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TENSOR PRODUCTS OF NON-SELF-ADJOINT OPERATOR ALGEBRAS

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1. Introduction. In this paper we study several norms that can be introduced on the algebraic tensor product of two, not necessarily self-adjoint, algebras of operators on a Hilbert space.

Following the work of Arveson [**2**], we know that if $\mathcal{A}$ is an algebra of operators on a Hilbert space $\mathcal{H}$ or, more generally, a subalgebra of a C^*-algebra $\mathcal{B}$, then to fully understand $\mathcal{A}$ we must also consider the whole family of norms on the k by k matrix algebras over $\mathcal{A}, \mathcal{M}_k(\mathcal{A})$. That is, we must regard $\mathcal{A}$ as a *matrix normed* space in the sense of Effros [**4**]. When $\mathcal{A}$ is an algebra of operators on $\mathcal{H}$, then $\mathcal{M}_k(\mathcal{A})$ is just the algebra of $k \times k$ matrices with entries from $\mathcal{A}$. This can be regarded as an algebra of operators on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (k times), denoted $\mathcal{H}^{(k)}$, and is endowed with the norm that it inherits as operators on $\mathcal{H}^{(k)}$. When $\mathcal{A}$ is a subalgebra of a C^*-algebra $\mathcal{B}$, then it is well-known that there is a unique norm on $\mathcal{M}_k(\mathcal{B})$ which makes it into a C^*-algebra, and we endow $\mathcal{M}_k(\mathcal{A})$ with the norm that it inherits as a subspace.

For the above reasons, if we are given an arbitrary complex algebra $\mathcal{A}$, then we shall call $\mathcal{A}$ an *operator algebra*, if it is endowed with a family of norms on $\mathcal{M}_k(\mathcal{A})$ and a representation ρ of $\mathcal{A}$ on some Hilbert space such that the norms on $\mathcal{M}_k(\mathcal{A})$ are induced by the representation. Thus

$$||(a_{ij})|| = ||(\rho(a_{ij}))||$$

for all (a_{ij}) in $\mathcal{M}_k(\mathcal{A})$ and all k. We call such a family of norms an *operator norm.*

Given two unital operator algebras, $\mathcal{A}_1$ and $\mathcal{A}_2$, we define a *complete operator cross-norm* to be any operator norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ which is a cross-norm, that is, $||a_1 \otimes a_2|| = ||a_1|| \cdot ||a_2||$, and which has the

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property that the natural inclusions of $\mathcal{A}_i$ into $\mathcal{A}_1 \otimes \mathcal{A}_2, i = 1, 2$, given by $a_1 \to a_1 \otimes 1$ and $a_2 \to 1 \otimes a_2$, induce isometries of $\mathcal{M}_k(\mathcal{A}_i)$ into $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$, for all k.

In this paper we introduce and study three natural complete operator cross-norms, which we call the spatial, minimal and maximal norms. We prove some elementary properties about these norms and analyze them for a variety of examples. In contrast to the C^*-algebra theory the general context is complicated by the fact that even for very simple finite dimensional algebras $\mathcal{A}_i \subseteq \mathcal{M}_n, i = 1, 2$, the minimal and maximal norms may differ. Establishing the equality of the minimal and maximal norms is usually equivalent to the ability to lift commuting contractive representations ρ_1 and ρ_2 of $\mathcal{A}_1$ and $\mathcal{A}_2$ to commuting unital dilations π_1 and π_2 of C^*-algebras $\mathcal{B}_i$ which contain $\mathcal{A}_i, i = 1, 2$. Ando's dilation theorem for commuting contractions [**1**], and the closely related commutant lifting theorem of Sz-Nagy and Foias [**13**], are key results that we need to obtain the dilations π_1 and π_2 in various contexts. For the upper triangular matrix subalgebra $\mathcal{T}(n)$ of $\mathcal{M}_n$ we need the lifting theorems for commuting contractive representations obtained in an earlier paper [**9**], or the new methods given below in §3.

It is well known that a triple of commuting contractions need not possess a dilating triple of commuting unitary operators (Parrott [**7**]). We find an analogue of this in §3 for a triple of commuting contractive representations of $\mathcal{T}(2)$, and this leads to the distinction of $|| \; ||_{\min}$ and $|| \; ||_{\max}$ on $\mathcal{T}(2) \otimes \mathcal{T}(2) \otimes \mathcal{T}(2)$. §2 contains some basic results about the three complete operator cross-norms we consider.

Before closing the section we comment on some of the similarities and distinctions between this theory and the theory for C^*-algebras. If $\mathcal{A}$ is a C^*-algebra, then any $*$-monomorphism ρ of $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$ is automatically a *complete isometry*, that is, $||(a_{ij})|| = ||(\rho(a_{ij}))||$ for all (a_{ij}) in $\mathcal{M}_k(\mathcal{A})$ and all k. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^*-algebras, then any norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$, such that the completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ in this norm is a C^*-algebra and such that the inclusions $a_1 \to a_1 \otimes 1$ and $a_2 \to 1 \otimes a_2$ are $*$-monomorphisms is called a C^*-norm. C^*-norms are automatically cross-norms [**14**], and so by the above remark are complete operator cross-norms.

Unlike the C^*-algebra case the cross-norm property does not come for free for operator algebras. That is, there are operator algebras $\mathcal{A}_1$ and

$\mathcal{A}_2$ and a representation ρ of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on a Hilbert space such that the operator norm induced by ρ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ has the property that each of the inclusion maps is a complete isometry, but $||\rho(a_1 \otimes a_2)|| < ||a_1|| \cdot ||a_2||$.

For a simple example of this phenomenon, let $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{T}(2)$, fix r and s, $0 \leq r, s < 1$ and consider the maps $\rho_1, \rho_2 : \mathcal{T}(2) \to \mathcal{M}_8$ given by

$$\rho_1\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & rb & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & rb \\ 0 & 0 & 0 & c \end{pmatrix} \oplus \begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & c \end{pmatrix},$$

$$\rho_2\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \oplus \begin{pmatrix} a & 0 & sb & 0 \\ 0 & a & 0 & sb \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}.$$

It is easily checked that ρ_1 and ρ_2 are completely isometric isomorphisms and that their ranges commute. Hence, there is an induced representation ρ of $\mathcal{A}_1 \otimes \mathcal{A}_2$ with $\rho(a_1 \otimes a_2) = \rho_1(a_1)\rho_2(a_2)$ and each of the inclusion maps is a complete isometry. However, if $\{e_{ij}\}$ denotes the usual matrix units in M_2, then $||\rho(e_{12} \otimes e_{12})|| = \max\{r, s\} \cdot ||e_{12}|| \cdot ||e_{12}||$. Thus, ρ does not induce a cross norm, except when r or s is 1.

There are a number of fundamental questions that one could consider about complete operator cross-norms. We focus, instead, on these three particular cross-norms. However, it is clear that, before a development can proceed which parallels somewhat the theory of C^*-norms, one needs an abstract characterization of operator algebras. We are restricted by the fact that operator algebra norms can only be defined via representations.

Recently, Ruan [**12**] has given an abstract characterization of those matrix normed spaces which have linear embeddings as spaces of operators which are completely isometric. Hopefully, this result can lead to a characterization of operator algebras. For an example of some of the difficulties, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be C^*-algebras and consider the Haagerup matrix norm on $\mathcal{B}_1 \otimes \mathcal{B}_2$ introduced in [**5**] (see also [**11**]). This algebra is an operator space, that is, there is a linear map $\psi : \mathcal{B}_1 \otimes \mathcal{B}_2 \to \mathcal{L}(\mathcal{H})$ which is a complete isometry. However, any completely contractive homormorphism $\pi : \mathcal{B}_1 \otimes \mathcal{B}_2 \to \mathcal{L}(\mathcal{H})$ is necessarily a $*$-homomorphism and hence is contractive in the maximal

C^*-norm on $\mathcal{B}_1 \otimes \mathcal{B}_2$. This norm is strictly smaller than the Haagerup norm. Thus $\mathcal{B}_1 \otimes \mathcal{B}_2$ together with the Haagerup norms on $\mathcal{M}_k(\mathcal{B}_1 \otimes \mathcal{B}_2)$ is an operator space in the sense we have defined above but it is not an operator algebra.

This is perhaps not a good example, for while $\mathcal{B}_1 \otimes \mathcal{B}_2$ can be shown to be a Banach algebra in the Haagerup norm, the Haagerup norm on even $\mathcal{M}_2(\mathcal{B}_1 \otimes \mathcal{B}_2)$ can fail to be a Banach algebra norm. For an example let $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{M}_2$.

2. The minimal, maximal and spatial complete operator cross-norms. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital operator algebras on the complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We do not assume that these algebras are self-adjoint or closed in any particular topology. In this section we introduce the spatial minimal and maximal complete operator cross-norms and prove various results about these norms. We assume throughout that our algebras have units and that all maps are unital.

Recall that if A_1 and A_2 are operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then we have an operator $A_1 \otimes A_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $A_1 \otimes A_2(h_1 \otimes h_2) = (A_1 h_1) \otimes (A_2 h_2)$. Thus, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then we may regard $\mathcal{A}_1 \otimes \mathcal{A}_2$ as an algebra of operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and this identification endows $\mathcal{A}_1 \otimes \mathcal{A}_2$ with an operator norm, which is clearly a complete operator cross-norm. We call this norm the *spatial operator norm*, and, for U in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$, we denote this norm by $||U||_{\text{spat}}$.

More generally, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are operator algebras and $\rho_i : \mathcal{A}_i \to \mathcal{L}(\mathcal{H}_i)$, $i = 1, 2$, are completely contractive homomorphisms, then we obtain a homomorphism $\rho_1 \otimes \rho_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\rho_1 \otimes \rho_2(A_1 \otimes A_2) = \rho_1(A_1) \otimes \rho_2(A_2)$, for A_1 in $\mathcal{A}_1$ and A_2 in $\mathcal{A}_2$. We let $\mathcal{F}_{\min}$ denote the family of representations of $\mathcal{A}_1 \otimes \mathcal{A}_2$ which can be obtained in this manner. For $U = (U_{ij})$ in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$, we set

$$||U||_{\min} = \sup\{||(\rho_1 \otimes \rho_2(U_{ij}))|| : \rho_1 \otimes \rho_2 \in \mathcal{F}_{\min}\}.$$

We call this family of norms the *minimal operator norm* on $\mathcal{A}_1 \otimes \mathcal{A}_2$, and let $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ denote $\mathcal{A}_1 \otimes \mathcal{A}_2$ endowed with these norms.

If $\rho_1 : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $\rho_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ are completely contractive homomorphisms into the algebra of operators on the same Hilbert space and if the range of ρ_1 commutes with the range of ρ_2, then we obtain a homomorphism $\rho_1 \odot \rho_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ by setting $\rho_1 \odot \rho_2(A_1 \otimes A_2) = \rho_1(A_1)\rho_2(A_2)$. We let $\mathcal{F}_{\max}$ denote the family of representations of $\mathcal{A}_1 \otimes \mathcal{A}_2$ which can be obtained in this manner. For all $U = (U_{ij})$ in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$, we set

$$||U||_{\max} = \sup\{||(\rho_1 \odot \rho_2(U_{ij}))|| : \rho_1 \odot \rho_2 \in \mathcal{F}_{\max}\},$$

and we call this family of norms the *maximal operator norm*, and let $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$ denote $\mathcal{A}_1 \otimes \mathcal{A}_2$ endowed with these norms.

Finally, if $\rho : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ is any homomorphism, for $U = (U_{ij})$ in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$, we set

$$||U||_\rho = ||(\rho(U_{ij}))||.$$

LEMMA 2.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be operator algebras. Then the minimal and maximal operator norms are complete operator cross-norms on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Moreover, for any U in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$,*

$$||U||_{\min} \leq ||U||_{\max}.$$

PROOF. The inequality is true since $\mathcal{F}_{\max}$ contains $\mathcal{F}_{\min}$. Clearly each of these norms has the property that the inclusion maps, $\mathcal{A}_i \to \mathcal{A}_1 \otimes \mathcal{A}_2$ are completely contractive.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are concrete algebras of operators on Hilbert spaces, then since clearly $||U||_{\text{spat}} \leq ||U||_{\min}$ for U in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$, we have that the inclusions are completely isometric and that these are cross-norms. More generally, if $\mathcal{A}_i$ are contained in $\mathcal{B}_i$, we may take $*$-monomorphisms, $\pi_i : \mathcal{B}_i \to \mathcal{L}(\mathcal{H}_i)$ and let ρ_i denote the restriction of π_i to $\rho_i, i = 1, 2$. Then $|| \cdot ||_{\rho_1 \otimes \rho_2}$ is a complete operator cross-norm and is dominated by the minimal norm.

Finally, to see that the minimal and maximal norms are operator norms, it is enough to observe that if one chooses sufficiently large subsets of $\mathcal{F}_{\min}$ and $\mathcal{F}_{\max}$ and considers, respectively, the direct sums of the representations in these subsets, then one can obtain representations of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on two Hilbert spaces which are completely isometric in the minimal and maximal norms, respectively. □

THEOREM 2.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be operator algebras, and let $\rho_i : \mathcal{A}_i \to \mathcal{L}(\mathcal{H}_i)$ be completely isometric isomorphisms, $i = 1, 2$. Then the minimal operator norm coincides with the operator norm induced by $\rho_1 \otimes \rho_2$.*

PROOF. We use the fact that if $\mathcal{B}_i$ are C^*-subalgebras of $\mathcal{L}(\mathcal{H}_i), i = 1, 2$, then the minimal operator norm on $\mathcal{B}_1 \otimes \mathcal{B}_2$ coincides with the spatial norm [**14**]. Thus, if we let $\mathcal{B}_i$ denote the C^*-subalgebra of $\mathcal{L}(\mathcal{H}_i)$ generated by $\rho_i(\mathcal{A}_i)$, then the operator norm induced by $\rho_1 \otimes \rho_2$ is the norm induced by the inclusion

$$\rho_1 \otimes \rho_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2.$$

Now, let $\psi_i : \mathcal{A}_i \to \mathcal{L}(\mathcal{K}_i), i = 1, 2$, be completely contractive homomorphisms. By Arveson's extension theorem there exist unital completely positive maps, $\theta_i : \mathcal{B}_i \to \mathcal{L}(\mathcal{K}_i)$ which extend $\psi_i \circ \rho_i^{-1} : \rho_i(\mathcal{A}_i) \to \mathcal{L}(\mathcal{K}_i), i = 1, 2$. These unital completely positive maps define a unital completely positive map $\theta_1 \otimes \theta_2 : \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2 \to \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ (see [**8**, Theorem 10.3] for example). Since $\theta_1 \otimes \theta_2$ is unital, it is completely contractive and, thus for any $U = (U_{ij})$ in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$,

$$||(\psi_1 \otimes \psi_2(U_{ij}))|| = ||(\theta_1 \otimes \theta_2(\rho_1 \otimes \rho_2(U_{ij})))|| \leq ||(\rho_1 \otimes \rho_2(U_{ij}))||,$$

where the later norm can be taken in $\mathcal{M}_k(\mathcal{B}_1 \otimes_{\min} \mathcal{B}_2)$ or spatially.

This inequality shows that $||U||_{\min} \leq ||U||_{\rho_1 \otimes \rho_2}$, and, since the reverse inequality is obvious, we have that the minimal operator norm and the operator norm induced by $\rho_1 \otimes \rho_2$ agree. □

COROLLARY 2.3. *Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be operator algebras with $\mathcal{A}_i$ a subalgebra of $\mathcal{B}_i, i = 1, 2$. Then the inclusion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ into $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a complete isometry when both algebras are endowed with their minimal norms. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are operator algebras and $\rho_i : \mathcal{A}_i \to \mathcal{C}_i$ are completely contractive homomorphisms, then $\rho_1 \otimes \rho_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$ is a completely contractive homomorphism when both algebras are endowed with their minimal norms.*

Clearly, the maximal operator norm is the maximum of all complete operator cross-norms. When both algebras are C^*-algebras then it is known that the minimal norm is the minimum of all complete operator cross-norms [**14**]. We do not know if the minimal norm is the minimum of all complete operator norms for general operator algebras. The example of §1 leads us to believe that this question is quite hard.

Unlike C^*-algebras, the minimal and maximal operator norms can differ even for finite dimensional algebras.

EXAMPLE 2.4. Let $\mathcal{A} \subseteq \mathcal{M}_2$ be the two dimensional operator algebra spanned by the identity and the matrix unit e_{12}. Let $\rho_1 = \rho_2$ be the identity representation and note that the matrix $e_{12} \otimes I + I \otimes e_{12}$ has the form

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which has norm $\sqrt{2}$. On the other hand, if ρ is the representation of $\mathcal{A} \otimes \mathcal{A}$ such that at $\rho(A_1 \otimes A_2) = \rho_1(A_1)\rho_2(A_2)$, for A_1, A_2 in $\mathcal{A}$, then the image of this matrix under ρ is

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},$$

which has norm 2. In particular, the spatial norm and the maximal norm differ, and so the minimal and maximal norms do not even agree on $\mathcal{A} \otimes \mathcal{A}$.

The equality of the minimal and maximal norms is closely related to the ability to lift each pair of commuting completely contractive representations of the coordinate algebras, to commuting $*$-representations of containing C^*-algebras. The next proposition states this more precisely. We write $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 = \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$ to mean that $||U||_{\min} = ||U||_{\max}$ for all U in $\mathcal{M}_k(\mathcal{A}_1 \otimes \mathcal{A}_2)$, for all k.

PROPOSITION 2.5. *Let $\mathcal{B}_i$ be C^*-algebras and $\mathcal{A}_i$ unital subalgebras, $i = 1, 2$. Then the following are equivalent*:

(i) $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 = \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$,

(ii) *for any pair of commuting completely contractive homomorphisms $\rho_i : \mathcal{A}_i \to \mathcal{L}(\mathcal{H})$, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a $*$-homomorphism $\pi : \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2 \to \mathcal{L}(\mathcal{K})$ such that*

$$\rho_1(a_1)\rho_2(a_2) = P_{\mathcal{H}}\rho(a_1 \otimes a_2)|_{\mathcal{H}},$$

where $P_{\mathcal{H}}$ denotes projection onto $\mathcal{H}$.

PROOF. Assume (i). Then the map $\rho_1 \odot \rho_2$ is completely contractive on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$. By Corollary 2.3 and Arveson's extension theorem, $\rho_1 \odot \rho_2$ can be extended to a unital completely positive map $\theta : \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2 \to \mathcal{L}(\mathcal{H})$. Apply Stinespring's dilation theorem to θ to deduce (ii).

Conversely, if we assume (ii) then we see that the operator norm induced by $\rho_1 \odot \rho_2$ is dominated by the restriction of the operator norm on $\mathcal{B}_1 \otimes_{\min} \mathcal{B}_2$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$. Again by Corollary 2.3, this restricted norm coincides with the minimal operator norm. Thus, the operator norm induced by any homomorphism in $\mathcal{F}_{\max}$ is dominated by the minimal operator norm, and so the maximal and minimal operator norms coincide. □

Let $\mathcal{P}(\mathbf{D})$ be the usual algebra of complex polynomials on the unit disk, normed by the supremum norm. We may regard it as an operator algebra by viewing it as a subalgebra of $C(T)$, the continuous functions on the circle. We know that any contraction operator T, gives rise to a completely contractive representation of $\mathcal{P}(\mathbf{D})$. Note that $C(T) \otimes_{\min} C(T) = C(T^2)$, the continuous functions on the torus, and any $*$-homomorphism of this algebra is determined by a pair of commuting unitaries. Thus Ando's theorem [**1**] that pairs of commuting contractions dilate to pairs of commuting unitaries is, by Proposition 2.5, the statement that the minimal and maximal operator norms coincide for $\mathcal{P}(\mathbf{D}) \otimes \mathcal{P}(\mathbf{D})$. Thus we may identify $\mathcal{P}(\mathbf{D}) \otimes_{\max} \mathcal{P}(\mathbf{D})$, $\mathcal{P}(\mathbf{D}) \otimes_{\min} \mathcal{P}(\mathbf{D})$, and $\mathcal{P}(\mathbf{D}^2)$ as operator algebras.

We may also identify $\mathcal{P}(\mathbf{D}^2) \otimes_{\min} \mathcal{P}(\mathbf{D})$ with $\mathcal{P}(\mathbf{D}^3)$, regarded as a subalgebra of $C(T^3)$ The examples of Crabbe-Davie [**3**] and Varopolos [**15**] of three contractions which violate von Neumann's inequality, show that commuting completely contractive representations of $\mathcal{P}(\mathbf{D}^2)$ and $\mathcal{P}(\mathbf{D})$ need not be contractive on $\mathcal{P}(\mathbf{D}^2) \otimes_{\min} \mathcal{P}(\mathbf{D})$. Parrott's example

[**7**] shows that three commuting contractions can be found such that the operator norm they induce on $\mathcal{P}(\mathbf{D}^2) \otimes \mathcal{P}(\mathbf{D})$ agrees with the minimal for scalar matrices, but is still larger than the minimal operator norm on $\mathcal{M}_k(\mathcal{P}(\mathbf{D}^2) \otimes \mathcal{P}(\mathbf{D}))$, for some k. Thus, as ordinary norms this induced norm and the minimal norm would agree, but not as operator norms.

From Proposition 2.5 we see that the statement that the minimal and maximal operator norms coincide on $\mathcal{A} \otimes \mathcal{P}(\mathbf{D})$ is equivalent to a statement about being able to lift a contraction that commutes with a completely contractive representation to a unitary that commutes with *some* dilation of the representation. Furthermore, this lifting also works for all powers of the contraction.

There is a result very closely related to Ando's theorem, the Sz-Nagy-Foias commutant lifting theorem [**13**]. In fact, for two contractions, these results are equivalent. We wish to give an operator algebra interpretation of this result as well. We shall see that it and its generalizations are best construed as statements that the minimal and maximal operator norms agree on $\mathcal{A} \otimes \mathcal{T}(2)$, where $\mathcal{T}(2)$ denotes the upper triangular 2×2 matrices.

PROPOSITION 2.6. *Let $\mathcal{A}$ be a subalgebra of the C^*-algebra $\mathcal{B}$. Then the following are equivalent*:

(i) $\mathcal{A}_1 \otimes_{\min} \mathcal{T}(2) = \mathcal{A} \otimes_{\max} \mathcal{T}(2)$,

(ii) *for any pair of completely contractive homomorphisms $\rho_i : \mathcal{A} \to \mathcal{L}(\mathcal{H}_i), i = 1, 2$, and contraction $T : \mathcal{H}_2 \to \mathcal{H}_1$ with $\rho_1(A)T = T\rho_2(A)$ for all A in $\mathcal{A}$, there exist $*$-homomorphisms, $\pi_i : \mathcal{B} \to \mathcal{L}(\mathcal{K}_i)$ with $\mathcal{H}_i$ contained in $\mathcal{K}_i$ and a unitary $U : \mathcal{K}_2 \to \mathcal{K}_1$ such that*

$$\rho_1(a) = P_{\mathcal{H}_i}\pi_i(a)|_{\mathcal{H}_i}, \quad i = 1, 2,$$

and

$$\rho_1(A)T = P_{\mathcal{H}_1}\pi_1(a)U|_{\mathcal{H}_2} = P_{\mathcal{H}_1}U\pi_2(A)|_{\mathcal{H}_2},$$

for all A in $\mathcal{A}$.

PROOF. Assuming (i), let $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $\gamma : \mathcal{T}(2) \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be defined by

$$\rho(A) = \begin{pmatrix} \rho_1(A) & 0 \\ 0 & \rho_2(A) \end{pmatrix}$$

and

$$\gamma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} aI_{\mathcal{H}_1} & bT \\ 0 & cI_{\mathcal{H}_2} \end{pmatrix}.$$

It is easily checked that ρ is completely contractive, and that the ranges of ρ and γ commute. The fact that γ is completely contractive was proved in [**9**]. Thus, applying Proposition 2.5, we have a $*$-homomorphism $\pi : \mathcal{B} \otimes \mathcal{M}_2 \to \mathcal{L}(\mathcal{K})$ with $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \subseteq \mathcal{K}$ such that $\gamma \odot \rho(U) = P_{\mathcal{H}}\pi(U)|_{\mathcal{H}}$ for all U in $\mathcal{A} \otimes \mathcal{T}(2)$.

Let $\{e_{ij}\}$ be the matrix units in $\mathcal{M}_2$, and let $\mathcal{K}_i = \pi(I \otimes e_{ii})\mathcal{K}, i = 1, 2$. Then $\pi(I \otimes e_{12})$ is determined by a unitary $U : \mathcal{K}_2 \to \mathcal{K}_1$, and $\pi : \mathcal{B} \otimes I \to \mathcal{L}(\mathcal{K})$ is given by $\pi(\mathcal{B}) = \pi_1(\mathcal{B}) \oplus \pi_2(\mathcal{B})$ where $\pi_i : \mathcal{B} \to \mathcal{L}(\mathcal{K}_i), i = 1, 2$.

It is now easily checked that π_1, π_2 and U have the desired properties.

Conversely, given any pair of commuting completely contractive maps $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \gamma : \mathcal{T}(2) \to \mathcal{L}(\mathcal{H})$, there is a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that ρ and γ have the above form. Thus we have π_1, π_2 and U as in (ii).

Defining $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2, \pi : \mathcal{B} \to \mathcal{L}(\mathcal{K})$ via $\pi = \pi_1 \oplus \pi_2$ and $\sigma : \mathcal{M}_2 \to \mathcal{L}(\mathcal{K})$ via

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} aI_{\mathcal{K}_1} & bU \\ cU^* & dI_{\mathcal{K}_2} \end{pmatrix},$$

we have that $\rho \odot \gamma(V) = P_{\mathcal{H}}\pi \odot \sigma(V)|_{\mathcal{H}}$ for any V in $\mathcal{A} \otimes \mathcal{T}(2)$. Thus, we have that the norm on $\mathcal{A} \otimes \mathcal{T}(2)$ induced by ρ and γ is dominated by the one induced by π and σ. But since the minimum and maximum norms agree on $\mathcal{B} \otimes \mathcal{M}_2$, the norm induced by π and σ is the minimum norm on $\mathcal{A} \otimes \mathcal{T}(2)$. Hence, the minimum and maximum operator norms coincide. □

In [**9**] the authors proved a number of theorems which are equivalent to the assertion that the minimal and maximal operator norms agree for certain algebras. We summarize those here.

By a *nest algebra* we mean a subalgebra of $\mathcal{L}(\mathcal{H})$ consisting of all operators which leave invariant the subspaces in a preassigned nest of subspaces of $\mathcal{H}$. We call it finite dimensional if $\mathcal{H}$ is finite dimensional. Thus, with respect to some basis and block structure, a finite dimensional nest algebra consists of all the block upper triangular matrices.

THEOREM 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite dimensional nest algebras, then $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ and $\mathcal{A} \otimes_{\min} \mathcal{P}(\mathbf{D}) = \mathcal{A} \otimes_{\max} \mathcal{P}(\mathbf{D})$.*

PROOF. These are restatements of [**9**, Theorem 2.1] and [**9**, Theorem 1.2], respectively. □

We give a new proof of this result in §3.

If the algebras in question are also endowed with some weaker topology, then one may wish to restrict attention to completely contractive representations which are continuous in that weaker topology. If $\mathcal{A}_i$ are subalgebras of von Neumann algebras, $i = 1, 2$, then we write $\mathcal{A}_1 \otimes^{\sigma}_{\min} \mathcal{A}_2$ and $\mathcal{A}_1 \otimes^{\sigma}_{\max} \mathcal{A}_2$ to denote the complete operator cross-norms that one obtains by restricting the homomorphisms used in the definitions to be continuous from the weak* to σ-weak topologies.

Thus, [**9**, Theorem 3.3] and [**9**, Theorem 3.1] imply

THEOREM 2.8. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be nest algebras on separable Hilbert spaces, then*

$$\mathcal{A}_1 \otimes^{\sigma}_{\min} \mathcal{A}_2 = \mathcal{A}_1 \otimes^{\sigma}_{\max} \mathcal{A}_2 \text{ and } \mathcal{A}_1 \otimes^{\sigma}_{\min} \mathcal{P}(\mathbf{D}) = \mathcal{A}_1 \otimes^{\sigma}_{\max} \mathcal{P}(\mathbf{D}).$$

For the second result, it was only necessary to assume that the representation of the nest algebra is σ-weakly continuous, the representation of $\mathcal{P}(\mathbf{D})$ need not be σ-weakly continuous.

We close this section with one last result which relates the C^*-algebra theory to the cases that we are interested in. A C^*-algebra $\mathcal{B}$ is called *nuclear* if, for every C^*-algebra $\mathcal{A}$, $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$.

PROPOSITION 2.9. *Let $\mathcal{A}$ be an operator algebra, and let $\mathcal{B}$ be a nuclear C^*-algebra, then $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$.*

PROOF. Let $\mathcal{A}$ be contained in a C^*-algebra $\mathcal{C}$. Given a pair of commuting completely contractive homomorphisms, $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H}), \pi : \mathcal{B} \to \mathcal{L}(\mathcal{H})$, we have that $\pi(\mathcal{B})$ is a nuclear C^*-subalgebras of $\mathcal{L}(\mathcal{H})$, since quotients of nuclear C^*-algebras are nuclear [**6**]. This implies that the commutant $\pi(\mathcal{B})'$ is injective [**6**], which means that the map ρ can be extended to a completely positive map $\theta : \mathcal{C} \to \pi(\mathcal{B})'$.

It follows that there is a unital completely positive map $\theta \odot \pi : \mathcal{C} \otimes_{\max} \mathcal{B} \to \mathcal{L}(H)$ with $\theta \odot \pi(c \otimes b) = \theta(c)\pi(b)$ [**8**]. Since $\mathcal{B}$ is nuclear, $\mathcal{C} \otimes_{\max} \mathcal{B} = \mathcal{C} \otimes_{\min} \mathcal{B}$, which contains $\mathcal{A} \otimes_{\min} \mathcal{B}$ as a subalgebra. Thus, $\theta \odot \pi$ is completely contractive on $\mathcal{A} \otimes_{\min} \mathcal{B}$, which implies that the operator norm induced by π and ρ is dominated by the minimum operator norm. □

3. Further results. In this section we consider some further results on the minimal and maximal operator norms. We first show that the minimal and maximal operator norms do not agree on $\mathcal{T}(2) \otimes \mathcal{T}(2) \otimes \mathcal{T}(2)$. This is a discrete version of Parrott's example [**7**] of three commuting contractions with no simultaneous commuting unitary dilation. We also prove a general result on tensor norms which implies, in particular, that $\mathcal{P}(\mathbf{D}) \otimes_{\min} \mathcal{T}(m) = \mathcal{P}(\mathbf{D}) \otimes_{\max} \mathcal{T}(n)$ and $\mathcal{T}(n) \otimes_{\min} \mathcal{T}(m) = \mathcal{T}(n) \otimes_{\max} \mathcal{T}(m)$, for all n and m, where $\mathcal{T}(n)$ denotes the upper triangular $n \times n$ matrices. These results are special cases of Theorem 2.7, which was proved in [**9**]. The proof in [**9**] was constructive, while the above results follow from general tensor product considerations. We should also point out that the proof of Theorem 2.7 in [**9**] reduced the case of a general finite dimensional nest algebra to the case of $\mathcal{T}(n)$. Here we perform this reduction again, in Theorem 3.4, with different arguments.

The maximal and minimal complete operator cross-norms are defined on multiple tensor products in a manner analogous to the case of two algebras. In analogy with Parrott's example [**7**] of a triple of commuting contractions which does not admit a triple of commuting unitary dilations we have the following theorem.

THEOREM 3.1. *The maximal and minimal complete operator cross norms differ for the algebra* $\mathcal{T}(2) \otimes \mathcal{T}(2) \otimes \mathcal{T}(2)$.

PROOF. Let U and V be unitary operators in $\mathcal{M}_2$ and consider the operators

$$R = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$$

in $\mathcal{M}_4$. Let ρ_R, ρ_S, ρ_T be the contractive representations of $\mathcal{T}(2)$ into $\mathcal{M}_{32} = \mathcal{M}_2 \otimes \mathcal{M}_2 \otimes \mathcal{M}_2 \otimes \mathcal{M}_4$ given by

$$\begin{aligned}
\rho_R(e_{12}) &= e_{12} \otimes I \otimes I \otimes R \\
\rho_R(e_{11}) &= e_{11} \otimes I \otimes I \otimes I \\
\rho_R(e_{22}) &= e_{22} \otimes I \otimes I \otimes I \\
\rho_S(e_{12}) &= I \otimes e_{12} \otimes I \otimes S \\
\rho_S(e_{11}) &= I \otimes e_{11} \otimes I \otimes I \\
\rho_S(e_{22}) &= I \otimes e_{22} \otimes I \otimes I \\
\rho_T(e_{12}) &= I \otimes I \otimes e_{12} \otimes T \\
\rho_T(e_{11}) &= I \otimes I \otimes e_{11} \otimes I \\
\rho_T(e_{22}) &= I \otimes I \otimes e_{22} \otimes I.
\end{aligned}$$

Then ρ_R, ρ_S, ρ_T are contractive representations and are mutually commuting since all products of R, S, T are zero. Furthermore, $\rho_R \odot \rho_S \odot \rho_T$ can be interpreted as the mapping which transports the 8×8 matrix (a_{ij}) in $\mathcal{T}(2) \otimes \mathcal{T}(2) \otimes \mathcal{T}(2)$ to the inflated Schur product

$$(a_{ij}I) \circ \begin{bmatrix} I & T & S & 0 & R & 0 & 0 & 0 \\ & I & & S & & R & & 0 \\ & & I & T & & & R & 0 \\ & & & I & & & & R \\ & & & & I & T & S & 0 \\ & & & & & I & & S \\ & & & & & & I & T \\ & & & & & & & I \end{bmatrix},$$

where I is the 4×4 identity matrix and where undefined entries are also zero. Notice that the inflated Schur product map has norm dominating the norm of the submap

$$\begin{bmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{bmatrix} \to \begin{bmatrix} aS & bR & 0 \\ cT & 0 & dR \\ 0 & eT & tS \end{bmatrix}.$$

Considering the special form of R, S, T, this submap has norm agreeing with the norm of the inflated Schur map

$$\begin{bmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{bmatrix} \to \begin{bmatrix} aI & bU & 0 \\ cV & 0 & dU \\ 0 & eV & fI \end{bmatrix}.$$

The norm of the image matrix agrees with the norm of

$$\begin{bmatrix} aI & bI & 0 \\ cI & 0 & dI \\ 0 & cI & fUV^*U^*V \end{bmatrix}.$$

(Multiplying left and right by appropriate diagonal unitaries.) Now make the choice

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and note that

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} I & I & 0 \\ I & 0 & I \\ 0 & I & +I \end{bmatrix}.$$

The first matrix has norm $\sqrt{2}$ while the latter has norm 2. Hence $\rho_R \otimes \rho_S \otimes \rho_T$ is not contractive. □

We thank Ken Davidson for simplifying an earlier proof of Theorem 3.1.

THEOREM 3.2. *Let $\mathcal{A}$ be an operator algebra. Then $\mathcal{A} \otimes_{\min} \mathcal{P}(\mathbf{D}) = \mathcal{A} \otimes_{\max} \mathcal{P}(\mathbf{D})$ if and only if $\mathcal{A} \otimes_{\min} \mathcal{T}(n) = \mathcal{A} \otimes_{\max} \mathcal{T}(n)$, for all n.*

PROOF. Assume the latter condition, and let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{K})$. Since $\mathcal{A} \otimes_{\min} \mathcal{P}(\mathbf{D})$ inherits its norm from $\mathcal{A} \otimes_{\min} \mathcal{C}(T)$, we see that, for $U = \sum_{i=0}^{m} a_i \otimes z^i$ in $\mathcal{A} \otimes_{\min} \mathcal{P}(\mathbf{D})$, we have that $||U||_{\min} = \sup\{|| \sum_{i=0}^{m} \lambda^i a_i||_{\mathcal{K}} : |\lambda| = 1\}$, where $|| \cdot ||_{\mathcal{K}}$ is the operator norm on $\mathcal{K}$. Let $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and let $T \in \mathcal{L}(\mathcal{H}), ||T|| \leq 1$, which commutes with $\rho(\mathcal{A})$; then ρ and T determines a representation $\rho_T : \mathcal{A} \otimes \mathcal{P}(\mathbf{D}) \to \mathcal{L}(\mathcal{H})$ and $||U||_{\max}$ is the supremum of $||\rho_T(U)||$ over all such ρ and T. We must show that, for $U = (U_{ij})$ in $\mathcal{M}_k(\mathcal{A} \otimes \mathcal{P}(\mathbf{D}))$, $||(\rho_T(U_{ij}))|| \leq ||U||_{\min}$. We only argue the case of $k = 1$.

For U as above, $\rho_T(U) = \sum_{i=0}^{m} \rho(a_i) T^i$. Let S denote the bilateral shift on ℓ_2. Identifying S with the operator M_z of multiplication by z on $L^2(T)$ we have that,

$$\begin{aligned} ||\rho_T(U)|| &\leq \sup\left\{\Big|\Big| \sum_{i=0}^{m} \rho(a_i) z^i T^i \Big|\Big| : |z| = 1\right\} \\ &= \Big|\Big| \sum_{i=0}^{m} \rho(a_i)(T \otimes S)^i \Big|\Big|, \end{aligned}$$

where the latter norm is taken as an operator on $\mathcal{H} \otimes \ell_2$. The operator $\sum_{i=1}^{m} \rho(a_i)(T \otimes S)^i = X$ is an infinite operator-valued upper triangular Toeplitz matrix on $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$ (infinitely many copies). The i-th super-diagonal of this Toeplitz operator is constantly $\rho(a_i)T^i$. Let $n \geq m$, and let X_n be the Toeplitz operator on $\mathcal{H}^{(n)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n copies) whose i-th super-diagonal is $\rho(a_i)T^i$. Then we have that $||X|| = \lim_{n\to\infty} ||X_n||$.

Consider the representation $\gamma : \mathcal{T}(n) \to \mathcal{L}(\mathcal{H}^{(n)})$ defined by $\gamma((\lambda_{ij})) = (\lambda_{ij} T^{j-i}), j \geq i$, and (λ_{ij}) in $\mathcal{T}(n)$. Also, let $\rho^{(n)} : \mathcal{A} \to \mathcal{L}(\mathcal{H}^{(n)})$ be defined by $\rho^{(n)}(a) = \rho(a) \oplus \cdots \oplus \rho(a)$. Define U_n in $\mathcal{A} \otimes \mathcal{T}(n)$ by setting

$$U_n = \sum_{i=0}^{n} a_i \otimes S_n^i,$$

where S_n is the matrix whose $(i, i+1)$ entry is 1 for all i and 0 elsewhere. Since $\rho^{(n)}$ and γ commute we have a representation

$$\rho^{(n)} \odot \gamma : \mathcal{A} \otimes \mathcal{T}(n) \to \mathcal{L}(\mathcal{H}^{(n)}) \quad \text{and} \quad X_n = \rho^{(n)} \odot \gamma(U_n).$$

Hence, $||X_n|| \le ||U_n||_{\max} = ||U_n||_{\min} = ||U_n||_{\mathcal{K}^{(n)}}$, where the latter norm is the operator norm on $\mathcal{K}^{(n)}$. Clearly,

$$||U_n||_{\mathcal{K}^{(n)}} \le ||\sum_{i=0}^{n} a_i \otimes S^i||_{\mathcal{K}\otimes\ell_2} = \sup\{||\sum_{i=0}^{n} a_i z^i||_{\mathcal{K}}: |z| = 1\} = ||U||_{\min},$$

where the first equality is obtained by identifying S with M_z and $L^2(T)$. Thus, we have that $||\rho_T(U)|| \le ||X|| \le ||U||_{\min}$, proving the first implication.

Conversely, assume that $\mathcal{A}\otimes_{\min}\mathcal{P}(\mathbf{D}) = \mathcal{A}\otimes_{\max}\mathcal{P}(\mathbf{D})$. Let $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $\gamma : \mathcal{T}(n) \to \mathcal{L}(\mathcal{H})$ be commuting completely contractive homomorphisms. Set $\mathcal{H}_i = \gamma(e_{ii})\mathcal{H}$, so that $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$. Relative to this decomposition there exist $\rho_i : \mathcal{A} \to \mathcal{L}(\mathcal{H}_i)$ so that $\rho(a) = \rho_1(a) \oplus \cdots \oplus \rho_n(a)$. Also, $\gamma(e_{i,i+1})$ is determined by contraction operators $X_i : \mathcal{H}_{i+1} \to \mathcal{H}_i, 1 \le i \le n-1$. The fact that $\rho(a)$ commutes with $\gamma(\mathcal{T}(n))$ is equivalent to the intertwining relations,

$$\rho_i(a)X_i = X_i\rho_{i+1}(a), \qquad 1 \le i \le n-1.$$

Set $X = \gamma(S_n)$, and define $\tilde{\gamma} : \mathcal{T}(n) \to \mathcal{L}(\mathcal{H}^{(n)})$ by $\tilde{\gamma}((\lambda_{ij})) = (\lambda_{ij}X^{j-1})$ for (λ_{ij}) in $\mathcal{T}(N)$. Also, let $\tilde{\rho} : \mathcal{A} \to \mathcal{L}(\mathcal{H}^{(n)})$ be defined by $\tilde{\rho}(a) = \rho(a) \oplus \cdots \oplus \rho(a)$, so that $\tilde{\rho}(a)$ commutes with $\tilde{\gamma}(\mathcal{T}(n))$ and defines $\tilde{\rho}\,\tilde{\odot}\,\tilde{\gamma} : \mathcal{A}\otimes\mathcal{T}(n) \to \mathcal{L}(\mathcal{H}^{(n)})$. Define $V : \mathcal{H} \to \mathcal{H}^{(n)}$ by $Vh = h_1 \oplus \cdots \oplus h_n$, where $h = h_1 + \cdots + h_n$ in the decomposition $\mathcal{H} = \mathcal{H}_1 + \cdots + \mathcal{H}_n$. It is easily checked that, for U in $\mathcal{A}\otimes\mathcal{T}(n), \rho\odot\gamma(U) = V^*(\tilde{\rho}\odot\tilde{\gamma}(U))V$. Thus, to prove that $||\rho\odot\gamma(U)|| \le ||U||_{\min}$, it will suffice to show that $||\tilde{\rho}\odot\tilde{\gamma}(U)|| \le ||U||_{\min}$.

Now if $U = \sum_{i\le j} a_{ij}\otimes e_{ij}$, then $\tilde{\rho}\odot\tilde{\gamma}(U) = (\rho(A_{ij})X^{j-i})$. Since X is a contraction which commutes with $\rho(\mathcal{A})$, we have that

$$\begin{aligned} ||\tilde{\rho}\odot\tilde{\gamma}(U)|| &\le ||(a_{ij}\otimes z^{j-i})||_{\max} = ||(a_{ij}\otimes z^{j-i})||_{\min} \\ &= \sup\{||(a_{ij}\lambda^{j-i})||_{M_n(\ )}|\lambda| = |\} \\ &= ||(a_{ij})||_{M_n(\mathcal{A})} = \Big|\Big|\sum_{i\le j} a_{ij}\otimes e_{ij}\Big|\Big|_{\min}, \quad \text{in } \mathcal{A}\otimes_{\min}\mathcal{T}(n). \end{aligned}$$

Since ρ and γ were arbitrary, these inequalities show that $||U||_{\max} \le ||U||_{\min}$ for U in $\mathcal{A}\otimes\mathcal{T}(n)$. The argument for U in $\mathcal{M}_k(\mathcal{A}\otimes\mathcal{T}(n))$ is identical. Thus, we have that $\mathcal{A}\otimes_{\min}\mathcal{T}(n) = \mathcal{A}\otimes_{\max}\mathcal{T}(n)$, for all n. □

We can now present a simpler proof of an important special case of Theorem 2.7.

COROLLARY 3.3. *We have that* $\mathcal{P}(\mathbf{D}) \otimes_{\min} \mathcal{T}(n) = \mathcal{P}(\mathbf{D}) \otimes_{\max} \mathcal{T}(n)$ *and* $\mathcal{T}(n) \otimes_{min} \mathcal{T}(m) = \mathcal{T}(n) \otimes_{\max} \mathcal{T}(m)$, *for all* m, n.

PROOF. By Ando's theorem $\mathcal{P}(\mathbf{D}) \otimes_{\min} \mathcal{P}(\mathbf{D}) = \mathcal{P}(\mathbf{D}) \otimes_{\max} \mathcal{P}(\mathbf{D})$. Thus by Theorem 3.2, we have that $\mathcal{P}(\mathbf{D}) \otimes_{\min} \mathcal{T}(n) = \mathcal{P}(\mathbf{D}) \otimes_{\max} \mathcal{T}(n)$, for all n. Applying the theorem to this last equality leads to $\mathcal{T}(n) \otimes_{\min} \mathcal{T}(m) = \mathcal{T}(n) \otimes_{\max} \mathcal{T}(m)$ for all n, m, $\square$

We can now use Theorem 3.2 to obtain the following more general result.

THEOREM 3.4. *Let* $\mathcal{A}$ *be an operator algebra. Then* $\mathcal{A} \otimes_{\min} \mathcal{P}(\mathbf{D}) = \mathcal{A} \otimes_{\max} \mathcal{P}(\mathbf{D})$ *if and only if* $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ *for every finite dimensional nest algebra.*

PROOF. Consider first the case of the finite dimensional nest algebra $\mathcal{B} = \mathcal{T}(n) \otimes \mathcal{M}_m$, and note that $\mathcal{T}(n) \otimes \mathcal{M}_m = \mathcal{T}(n) \otimes_{\min} \mathcal{M}_m = \mathcal{T}(n) \otimes_{\max} \mathcal{M}_m$. This follows from the fact that a representation ρ of $\mathcal{T}(n) \otimes \mathcal{M}_m$, with $||\rho(e_{ij} \otimes f_{k\ell})| \leq 1$ for each matrix unit $e_{ij} \otimes f_{k\ell}$, is automatically completely contractive (see [**10**, Proposition 1.1]). It is also a consequence of Proposition 2.9 above and the nuclearity of $\mathcal{M}_m$. Using Theorem 3.2 it follows that

$$\begin{aligned}
\mathcal{A} \otimes_{\max} (\mathcal{T}(n) \otimes \mathcal{M}_m) &= \mathcal{A} \otimes_{\max} (\mathcal{T}(n) \otimes_{\max} \mathcal{M}_m) \\
&= (\mathcal{A} \otimes_{\max} \mathcal{T}(n)) \otimes_{\max} \mathcal{M}_m \\
&= (\mathcal{A} \otimes_{\min} \mathcal{T}(n)) \otimes_{\max} \mathcal{M}_m \\
&= (\mathcal{A} \otimes_{\min} \mathcal{T}(n)) \otimes_{\min} \mathcal{M}_m \\
&= \mathcal{A} \otimes_{\min} (\mathcal{T}(n) \otimes \mathcal{M}_m).
\end{aligned}$$

To deal with the case of a general finite dimensional nest algebra $\mathcal{B}$ we show that there is a completely isometric (nonunital) homomorphism $\alpha : \mathcal{B} \to \mathcal{T}(n) \otimes \mathcal{M}_m$, for some n, m such that every contractive representation of $\mathcal{B}$ can be regarded as the restriction of a contractive representation of $\mathcal{T}(n) \otimes \mathcal{M}_m$.

As before, let $\mathcal{T}(n) \otimes \mathcal{M}_m$ be spanned by the matrix units $e_{ij} \otimes f_{k\ell}$, for $1 \le i \le j \le n$, and $1 \le k, \ell \le m$, and consider a block diagonal projection $\mathcal{P} = \sum_{i=1}^{n} e_{ii} \otimes q_i$, where q_i is a non zero diagonal projection in $\mathcal{M}_m$. Then the subalgebra $\mathcal{P}\mathcal{B}_1\mathcal{P}$, where $\mathcal{B}_1 = \mathcal{T}(n) \otimes \mathcal{M}_m$, is a typical finite dimensional nest algebra. That is, for some choice of n, m and $\mathcal{P}$ the algebras $\mathcal{B}$ and $\mathcal{P}\mathcal{B}_1\mathcal{P}$ are completely isometrically isomorphic. We assume then that $\mathcal{B} = \mathcal{P}\mathcal{B}_1\mathcal{P}$.

Let $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ be a unital completely contractive representation. The diagonal subalgebra $\mathcal{B} \cap \mathcal{B}^*$ has the form $\mathcal{M}_{r_1} \oplus \cdots \oplus \mathcal{M}_{r_n}$, where r_i is the rank of q_i, and we may assume that the restriction of ρ to $\mathcal{B} \cap \mathcal{A}$ is a direct sum of inflations, that is, there is a decomposition

$$\mathcal{H} = \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{r_i} \oplus \mathcal{H}_i \Big\} = \sum_{i=1}^{n} \oplus \mathcal{K}_i$$

such that, for $b = b_1 \oplus \cdots \oplus b_n$ in $\mathcal{B} \cap \mathcal{B}^*, \rho(b) = (b_1 \otimes I_{\mathcal{H}_1}) \oplus \cdots \oplus (b_n \otimes I_{\mathcal{H}_n})$. With this normalizing assumption it follows that one can identify the contractions in the set

$$S_{ij} = \{\rho(e_{ij} \otimes f_{k\ell}) : e_{ij} \otimes f_{k\ell} \in \mathcal{B}\}$$

for each pair i, j. That is, there is a contraction $X_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ such that $\rho(e_{ij} \otimes f_{k\ell}) = X_{ij} \otimes f_{k\ell}$ for all $e_{ij} \otimes f_{k\ell}$ in S_{ij}. For a matrix $(b_{ijk\ell})$ in $\mathcal{B}$ we now have

$$\rho((b_{ijk\ell})) = \sum_{ijk\ell} b_{ijk\ell}(X_{ij} \otimes f_{k\ell}).$$

With the natural identification of $\mathcal{H}$ as a subspace of $\tilde{\mathcal{H}} = \sum_{i=1}^{n} \{\sum_{i=1}^{m} \oplus \mathcal{H}_i\}$ it is clear that the formula above can be used to define a representation $\tilde{\rho} : \mathcal{B}_1 \to \mathcal{L}(\tilde{\mathcal{H}})$ such that ρ is the restriction of $\tilde{\rho}$.

Consider now a completely contractive representation $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ which commutes with ρ. Then, with respect to the decomposition of $\mathcal{H}$ above induced by ρ, we have $\sigma = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \oplus \sigma_i$ for some completely contractive representations $\sigma_i : \mathcal{A} \to \mathcal{L}(\mathcal{H}_i), 1 \le i \le n$.

It is easily verified that the representation $\tilde{\sigma} : A \to \mathcal{L}(\tilde{H})$ given by $\tilde{\sigma} = \sum_{i=1}^{n} \sum_{i=1}^{m} \oplus \sigma_i$ extends σ and commutes with $\tilde{\rho}$.

It now follows that the maximal norm on $\mathcal{A} \otimes \mathcal{B}$ is dominated by the maximal norm on the containing algebra $\mathcal{A} \otimes \mathcal{B}_1$. By our earlier observation $\mathcal{A} \otimes_{\max} \mathcal{B}_1 = \mathcal{A} \otimes_{\min} \mathcal{B}_1$ and so $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$, as required. $\square$

Thus we see that Theorem 2.7 can be deduced as a corollary of Theorem 3.4 in the same fashion the Corollary 3.3 followed from Theorem 3.2.

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