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ON NONVANISHING SOLUTIONS OF A CLASS OF FUNCTIONAL EQUATIONS

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ABSTRACT. We study some properties of the solutions of a class of functional equations. For example, we prove that if $w(z)$ is a function analytic in the closed unit disc $\{z \mid |z| \leq 1\}$ such that $w(0) = 0$, $0 < \varepsilon < 1$, and

$$|w'(z)| + \varepsilon |w(z)| = 1, \quad |z| = 1,$$

then $w(z) \neq 0$ if $|z| = 1$.

1. Introduction. Let $\mathbf{D}$ denote the unit disc $\{z \mid |z| < 1\}$ of the complex plane $\mathbf{C}$ and, for Ω a subset of $\mathbf{C}$, $\mathcal{H}(\Omega)$ the set of functions analytic on an open neighborhood of Ω. In what follows, we consider a function $\psi \in \mathcal{H}(\overline{\mathbf{D}})$ positive and increasing on $[0, 1]$, such that

$$|\psi'(z)| \leq \psi'(|z|), \quad z \in \mathbf{D} \text{ and } \psi(0) \geq 0, \tag{1}$$

and the class

$$\mathcal{W}_\psi = \{w \in \mathcal{H}(\mathbf{D}) \mid w(0) = 0 \text{ and } \lim_{z \to \xi} |w'(z)| + \psi(|w(z)|) = 1, \xi \in \partial \mathbf{D}\}.$$

It is easily seen from (1) and the maximum principle that, for $w \in \mathcal{W}_\psi$,

$$|w'(z)| + \psi(|w(z)|) \leq 1, \quad z \in \mathbf{D}.$$

We may assume that $\psi(0) < 1$.

The existence of univalent members of classes similar to $\mathcal{W}_\psi$ has been studied by Beurling [**2**] and Avhadiev [**1**]. For any $n = 1, 2, \ldots$, the equation

$$nt + \psi(t) = 1$$

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has a unique solution $t_n \in (0,1)$ and the functions $w_n(z) = t_n e^{i\theta} z_n$ belong to $\mathcal{W}_\psi$ for any $\theta \in [0, 2\pi]$. However, the classes $\mathcal{W}_\psi$ contain more exotic members, as the following argument shows.

Let us assume for the rest of the paper that

$$\psi([0,1]) \subset [0,1). \tag{2}$$

We look at $\mathcal{H}(\mathbf{D})$ as a topological vector space endowed with the locally convex topology of uniform convergence over compact subsets of $\mathbf{D}$. The "ball"

$$\mathcal{B} = \{w \in \mathcal{H}(\mathbf{D}) \mid |w(z)| \le |z|, z \in \mathbf{D}\}$$

is a compact and convex subset of $\mathcal{H}(\mathbf{D})$. Let $0 < r < 1$ and $b(z)$ be a finite Blaschke product. We define an operator T_r over $\mathcal{B}$ by

$$T_r(w)(z) = \int_0^z b(\xi) \exp\left\{\frac{1}{2}\pi \int_0^{2\pi} \frac{1+\xi e^{-i\theta}}{1-\xi e^{-i\theta}} \ln(1-\psi(|w(re^{i\theta})|))\, d\theta\right\} d\xi.$$

Since, for $z \in \mathbf{D}$,

$$|T_r(w)'(z)| = |b(z)| \exp\left\{\frac{1}{2}\pi \int_0^{2\pi} \operatorname{Re}\left[\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} \ln(1-\psi(|w(re^{i\theta})|))\right] d\theta\right\}$$

we have $T_r(\mathcal{B}) \subseteq \mathcal{B}$ and clearly T_r is continuous over $\mathcal{B}$. By Tychonoff's fixed point theorem, [**3**, p. 414], there exists for any $r \in (0,1)$, a function $w_r \in \mathcal{B}$ with $T_r(w_r) = w_r$, i.e.,

$$w_r'(z) = b(z) \exp\left\{\frac{1}{2}\pi \int_0^{2\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} \ln(1-\psi(|w_r(re^{i\theta})|))\, d\theta\right\}. \tag{3}$$

The family $\{w_r\}_{0<r<1}$ is a normal family: there must exist a sequence $\{r_n\}$, $r_n \to 1$ and a function $w \in \mathcal{B}$ such that $w_{r_n} \to w$ in $\mathcal{H}(\mathbf{D})$. It is easily seen that each w_{r_n} has a continuous extension to $\overline{\mathbf{D}}$ ($|w_{r_n}'(z)|$ is bounded over $\mathbf{D}$!) and the convergence of $\{w_{r_n}\}$ is indeed uniform over $\overline{\mathbf{D}}$. By taking limits in (3) we obtain, for $z \in \mathbf{D}$,

$$|w'(z)| = |b(z)| \exp\left\{\frac{1}{2}\pi \int_0^{2\pi} \operatorname{Re} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} \ln(1-\psi(|w(e^{i\theta})|))\, d\theta\right\}$$

and clearly

$$\lim_{z\to e^{i\theta}} |w'(z)| = 1 - \psi(|w(e^{i\theta})|), \quad \theta \in [0, 2\pi],$$

i.e., $w \in \mathcal{W}_\psi$. Under the additional assumption $\psi(z) \equiv \psi_0(z^2)$ where $\psi_0 \in \mathcal{H}(\overline{\mathbf{D}})$, it has been shown in [**4**] that some w obtained as above is indeed analytic in $\overline{\mathbf{D}}$. The case where $\psi(z) \equiv z^2$ (of particular interest due to its obvious relation with the hyperbolic metric) has been thoroughly discussed in [**5**]. We wish to stress that, except for the monomials w_n, no other nontrivial members of $\mathcal{W}_\psi$ are explicitly known or can be numerically constructed. This is a justification for the main result of this paper.

Theorem 1. *Let $\psi \in \mathcal{H}(\mathbf{D})$ satisfy* (1) *and* (2) *with $\psi'(0) > 0$. Let also $\theta_0 \in [0, 2\pi]$. Then*

(a) *If $w \in \mathcal{W}_\psi$ is analytic in a neighborhood of $e^{i\theta_0}$, then $w(e^{i\theta_0}) \neq 0$.*

(b) *If $w \in \mathcal{W}_\psi$ and $w(e^{i\theta_0}) = 0$, then there does not exist a constant $\kappa > 0$ for which*

$$|w(e^{i\theta})| \leq \kappa |e^{i\theta} - e^{i\theta_0}|^2, \quad \theta \in [0, 2\pi].$$

It is not clear whether the condition $\psi'(0) > 0$ is necessary for Theorem 1 to hold. Let us remark however that part (a) of the result won't be valid for constant functions $\psi(z) \equiv k \in (0, 1)$; the function $w(z) = (1-k)\int_1^z (t-a)/(1-at)\, dt$ belongs to $\mathcal{W}_\psi$ if $a \in (0, 1)$ is chosen properly and $w(1) = 0$.

Proof of Theorem 1. It is sufficient to assume from now on that $\psi(0) = 0$ because, if $w \in \mathcal{W}_\psi$ with $\psi(0) > 0$, then $w^* \in \mathcal{W}_{\psi^*}$, with

$$w^*(z) := \frac{w(z)}{1 - \psi(0)} \text{ and } \psi^*(z) =: \frac{\psi((1 - \psi(0))z) - \psi(0)}{1 - \psi(0)}.$$

Clearly $\psi^*(0) = 0$, ψ^* verifies (1), $\psi^{*\prime}(0) > 0$ and $\psi^*([0, 1]) \subset [0, 1)$.

We prove part (a) by contradiction. For any real ϕ, the function

$$W_\phi(z) := w'(ze^{i\theta_0}) + e^{i\phi}\psi(w(ze^{i\theta_0}))$$

belongs to $\mathcal{H}(\mathbf{D} \cup \{1\})$ and, for $z \in \mathbf{D}$,

$$\begin{aligned} |W_\phi(z)| &\leq |w'(ze^{i\theta_0})| + \psi(|w(ze^{i\theta_0})|) \\ &\leq 1 = |W_\phi(1)|. \end{aligned}$$

Here we have used the fact that $\psi(0) = w(e^{i\theta_0}) = 0$. According to a result of Löwner [**7**, pp. 291–292], we have

$$0 < \frac{W'_\phi(1)}{W_\phi(1)} = e^{i\theta_0}\,\frac{w''(e^{i\theta_0})}{w'(e^{i\theta_0})} + e^{i(\phi+\theta_0)}\psi'(0).$$

This cannot hold for all ϕ because $\psi'(0) \neq 0$.

Our proof of part (b) of Theorem 1 is a bit more technical and we shall rely on a result due to Gaston Julia [**6**, pp. 43–45] which allows us to mimic the argument above in the situation where the function w is not necessarily analytic on $\overline{\mathbf{D}}$.

Theorem (Julia). *Let $F \in \mathcal{H}(\mathbf{D})$ and assume that F is nonconstant and $F(\mathbf{D}) \subseteq D$. Let*

$$M_F := \sup_{z\in\mathbf{D}} |1-F(z)|^2(1-|z|^2)/|1-z|^2(1-|F(z)|^2).$$

Then $0 < M_F \leq \infty$. If $M_F = \infty$, then

$$\lim_{n\to\infty}(1-|F(z_n)|)/(1-|z_n|) = \infty \tag{4}$$

for all sequences $\{z_n\} \subset \mathbf{D}$ such that $z_n \to 1$ and $F(z_n) \to 1$. In particular,

$$\lim_{z\to 1}(1-F(z))/(1-z) = \infty,$$

where $\lim_{z\to 1}$ means a limit as $z \to 1$ within a Stolz angle at 1 contained in $\mathbf{D}$. Alternatively, if $M_F < \infty$, then

$$\lim_{z\to 1}(1-F(z))/(1-z) = \lim_{z\to 1} F'(z) = M_F. \tag{5}$$

We consider two cases.

Case 1. $w(e^{i\theta_0}) = 0$ and $M_{w'(ze^{i\theta_0})} < \infty$. For W_ϕ as above we have

$$\frac{1 - W_\phi(z)}{1 - z} = -e^{i\phi}\,\frac{\psi(w(ze^{i\theta_0}))}{1 - z} + \frac{1 - w'(ze^{i\theta_0})}{1 - z}, \quad z \in \mathbf{D}.$$

Since

$$\begin{aligned}\left|\frac{\psi(w(ze^{i\theta_0}))}{1 - z}\right| &= \left|\frac{\psi(w(ze^{i\theta_0}))}{w(ze^{i\theta_0})}\right|\left|\frac{w(ze^{i\theta_0})}{1 - z}\right| \\ &= \left|\frac{\psi(w(ze^{i\theta_0}))}{w(ze^{i\theta_0})}\right|\frac{1}{|1 - z|}\left|\int_{e^{i\theta_0}}^{ze^{i\theta_0}} w'(x)\,dx\right| \\ &\le \sup_{|z|\le 1} |\psi(z)| := \|\psi\|_\infty < \infty,\end{aligned}$$

we obtain by (4) and (5) that $M_{W_\phi} < \infty$ and, by (5),

$$0 < \lim_{z\to 1} W'_\phi(z) = \lim_{z\to 1} e^{i\theta_0} w''(ze^{i\theta_0}) + e^{i(\phi+\theta_0)}\psi'(0)w'(ze^{i\theta_0}).$$

This must hold for all real ϕ and, in particular, because $\psi'(0) \ne 0$, we have $\lim_{z\to 1} w'(ze^{i\theta_0}) = 0$, which contradicts the definition of the class $\mathcal{W}_\psi$. Thus, Case 1 cannot occur.

Case 2. $w(e^{i\theta_0}) = 0$ and $M_{w'(ze^{i\theta_0})} = \infty$. The following lemmas will turn out to be useful in the process of deriving a crucial inequality below.

Lemma 1. *The function w' has at most a finite number of zeros in $\mathbf{D}$ and $|w'(e^{i\theta})| \ne 0$ for any real θ.*

Proof. Here $|w'(e^{i\theta})|$ means $\lim_{z\to e^{i\theta}} |w'(z)|$, which exists by definition of $\mathcal{W}_\psi$. Let $0 < r < 1$ and $m_r := \max_{|z|=r=|z_r|} |w(z)| = |w(z_r)|$. By Löwner's result, we have $z_r(w'(z_r))/(w(z_r)) \ge 1$ and

$$\begin{aligned}\left|\frac{w(z_r)}{z_r}\right| + \psi(|w(z_r)|) &\le z_r\,\frac{w'(z_r)}{w(z_r)}\left|\frac{w(z_r)}{z_r}\right| + \psi(|w(z_r)|) \\ &= |w'(z_r)| + \psi(|w(z_r)|) \le 1.\end{aligned} \tag{6}$$

If $m = \sup_{z\in\mathbf{D}} |w(z)|$, we obtain as $r \to 1$ in (6), $m + \psi(m) \le 1$. In particular $m < 1$ and $\psi(m) < 1$. If $e^{i\theta}$ were an accumulation point of zeros of w' or if $|w'(e^{i\theta})| = 0$, then we would obtain $1 = \psi(|w(e^{i\theta_0}|) \leq \psi(m) < 1$. The conclusion of Lemma 1 follows.

Lemma 2. *For any finite Blaschke product $b(z)$ we have*

$$c_b := \sup_{z \in \mathbf{D}} (1 - |b(z)|)/(|1 - z|) < \infty.$$

Proof. We have $(1 - |b(z)|)/(|1 - z|) \leq (1 - |b(z)|^2)/(|1 - z|)$, $z \in \mathbf{D}$, and $b(z) = (zb_1(z) + b(0))/(1 + \overline{b(0)} z b_1(z))$ where b_1 is a finite Blaschke product of degree less than the degree of b. Then

$$\begin{aligned} \frac{1 - |b(z)|}{|1 - z|} &\leq \frac{1 - |b(0)|^2}{|1 + \overline{b(0)} z b_1(z)|^2} \, \frac{1 - |z|^2 |b_1(z)|^2}{|1 - z|} \\ &\leq \frac{1 + |b(0)|}{1 - |b(0)|} \left(\frac{1 - |z|^2}{|1 - z|} + \frac{1 - |b_1(z)|^2}{|1 - z|} \right) \\ &\leq 2 \, \frac{1 + |b(0)|}{1 - |b(0)|} \left(1 + \frac{1 - |b_1(z)|}{|1 - z|} \right) \end{aligned} \tag{7}$$

and

$$\frac{1 - |(z - a)/(1 - \bar{a}z)|}{|1 - z|} \leq \frac{1 + |a|}{1 - |a|}, \quad a, z \in \mathbf{D}. \tag{8}$$

A proof by induction of Lemma 2 clearly follows from (7) and (8). □

Since $w(e^{i\theta_0}) = 0$, there exists a sequence $\{z_n\} \subset \mathbf{D}$ converging to 1 within a Stolz angle at 1 and a real number μ such that $\lim_{n \to \infty} e^{i\mu} w'(z_n e^{i\theta_0}) = 1$. Let us define $v(z) := e^{i\mu} w'(z e^{i\theta_0})$. Just as above, the assumption $M_v < \infty$ would lead to a contradiction. We therefore assume that $M_v = \infty$ and, by (4) in Julia's theorem, we obtain $\lim_{n \to \infty} (1 - |v(z_n)|)/(1 - |z_n|) = \infty$. There exists $k > 0$ such that, for all n,

$$\frac{1 - |v(z_n)|}{|1 - z_n|} = \frac{1 - |v(z_n)|}{1 - |z_n|} \, \frac{1 - |z_n|}{|1 - z_n|} \geq k \, \frac{1 - |v(z_n)|}{1 - |z_n|}$$

and, clearly

$$\overline{\lim_{z \to 1}} \, \frac{1 - |v(z)|}{|1 - z|} = \infty. \tag{9}$$

According to Lemma 1, v has at most a finite number of zeros in $\mathbf{D}$ and no zeros on $\partial\mathbf{D}$. Moreover, $|v(z)|$ is continuous over $\overline{\mathbf{D}}$ and $\min_{z\in\partial\mathbf{D}} |v(z)| > 0$. It follows from a standard factorization result that

$$v(z) = b(z) \exp\left\{\frac{1}{2}\,\pi \int_0^{2\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\, \ln|v(e^{i\theta})|\, d\theta\right\}, \quad z \in \mathbf{D},$$

where b is a finite Blaschke product. We obtain, after writing

$$d\nu(\theta) := \frac{1}{2\pi}\, \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}\, d\theta$$

for fixed $z \in \mathbf{D}$,

$$\begin{aligned}
\frac{1-|v(z)|}{|1-z|} &= \frac{1-|b(z)|\exp\{\int_0^{2\pi} \ln|v(e^{\theta})|\, d\nu(\theta)\}}{|1-z|} \\
&= \frac{1-|b(z)|}{|1-z|} + |b(z)|\, \frac{1-\exp\{\int_0^{2\pi} \ln|v(e^{i\theta})|\, d\nu(\theta)\}}{|1-z|} \\
&\le c_b + \frac{1-\exp\{-\int_0^{2\pi} \ln(1/[1-\psi(|w(e^{i(\theta+\theta_0)})|)])\, d\nu(\theta)\}}{|1-z|} \\
&\le c_b + \frac{1-(\int_0^{2\pi} (1-\psi(|w(e^{i(\theta+\theta_0)})|))^{-1}\, d\nu(\theta))^{-1}}{|1-z|} \\
&\le c_b + \frac{1}{|1-z|} \int_0^{2\pi} \frac{\psi(|w(e^{i(\theta+\theta_0)})|)}{1-\psi(|w(e^{i(\theta+\theta_0)})|)}\, d\nu(\theta) \\
&\le c_b + \frac{\|\psi\|_\infty}{1-\psi(m)}\, \frac{1}{|1-z|} \int_0^{2\pi} |w^(e^{i(\theta+\theta_0)})|\, d\nu(\theta).
\end{aligned}$$

In the above estimations we have made use of Lemma 2, the inequality between the arithmetic and geometric means and some elements of the proof of Lemma 1.

Assuming now that $|w(e^{i(\theta+\theta_0)})| \le \kappa|1-e^{i\theta}|^2$ for some $\kappa > 0$, we obtain

$$\begin{aligned}
\frac{1-|v(z)|}{|1-z|} &\le c_b + \frac{\kappa\|\psi\|_\infty}{1-\psi(m)} \int_0^{2\pi} \frac{|1-e^{i\theta}|^2}{|1-z|}\, d\nu(\theta) \\
&= c_b + \frac{2\kappa\|\psi\|_\infty}{1-\psi(m)}\, \frac{\mathrm{Re}\,(1-z)}{|1-z|} \\
&\le c_b + \frac{2\kappa\|\psi\|_\infty}{1-\psi(m)},
\end{aligned}$$

which of course contradicts (9). This completes our proof of Theorem 1. $\square$

It is not clear at all that functions w' with infinite angular derivatives and $w \in \mathcal{W}_\psi$ (for some admissible ψ) actually do exist. By suitably restricting the class $\mathcal{W}_\psi$, we may obtain a more universal version of Theorem 1. Let, for example,

$$\mathcal{W}'_\psi = \left\{ w \in \mathcal{H}(\mathbf{D}) \mid w(0) = 0 \text{ and } \lim_{z \to \xi} \frac{\psi(|w(z)|)}{1 - |w'(z)|} = 1, \quad \xi \in \partial \mathbf{D} \right\}.$$

It is easy to prove that $M_{w'(ze^{i\theta_0})} < \infty$ if $w \in \mathcal{W}'_\psi$ and $w(e^{i\theta_0}) = 0$. Our arguments then imply that the following version of Theorem 1 holds.

Theorem 2. *Let* $\psi \in \mathcal{H}(\overline{\mathbf{D}})$ *be as above and* $\psi'(0) > 0$. *Then* $w(z) \neq 0$ *if* $w \in \mathcal{W}'_\psi$ *and* $z \in \partial \mathbf{D}$.

Concluding remarks. We wish to make some observations concerning Blaschke products and our proof of Lemma 2. We believe this proof is not as transparent as it should be and can somehow be generalized.

Let us assume b is a Blaschke product whose zeros do not accumulate at $z = 1$. Then b is analytic in a neighborhood of $z = 1$ and

$$\overline{\lim_{z \to 1}} \frac{1 - |b(z)|}{|1 - z|} \leq \overline{\lim_{z \to 1}} \left| \frac{b(1) - b(z)}{1 - z} \right| = |b'(1)| < \infty$$

and the statement of Lemma 2 shall hold for such b. Conversely, if the zeros of a Blaschke product $b(z)$ accumulate at $z = 1$, there shall exist a sequence $\{z_n\} \subset \mathbf{D}$ with $z_n \to 1$ and

$$\frac{1 - |b(z_n)|}{|1 - z_n|} = \frac{1}{|1 - z_n|},$$

so that $\overline{\lim}_{z \to 1}(1 - |b(z)|)/(|1 - z|) = \infty$. We have thus proved that Lemma 2 holds in fact precisely for Blaschke products (finite or not) whose zeros do not accumulate at 1.

In the case where $\sup_{z \in \mathbf{D}}(1 - |b(z)|)/(|1 - z|) = \infty$, we may actually have $\lim_{z \to 1}(1 - |b(z)|)/(|1 - z|) = \infty$ for any Stolz angle at $z = 1$; let us consider the inner function $f(z) = e^{-(1+z)/(1-z)}$. We have

$$\frac{1-|f(z)|}{|1-z|} = \frac{1-e^{-\mathrm{Re}\,((1+z)/(1-z))}}{|1-z|} \tag{10}$$

and if z approaches 1 from within a Stolz angle at 1, z will eventually belong to the interior or an orocycle at 1 where $\mathrm{Re}\,((1+z)/(1-z)) > k$ for some $k > 0$; clearly $1 - e^{-\mathrm{Re}\,((1+z)/(1-z))} > 1 - e^{-k}$ and, by (10), $\lim_{z\to 1}(1-|f(z)|)/(|1-z|) = \infty$. According to a famous result of Frostman, $b_c(z) := (f(z)-c)/(1-\bar{c}f(z))$ is a Blaschke product for "most" values of $c \in \mathbf{D}$ and, since

$$\frac{1-|b_c(z)|}{|1-z|} \geq \frac{1-|c|}{2(1+|c|)}\,\frac{1-|f(z)|}{|1-z|}, \quad z \in \mathbf{D},$$

our claim follows.

Even in the case where $\sup_{z\in\mathbf{D}}(1-|b(z)|)/(|1-z|) < \infty$, the boundary behavior of the Blaschke product may be rather surprising. Using the inner functions $f(z) = e^{-(1+e^{i\theta}z)/(1-e^{i\theta}z)}$ with $0 < \theta < 2\pi$ and Frostman's result, one can create an infinite Blaschke product $b(z)$ such that $b(z)$ is analytic at $z = 1$ and the angular cluster set of $(1-|b(z)|)/(|1-z|)$ at 1 is the whole interval $(0, |b'(1)|]$.

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