2$ such that $\{a_1, a_2, a_3\} \subseteq X$, $|X| = |\mathbf{F}|$, and a_1, a_2, a_3 are the only three colinear points in X.*

Proof. Let

$$\mathcal{X} = \Big\{X \subseteq \mathbf{F}^2 :\ \{a_1, a_2, a_3\} \subseteq X \text{ and } a_1, a_2, a_3 \text{ are the only three colinear points in } X\Big\}.$$

Then $(\mathcal{X}, \subseteq)$ is a poset and every chain $\mathcal{C}$ in $(\mathcal{X}, \subseteq)$ has an upper bound $\cup_{X\in\mathcal{C}} X$. By Zorn's lemma, $(\mathcal{X}, \subseteq)$ has a maximal element X_0. We claim that $|X_0| = |\mathbf{F}|$. Suppose to the contrary that $|X_0| < |\mathbf{F}|$. Let $\mathcal{L}$ be the set of all lines in $\mathbf{F}^2$ passing through two points in X_0, and let S be the set of all slopes of lines in $\mathcal{L}$. Then $|S| \le |X_0|^2 < |\mathbf{F}|$. Hence there is a line M in $\mathbf{F}^2$ which is not parallel to any line in $\mathcal{L}$. Since $|M \cap L| = 1$ for each $L \in \mathcal{L}$, we have $|M \cap (\cup_{L\in\mathcal{L}} L)| \le |\mathcal{L}| \le |X_0|^2 < |\mathbf{F}| = |M|$. Let $z \in M \setminus (\cup_{L\in\mathcal{L}} L)$. Clearly, $X_0 \cup \{z\} \in \mathcal{X}$, which contradicts the maximality of X_0. $\square$

Corollary 2.2. *Let $\mathbf{F}$ be an ordered field. Then there is a map $f : \mathbf{F}^2 \to \mathbf{F}^2$ which preserves noncolinearity but does not preserve nonbetweenness.*

Proof. Suppose that $a_1, a_2, a_3 \in \mathbf{F}^2$ are colinear: assume that $a_2 \in \operatorname{seg}(a_1, a_3)$. Choose $X = X(a_1, a_2, a_3; \mathbf{F})$ from Lemma 2.1, and let $f : \mathbf{F}^2 \to X$ be any injection such that $f(a_1) = a_1$, $f(a_2) = a_3$, and $f(a_3) = a_2$. □

Now we turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. P1 $\Rightarrow$ P2 is obvious.

P2 $\Rightarrow$ P3. Suppose to the contrary that there exists an injection $f : \mathbf{F}^2 \to \mathbf{F}$ which preserves betweenness. Then $f : \mathbf{F}^2 \to \mathbf{F} \hookrightarrow \mathbf{F}^2$ preserves betweenness but does not preserve nonbetweenness, which is a contradiction.

P3 $\Rightarrow$ P2. Suppose to the contrary that there exists an injection $f : \mathbf{F}^2 \to \mathbf{F}^2$ which preserves betweenness but does not preserve nonbetweenness. Let $a, b, c \in \mathbf{F}^2$ be such that $b \notin \operatorname{seg}[a, c]$ but $f(b) \in \operatorname{seg}[f(a), f(c)]$. Since a, b, c affinely span $\mathbf{F}^2$, $f(\mathbf{F}^2) \subseteq \overline{f(a)f(c)}$. Thus, the map

$$\mathbf{F}^2 \xrightarrow{f} \overline{f(a)f(c)} \cong \mathbf{F}$$

is an injection that preserves betweenness. This contradicts P3.

P3 $\Rightarrow$ P4. Assume to the contrary that $\mathbf{F}$ is compressible. Let $g : \mathbf{F} \to \mathcal{I}(\mathbf{F})$ be an order preserving map. For each $\alpha \in \mathbf{F}$, we claim that there exists an injection $h_\alpha : \mathbf{F} \to g(\alpha)$ which preserves the order of $\mathbf{F}$. For example, if $g(\alpha) = (-1, 1)$, we can define

$$h_\alpha(\gamma) = \begin{cases} \gamma/(1+\gamma) & \text{if } \gamma \geq 0, \\ \gamma/(1-\gamma) & \text{if } \gamma < 0. \end{cases}$$

Note that $h_\alpha : \mathbf{F} \to (-1, 1)$ is an injection and preserves the order. Define $f : \mathbf{F}^2 \to \mathbf{F}$ by

$$f(\alpha, \gamma) = h_\alpha(\gamma), \quad \text{for all } (\alpha, \gamma) \in \mathbf{F}^2.$$

Then f is an injection which preserves betweenness, contradicting P3. □

We conclude this section with a look at compressibility.

Proposition 2.3. *If an ordered field* $\mathbf{F}$ *has a dense subset* D *such that* $|\mathbf{F}| > |D|$, $\mathbf{F}$ *is incompressible.*

Proof. Otherwise, let $f : \mathbf{F} \to \mathcal{I}(\mathbf{F})$ be an order preserving map. For each $\gamma \in \mathbf{F}$, choose $\delta \in D \cap f(\gamma)$. Then $\gamma \mapsto \delta$ is an injection from $\mathbf{F}$ to D. Thus, $|\mathbf{F}| \leq |D|$, which is a contradiction. $\square$

Proof of Theorem 1.6. We prove that if an ordered field $\mathbf{F}$ is countable, then it is compressible. Let $\mathbf{F} = \{\alpha_0, \alpha_1, \dots\}$. An order preserving map $f : \mathbf{F} \to \mathcal{I}(\mathbf{F})$ can be constructed inductively as follows. Choose $f(\alpha_0) \in \mathcal{I}(\mathbf{F})$ arbitrarily. Suppose that, for $i = 0, \dots, k$, $f(\alpha_i) = (\beta_i, \gamma_i) \in \mathcal{I}(F)$ have been chosen such that for $i, j \in \{0, \dots, k\}$, $\alpha_i < \alpha_j$ implies $\gamma_i < \beta_j$. Let

$$\alpha_{i_0} = \max\{\alpha_i : 0 \leq i \leq k,\ \alpha_i < \alpha_{k+1}\},$$
$$\alpha_{j_0} = \min\{\alpha_j : 0 \leq j \leq k,\ \alpha_j > \alpha_{k+1}\}.$$

Then $\gamma_{i_0} < \beta_{j_0}$. Choose $\beta_{k+1}, \gamma_{k+1} \in \mathbf{F}$ such that $\gamma_{i_0} < \beta_{k+1} < \gamma_{k+1} < \beta_{j_0}$ and define $f(\alpha_{k+1}) = (\beta_{k+1}, \gamma_{k+1}) \in \mathcal{I}(\mathbf{F})$. $\square$

Corollary 2.4. *Let* $\mathbf{F}$ *be a subfield of* $\mathbf{R}$. *Then* $\mathbf{F}$ *is incompressible if and only if* $\mathbf{F}$ *is uncountable.*

Proof. This follows from Propositions 2.3 and Theorem 1.6. $\square$

3. Proof of Theorem 1.4. We now prove Theorem 1.4: P3 holds if $\mathbf{F} = \mathbf{R}$. Suppose to the contrary that there is an injection $f : \mathbf{R}^2 \to \mathbf{R}$ which preserves betweenness. Then f gives each line L in $\mathbf{R}^2$ a well-defined orientation: the unit direction vector v of L such that $f(a+v) > f(a)$ for some (hence all) $a \in L$.

Lemma 3.1. *Within each parallel class of lines, except possibly one parallel class, all lines have the same orientation.*

Proof. We show that there cannot be two parallel classes with lines of opposing orientations. Otherwise, we will have a parallelogram in one of the two cases in Figure 1, where the arrows indicate the orientations.

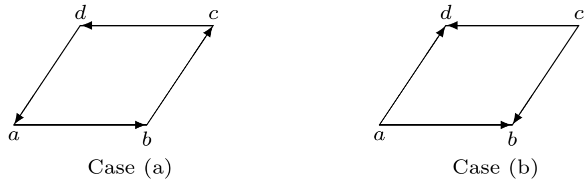


FIGURE 1. Two cases of oriented parallelograms.

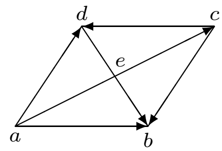


FIGURE 2. A parallelogram with diagonals.

In case (a), we have

$$f(a) < f(b) < f(c) < f(d) < f(a),$$

which is a contradiction. In case (b), without loss of generality, assume that the orientation of the diagonals are as in Figure 2. Then

$$f(e) < f(c) < f(d) < f(e),$$

which is again a contradiction. $\square$

So at most one parallel class can admit lines of opposing orientations. Without loss of generality, suppose that all horizontal lines are oriented to the right and all vertical lines upwards. Then observe that if L_1 and L_2 are two lines with slopes ρ_1, ρ_2 such that $-\infty < \rho_1 < \rho_2 < +\infty$ and L_1 is oriented to the right, then L_2 is also oriented to the right (cf. Figure 3), for if L_2 were oriented leftwards, there would be a cycle.

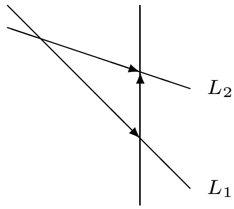


FIGURE 3. Orientations of L_1 and L_2.

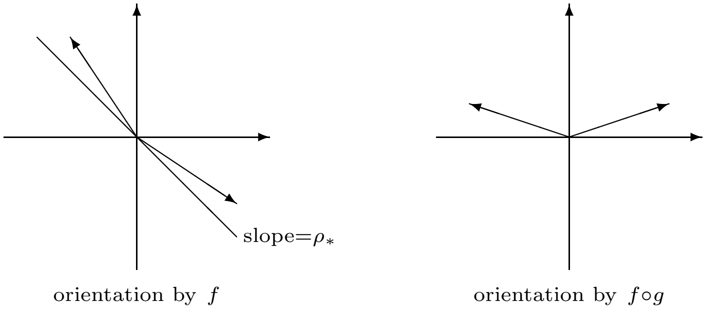


FIGURE 4. Orientations by f and $f \circ g$.

Put

$$P = \{\rho \in \mathbf{R} : \text{lines with slope } \rho \text{ are oriented to the right}\}.$$

From the above observation, P is an interval containing $[0, +\infty)$, and there exists

$$\rho_* = \inf P \in [-\infty, 0) \tag{3.1}$$

such that if $\rho > \rho_*$, then a line of slope ρ is oriented towards the right, while if $\rho < \rho_*$, a line of slope ρ is oriented towards the left, see Figure 4. (If $\rho_* = -\infty$, all nonvertical lines run to the right.)

If $\rho_* = -\infty$, let $g : \mathbf{R}^2 \to \mathbf{R}^2$ be $g(\phi, \psi) = (-\psi, \phi)$; otherwise, let $g(\phi, \psi) = (\phi,\, \psi - \rho_* \phi)$. Note that as g is an injection that preserves betweenness, so is $f \circ g$. Then, oriented by $f \circ g$, all nonhorizontal lines are oriented upwards, and

$$(f \circ g)(\phi_1, \psi_1) < (f \circ g)(\phi_2, \psi_2) \quad \text{whenever} \quad \psi_1 < \psi_2.$$

So if L_ψ is the horizontal line of vertical coordinate ψ, $\psi_1 < \psi_2$ implies that all of $(f \circ g)(L_{\psi_1})$ precedes all of $(f \circ g)(L_{\psi_2})$, i.e., the map $\psi \to (f \circ g)(L_\psi)$ compresses $\mathbf{R}^2$, which is a contradiction to Corollary 2.4.

Remark 3.2. The above proof almost works for any incompressible field. Unfortunately, the critical step (3.1) depends on the completeness of $\mathbf{R}$.

4. Proof of Theorem 1.5. We now prove Theorem 1.5: if $\mathbf{F} \subseteq \mathbf{R}$, then P2 $\Rightarrow$ P1. This will follow from Theorem 4.5.

In this section, we assume that $\mathbf{F}$ is a subfield of $\mathbf{R}$ and $f : \mathbf{F}^2 \to \mathbf{F}^2$ a map which preserves betweenness and nonbetweenness.

Lemma 4.1. *Let L be any line in $\mathbf{F}^2$, and let M be the line in $\mathbf{F}^2$ which contains $f(L)$. Then $f|_L : L \to M$ is continuous.*

Proof. We may assume that both L and M are the horizontal axis. Write $f(\gamma, 0) = (g(\gamma), 0)$, $\gamma \in \mathbf{F}$. It suffices to prove that $g : \mathbf{F} \to \mathbf{F}$ is continuous. Note that since f preserves betweenness, g is monotone; without loss of generality, suppose that g is increasing.

Suppose to the contrary that g is not continuous. Without loss of generality, suppose that g is not right continuous at 0. Assume $g(0) = 0$ and let

$$\omega = \lim_{\substack{\gamma \to 0^+ \\ \gamma \in \mathbf{F}}} g(\gamma). \tag{4.1}$$

Then $\omega \in \mathbf{R}$ and $\omega > 0$.

By (4.1), we can choose $\beta \in \mathbf{F}$, $\beta > 0$, such that $\beta' := g(\beta)$ is sufficiently close to ω for the lines $\overline{f(0,1)(0,\omega)}$ and $\overline{f(0,2)(\beta',0)}$ to intersect at some point r such that $(\omega, 0)$ is between $f(0,1)$ and r and $(\beta', 0)$ is between $f(2,0)$ and r. (See Figure 5.)

Now choose $p \in \mathbf{F}^2$ on $\overline{(0,2)(\beta,0)}$ so that $(0,2)$ lies between p and $(\beta, 0)$. Let $(\alpha, 0)$ be the intersection of $\overline{p(0,1)}$ and L, and let $f(\alpha, 0) = (\alpha', 0)$. As the intersection of $\overline{(0,1)(\alpha,0)}$ and $\overline{(0,2)(\beta,0)}$ is p,

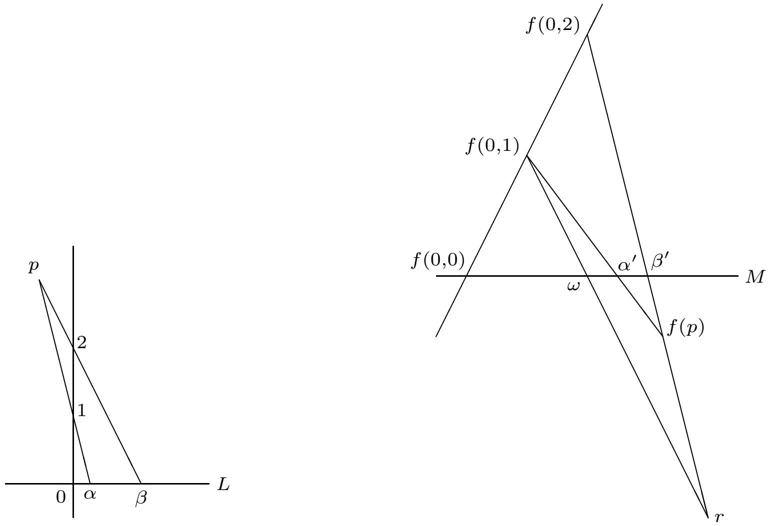


FIGURE 5. Proof of Lemma 4.1.

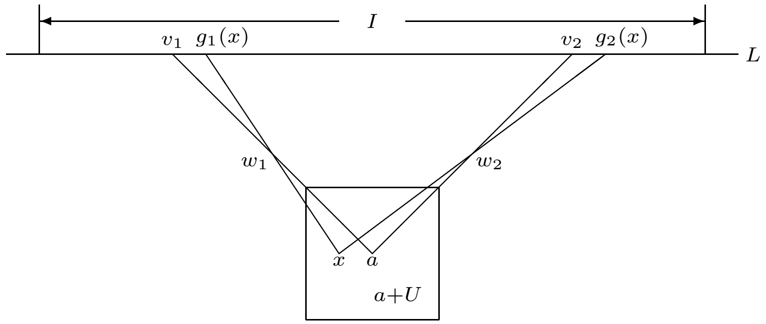
the intersection of $\overline{f(0,1)(\alpha',0)}$ and $\overline{f(0,2)(\beta',0)}$ is $f(p)$. As $0 < \alpha < \beta$, $\omega < \alpha' < \beta'$ and $f(p)$ is between $f(\beta,0) = (\beta',0)$ and r (see Figure 5). Then $f(0,2)$ is not between $f(\beta,0)$ and $f(p)$, contradicting the fact that $(0,2)$ is between $(\beta,0)$ and p and that f preserves betweenness. □

Remark 4.2. As g in Lemma 4.1 is monotone and continuous on a dense subset of $\mathbf{R}$, it can be uniquely extended to a continuous monotone function $g^* : \mathbf{R} \to \mathbf{R}$, which is thus uniformly continuous on bounded closed intervals of $\mathbf{R}$. Thus, g itself is uniformly continuous on bounded closed intervals of $\mathbf{F}$.

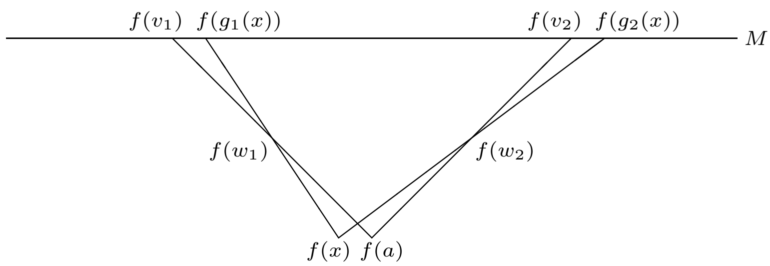
Lemma 4.3. *Let $U = [-1/4, 1/4] \times [-1/4, 1/4] \subseteq \mathbf{F}^2$. Then for every $a \in \mathbf{F}^2$, f is uniformly continuous on $a + U$.*

Proof. Without loss of generality, assume $a = (0,0)$. Let

$$v_1 = (-2,2), \quad w_1 = (-1,1), \quad v_2 = (2,2), \quad w_2 = (1,1).$$



(a) the domain $\mathbb{F}^2$



(b) the target $\mathbb{F}^2$

FIGURE 6. Proof of Lemma 4.3.

Put $L = \overline{v_1 v_2}$ and $M = \overline{f(v_1)f(v_2)}$. Identify both L and M with $\mathbf{F}$. By Lemma 4.1, $f|_L : \mathbf{F} \to \mathbf{F}$ is monotone and continuous. By the above remark, $f|_L$ is uniformly continuous on every bounded interval of $\mathbf{F}$.

For every $x \in a+U$, and for $i = 1, 2$, let $g_i(x)$ be the intersection point of $\overline{xw_i}$ and L. Then $f(x)$ is the intersection point of $\overline{f(g_i(x))f(w_i)}$, $i = 1, 2$. (See Figure 6.) Let I be a bounded closed interval on L such that $I \supseteq g_i(a+U)$, $i = 1, 2$. Clearly, g_i is uniformly continuous on $a+U$. Since f is uniformly continuous on I, $f(g_i(x))$ is uniformly continuous on $a+U$. It follows that $f(x)$ is uniformly continuous on $a+U$. (See Figure 6 (b).) $\square$

Lemma 4.4. *The map f has a unique continuous extension $f^* : \mathbf{R}^2 \to \mathbf{R}^2$. Moreover, f^* preserves betweenness and nonbetweenness.*

Proof. Since f is locally uniformly continuous on $\mathbf{F}^2$ (Lemma 4.3) and since $\mathbf{F}^2$ is dense in $\mathbf{R}^2$, the existence and uniqueness of the continuous extension f^* follow from a simple fact in analysis. (See, e.g., [**2**, Chapter 7, Proposition 11].)

To prove that f^* preserves betweenness, let $a, b, c \in \mathbf{R}^2$ be such that $b = \tau a + (1-\tau)c$ for some $\tau \in \mathbf{R}$, $0 \le \tau \le 1$. Choose sequences a_n, c_n in $\mathbf{F}^2$ such that $a_n \to a$, $c_n \to c$, and choose a sequence τ_n in $\mathbf{F}$ such that $0 \le \tau_n \le 1$ and $\tau_n \to \tau$. Put $b_n = \tau_n a_n + (1-\tau_n)c_n \in \mathbf{F}^2$. Then $b_n \to b$. Since f preserves betweenness, we have:

(1) as the slope of $\overline{f(a_n)f(b_n)}$ is the same as $\overline{f(b_n)f(c_n)}$,

$$\det \begin{bmatrix} f(a_n) - f(b_n) \\ f(c_n) - f(b_n) \end{bmatrix} = 0$$

and

(2) as the vectors $f(a_n) - f(b_n)$ and $f(c_n) - f(b_n)$ point in opposite directions,

$$\big[f(a_n) - f(b_n)\big]\big[f(c_n) - f(b_n)\big]^T \le 0.$$

Letting $n \to \infty$, we have, by the continuity of matrix operations,

$$\det \begin{bmatrix} f^*(a) - f^*(b) \\ f^*(c) - f^*(b) \end{bmatrix} = 0$$

and

$$\big[f^*(a) - f^*(b)\big]\big[f^*(c) - f^*(b)\big]^T \le 0.$$

Hence, $f^*(b)$ is between $f^*(a)$ and $f^*(c)$.

To prove that f^* preserves nonbetweenness, by Theorems 1.3 and 1.4, it suffices to show that f^* is one-to-one. Assume to the contrary that $f^*(a) = f^*(b)$ for some $a, b \in \mathbf{R}^2$ with $a \ne b$. Since f^* preserves betweenness, $f^*(c) = f^*(a)$ for all $c \in \operatorname{seg}[a, b]$. Choose $u, v, w \in \mathbf{F}^2 \setminus \overline{ab}$ not colinear such that $\overline{uv}$ and $\overline{uw}$ (viewed as lines in $\mathbf{R}^2$) intersect $\operatorname{seg}[a, b]$ at v' and w'. (See Figure 7.) Since $\overline{f(u)f(v)}$ and $\overline{f(u)f(w)}$ are nonparallel lines in $\mathbf{F}^2$, they are also nonparallel

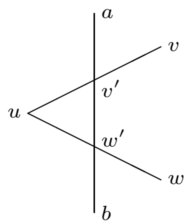


FIGURE 7. Proof of Lemma 4.4.

lines in $\mathbf{R}^2$. Clearly, $f(u)$ is the intersection point of $\overline{f(u)f(v)}$ and $\overline{f(u)f(w)}$ in $\mathbf{R}^2$. On the other hand, $f^*(a) = f^*(v') \in \overline{f(u)f(v)}$ and $f^*(a) = f^*(w') \in \overline{f(u)f(w)}$. Hence, $f^*(a)$ is another intersection point of $\overline{f(u)f(v)}$ and $\overline{f(u)f(w)}$ in $\mathbf{R}^2$. Therefore, $\overline{f(u)f(v)} = \overline{f(u)f(w)}$, which is impossible since $f : \mathbf{F}^2 \to \mathbf{F}^2$ is one-to-one. □

To establish Theorem 1.5, it suffices to prove the following theorem.

Theorem 4.5. *Let* $\mathbf{F}$ *be a subfield of* $\mathbf{R}$ *and* $f : \mathbf{F}^2 \to \mathbf{F}^2$ *a map which preserves betweenness and nonbetweenness. Then* f *is affine.*

Proof. By Lemma 4.4, f has a unique continuous extension $f^* : \mathbf{R}^2 \to \mathbf{R}^2$ which preserves betweenness and nonbetweenness. Note that f^* maps every line segment onto a line segment.

We claim that f^* maps every line L in $\mathbf{R}^2$ onto a line in $\mathbf{R}^2$. Choose two distinct points $a_1, a_2 \in L$ and $b \in \mathbf{R}^2 \setminus L$. We claim that $\overline{f^*(a_1)f^*(a_2)} = f^*(L)$; as f^* preserves betweenness, $f^*(L) \subseteq \overline{f^*(a_1)f^*(a_2)}$, and it suffices to prove that $\overline{f^*(a_1)f^*(a_2)} \subseteq f^*(L)$. We already have

$$\operatorname{seg}\left[f^*(a_1), f^*(a_2)\right] = f^*\big(\operatorname{seg}[a_1, a_2]\big) \subseteq f^*(L).$$

If $u \in \overline{f^*(a_1)f^*(a_2)} \setminus \operatorname{seg}\left[f^*(a_1), f^*(a_2)\right]$, let M be a line through u which intersects both $\operatorname{seg}(f^*(a_1), f^*(b))$ and $\operatorname{seg}(f^*(a_2), f^*(b))$, say at

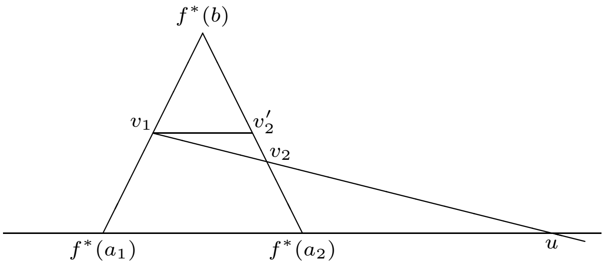


FIGURE 8. Proof of Theorem 4.5.

v_1 and v_2, respectively. (See Figure 8.) Note that seg $[f^*(a_i), f^*(b)] = f^*(\text{seg}\,[a_i, b]) \subseteq f^*(\mathbf{R}^2)$, $i = 1, 2$. We claim that $\overline{(f^*)^{-1}(v_1)(f^*)^{-1}(v_2)}$ is not parallel to L. Let $v_2' \in$ seg $[f^*(a_2), f^*(b)]$ be such that $\overline{v_1 v_2'}$ is parallel to $\overline{f^*(a_1)f^*(a_2)}$, so the line through $(f^*)^{-1}(v_1)$ and parallel to L is $\overline{(f^*)^{-1}(v_1)(f^*)^{-1}(v_2')}$, which is different from $\overline{(f^*)^{-1}(v_1)(f^*)^{-1}(v_2)}$. Let

$$\overline{(f^*)^{-1}(v_1), (f^*)^{-1}(v_2)} \cap L = \{c\}.$$

Then $u = f(c) \in f^*(L)$.

From the fact that f^* preserves betweenness and nonbetweenness and maps lines onto lines, its follows immediately that $f^* : \mathbf{R}^2 \to \mathbf{R}^2$ is onto.

It is a classical result, see e.g., [**3**, Proposition 69.1], that if V is an $n(\geq 2)$-dimensional vector space over a field with at least three elements, then any bijection $g : V \to V$ which maps lines to lines is semi-affine. Consequently, $f^* : \mathbf{R}^2 \to \mathbf{R}^2$ is semi-affine. Since the automorphism group of $\mathbf{R}$ is trivial, f^* is affine. It follows immediately that $f = f^*|_{\mathbf{F}^2} : \mathbf{F}^2 \to \mathbf{F}^2$ is affine. □

Corollary 4.6. P1 *holds for* $\mathbf{F} = \mathbf{R}$.

Proof. Simply combine Theorems 1.3, 1.4 and 1.5. □

REFERENCES

1. D. Gans, *Transformations and geometries*, Appleton-Century-Crofts, New York, 1969.

2. H.L. Royden, *Real analysis*, The Macmillan Company, New York, 1968.

3. E. Snapper and R.J. Troyer, *Metric affine geometry*, Academic Press, New York, 1971.

4. B.L. van der Waerden, *Algebra*, 7th ed., Frederick Ungar Publishing Co., New York, 1970.

5. J. Wetzel, private communication, 2004.

Department of Mathematics, University of South Florida, Tampa, FL 33620
Email address: xhou@math.usf.edu

Department of Mathematics, University of South Florida, Tampa, FL 33620
Email address: mccolm@cas.usf.edu