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INVARIANT SUBSPACES AND THIN SETS

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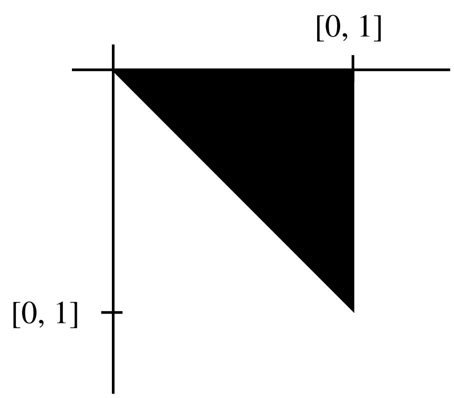
This expository article will outline some connections between the existence of compact operators in reflexive operator algebras with a commutative subspace lattice (CSL algebras) and the theory of "thin sets" in harmonic analysis. Full details will appear elsewhere [**5**].

Let X be a compact metric space, μ a finite Borel measure on X and $\leq$ a closed partial-order on X. The operator algebra $\mathrm{Alg}\,(X, \leq, \mu)$ is described in [**1**] where its main properties are developed. We mention that

$$\mathrm{Lat}\,(\mathrm{Alg}\,(X, \leq, \mu) = \mathcal{L}\,(X, \leq) = \{P_E : E \text{ is a decreasing Borel set}\}.$$

We are concerned with the existence of compact operators in $\mathrm{Alg}\,(X, \leq, \mu)$.

Example 1. Let $X = [0, 1]$, with Lebesgue measure dx and the usual linear order. Then $A = \mathrm{Alg}\,([0, 1], \leq, dx)$ is a nest algebra consisting of all operators on $L^2[0, 1]$ "supported" on the graph of the linear order



$$\mathrm{Lat}\,(A) = \{P_{[0,r]} : 0 \leq r \leq 1\}.$$

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To obtain nonzero compact operators in A take Hilbert-Schmidt integral operators whose kernels are concentrated on the graph of $\leq$. For example, the integration operator

$$f \mapsto \int_x^1 f(t)\,dt \in A$$

and is compact.

In general $\operatorname{Alg}(X, \leq, \mu)$ contains a nonzero Hilbert-Schmidt operator iff the graph of the partial order has positive $\mu \times \mu$ measure.

Example 2. Let $X = 2^\infty = \{0,1\} \times \{0,1\} \times \cdots$ be the Cantor group with the product topology. Define $\leq$ by $(x_1, x_2, \dots) \leq (y_1, y_2, \dots)$ iff $x_n \leq y_n$ for all n and take $m_{1/2}$ to be the infinite product measure $\mu \times \mu \times \cdots$, where μ is the measure on $\{0,1\}$ which assigns $\mu(0) = 1/2, \mu(1) = 1/2$. One calculates that the product measure of the graph of the partial order is 0. In [**4, 5**] we showed that $\operatorname{Alg}(X, \leq, m_{1/2})$ contains no nonzero compact operators. We now outline a "thin set" proof of this result.

Definition. Let T be the circle group. A closed set $E \subseteq T$ is called a set of uniqueness if any trigonometric series

$$\sum_{-\infty}^{\infty} c_n e^{inx}, \qquad c_{|n|} \to 0 \text{ as } |n| \to \infty,$$

which vanishes off E must have $c_n = 0$ for all n. Otherwise E is called a set of multiplicity.

Example 3. The Riemann-Lebesgue lemma implies that any set of positive measure is a set of multiplicity. Cantor proved that any finite set is a set of uniqueness and W.H. Young generalized this to countable sets. In 1937 Nina Bary proved that Cantor's middle third set is a set of uniqueness. There exist sets of multiplicity having zero Lebesgue measure.

For a general compact abelian group the definition must be phrased in terms of distributions (pseudomeasures) [**7**].

Let $X = G$ be a compact abelian group with dg the Haar measure and $\leq$ a closed partial order on G.

THEOREM 1. *If* $\mathrm{Alg}\,(X, \leq, dg)$ *contains a nonzero compact operator, then the graph of* $\leq$ *is a set of multiplicity in the group* $G \times G$.

THEOREM 2. *The graph of the partial order in Example* 2 *is a set of uniqueness in the group* $2^{\infty} \times 2^{\infty}$.

COROLLARY 1. *The operator algebra* $\mathrm{Alg}\,(2^{\infty}, \leq, m_{1/2})$ *contains no nonzero compact operators.*

COROLLARY 2. *The lattice* $\mathcal{L}\,(2^{\infty}, \leq)$ *is not attainable by a nonzero compact operator.*

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