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THE USE OF REPULSIVE FIXED POINTS TO ANALYTICALLY CONTINUE CERTAIN FUNCTIONS

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Dedicated to Professor W.J. Thron on the occasion of his 70th birthday

Introduction. Attractive and repulsive fixed points can be used to enhance the convergence behavior of certain sequences of analytic functions that display the following compositional structure:

Let $\{f_n(\zeta, z)\}$ be a sequence of functions that are analytic in both variables (and continuous, along with their partials) in $S \times D$, where S and D are regions (not necessarily bounded) and, for each n, $D \supset f_n(S, D)$. Suppose that $f_n \to f$ on $S \times D$. Set

$$F_1(\zeta, z) = f_1(\zeta, z) \text{ and } F_n(\zeta, z) = F_{n-1}(\zeta, f_n(\zeta, z)) \text{ for } n > 1. \tag{1}$$

The sequence

$$\{F_n(\zeta, z)\} \tag{2}$$

may be called "limit periodic," an expression widely used to designate an important class of continued fractions that can be interpreted in this fashion.

The first investigation of the convergence behavior of such sequences (with regard to a fixed ζ) appears to have been a paper by Magnus and Mandell [**10**] on limit periodic compositions of linear fractional transformations (LFTs). They deduced that, in the most common circumstances, the sequence (2), in effect, converges to a common function $\lambda(\zeta)$ for all values of z except the repulsive fixed point (β) of the limit function. In particular, (2) converges for $z = \alpha$, the attractive fixed point of f. The author carried on these investigations by focusing on sequences $\{f_n\}$ of more esoteric varieties of LFTs [**1**]. Later, the author described the use of $z = \alpha$ to accelerate the convergence of certain limit periodic continued fractions (that may

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be interpreted as compositions of special LFTs) [**2, 4**]. Thron and Waadeland wrote a comprehensive paper detailing the accelerative effect for more general limit periodic continued fractions in 1979 [**11**]. Later, the author described a general theory of limit periodic iteration in complete metric spaces [**6**], including a general acceleration result similar to that reported in [**11**]. A second paper concerning the more general structure (1) in the specific setting of the complex plane followed [**7**]. Here the author describes conditions on $\{f_n(\zeta, z)\}$ that imply $\text{Lim}_{n\to\infty} F_n(\zeta, z) = \lambda(\zeta)$, a function independent of z, for z and ζ belonging to certain subsets of $\mathbf{C}$.

At this point, there is a fairly complete theory of acceleration of (2) involving the attractor $z = \alpha$. However, the more interesting and less obvious use of a repulsive fixed point to analytically continue the function $\lambda(\zeta)$—for a *more general* sequence (2)—has received no attention. As we shall see, having established the convergence results in [**7**], the question of analytic continuation is not difficult to answer.

First, however, let us look briefly at work leading up to this point. In [**13**], Waadeland employed what he referred to as the "wrong modification" of a T-fraction to increase the region of convergence of this particular infinite expansion. In fact, he used the repulsive fixed point of a linear fractional transformation to accomplish the feat. A short time later, the author described the possibility for using a repulsor in a less restrictive setting involving LFTs [**3**]. Thron and Waadeland, deliberating on specific LFTs, produced a lemma that allowed them to analytically continued certain limit periodic continued fractions [**12**]. The author discussed a similar (and somewhat restrictive) result for certain power series in [**5**]. Jacobsen has investigated analytic extension for more sophisticated limit k-periodic fractions in at least two papers [**8, 9**]. In the present paper the author provides more general results in the setting of analytic functions $\{f_n\}$—results that apply to power series and continued fractions, as well as to a host of other infinite expansions of the form (1). As the reader will discern, Theorem 2 builds upon the work of Thron, Waadeland, and others.

The two examples that follow provide, in very simple settings, the motivation for the use of attractive/repulsive fixed points in enriching the convergence of limit periodic structures.

Example 1. The power series $P(\zeta) = 1 + \zeta + \zeta^2 + \cdots$ has an elementary limit periodic structure in which $f_n(\zeta, z) \equiv 1 + \zeta z$; $\alpha(\zeta) = 1/(1-\zeta)$ is an *attractor* if $|\zeta| < 1$ (i.e., $|f_n(\zeta, z) - \alpha(\zeta)| < |z - \alpha(\zeta)|$) and a *repulsor* if $|\zeta| > 1$ (i.e., $|f_n(\zeta, z) - \alpha(\zeta)| > |z - \alpha(\zeta)|$). Observe that $F_n(\zeta, \alpha(\zeta)) \equiv \alpha(\zeta)$, so that, as an attractor, α gives maximum acceleration to the value of $P(\zeta)$, and, as a repulsor, gives complete analytic continuation of $P(\zeta)$.

Example 2. Let us expand the number $\zeta \neq 1$ as a periodic continued fraction by writing $\zeta = \zeta(1-\zeta)/(1-\zeta) = \cdots = -[-\zeta(1-\zeta)/1]$. If $\operatorname{Re} \zeta < 1/2$, the attractor of $f_n(\zeta, z) \equiv \zeta(1-\zeta)/(1-z)$ is $\alpha(\zeta) = \zeta$, and the repulsor is $\beta(\zeta) = 1 - \zeta$. For $\operatorname{Re} \zeta > 1/2$, the roles of α and β are reversed. Formally, the continued fraction converges to ζ in the first case and to $1-\zeta$ in the second. Thus, instant convergence to the proper limit occurs if, at all times, one uses $z = \zeta$ in $\{F_n(\zeta, z)\}$, i.e., using the repulsor when $\operatorname{Re} \zeta > 1/2$ analytically continues the continued fraction to its proper value.

The continuation theorem. Our result pertains to the "tail end" of $F_n(\zeta, z)$. For brevity, let us set

$$f_n(\zeta, z) = f_n(z), \quad f(\zeta, z) = f(z), \quad \alpha(\zeta) = \alpha,$$
$$F_{n,n+m}(\zeta, z) = f_n \circ f_{n+1} \circ \cdots \circ f_{n+m}(z),$$
$$\lambda_n(\zeta) = \operatorname{Lim}_{m\to\infty} F_{n,n+m}(\zeta, z).$$

In order to analytically continue $\lambda(\zeta) = \operatorname{Lim}_{n\to\infty} F_n(\zeta, z)$, it is convenient to have at hand a "nucleus" (see [**7**]) of the following form:

Let us assume that there exist compact regions Δ and Z in the complex plane, such that $S \supset \Delta$ and $D \supset Z$, and

(i) for each $\zeta \in \Delta$, there exists a fixed point $z = \alpha(\zeta)$ of $f(\zeta, z)$ with $\alpha(\zeta) \in Z$,

(ii) $\lambda_n(\zeta)$ is defined and analytic on Δ for all $z \in Z$ and all $n \geq 0$ ($\lambda(\zeta) = \lambda_1(\zeta)$).

The following theorem summarizes the results in [**7**] in this connection.

Theorem 1. *Suppose there exists a compact region Δ and a simple closed contour Γ, with $\Omega = \Gamma \cup \operatorname{Int} \Gamma$, such that*

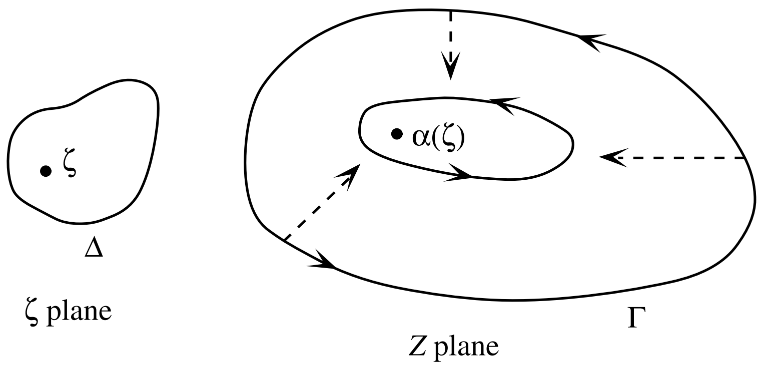

FIGURE 1.

(i) $f_n(\zeta, z)$ *is analytic on* $\Delta \times \Omega$,

(ii) $f_n \to f$ *uniformly on* $\Delta \times \Omega$,

(iii) $\mathrm{Int}\,\Omega \supset f(\Delta, \Omega)$ $(\Rightarrow \ni \alpha(\zeta) = f(\zeta, \alpha(\zeta)))$,

(iv) $|f(\zeta, z) - \alpha(\zeta)| < |z - \alpha(\zeta)| \forall \zeta \in \Delta,\ \forall z \in \Gamma$.

Then there exists an n *such that* $\mathrm{Lim}_{m\to\infty} F_{n,n+m}(\zeta, z) = \lambda_n(\zeta)$ *analytic on* Δ *for every* $z \in \Omega$, *including* $\alpha(\zeta)$. *Furthermore, if* f_n *is analytic on* $\Delta \times D_n$ *where* $D_{n-1} \supseteq f_n(\Delta, D_n)$ *and the* D_n *are regions with* $\cap D_n \supset \Omega$, *then* $\mathrm{Lim}_{n\to\infty} F_n(\zeta, z) = \lambda(\zeta)$, *analytic on* Δ. (See Figure 1.)

We now turn to analytic continuation. (Refer to Figure 2.)

Theorem 2. *Suppose there exists a compact region* $E(\Delta)$ *such that* $S \supset E(\Delta) \supset \Delta$, *in which, for all* $\zeta \in E(\Delta)$,

(a) $\alpha(\zeta)$ *is analytic, with* $f(\zeta, \alpha(\zeta)) = \alpha(\zeta)$ *and* $D \supset \alpha(E(\Delta))$,

(b) $|f_n(\zeta, \alpha(\zeta)) - \alpha(\zeta)| < cr^n$, $0 \le r < 1$,

(c) *For* n *sufficiently large, the partial derivatives satisfy* $|D_z f_n(\zeta, \alpha(\zeta))| < K$, *where* $1 < K < 1/r$.

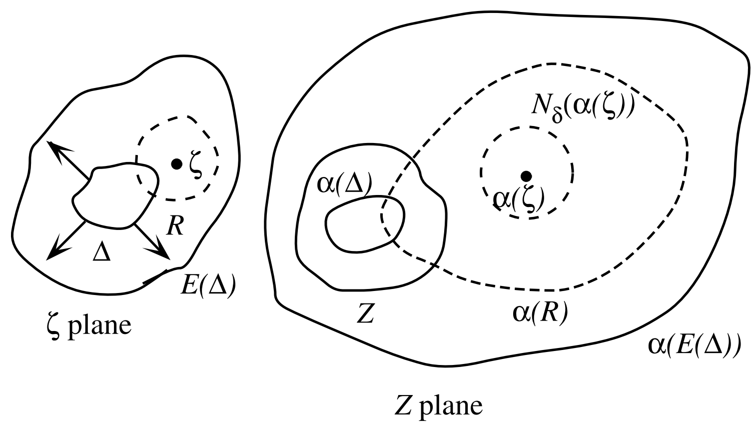

FIGURE 2.

Then, on each (bounded) open region R of $E(\Delta)$, with $\partial(R) \cap \partial(E(\Delta)) = \varnothing$ and $\mathrm{Int}\,(R \cap \Delta) \neq \varnothing$, there exists $n(R)$ such that $\mathrm{Lim}_{\,m\to\infty} F_{n,n+m}$ $(\zeta, \alpha(\zeta)) = \lambda_n(\zeta)$, analytic for all $\zeta \in R \cup \Delta$ and all $n \geq n(R)$.

Proof. Assuming the existence of the nucleus described above, it is sufficient to show that, for some fixed n, $\{F_{n,n+m}(\zeta, \alpha(\zeta))\}$ is uniformly bounded on compact subsets of R (which overlaps Δ, where this sequence converges), then apply the Stieltjes-Vitali theorem to R.

We first observe that (c) implies the existence of $d > 0$ such that, for all $\zeta \in E(\Delta)$, $[t \in \alpha(E(\Delta))$ and $|t - \alpha(\zeta)| < d] \Rightarrow |D_t f_n(\zeta, t)| < K$. This is due to the continuity of the absolute value of the derivative on $E(\Delta) \times \alpha(E(\Delta))$. Because of the boundary conditions on R and $E(\Delta)$, there exists a number $0 < \delta = \delta(R) < d$ such that $N_\delta(\alpha(\zeta))$ is contained in $\alpha(E(\Delta))$ for each $\zeta \in \Omega$, where Ω is *any* compact subset of R. We shall now show that, for n sufficiently large and all m, $|F_{n,n+m}(\zeta, \alpha) - \alpha| < \delta$ for all $\zeta \in \Omega$. Since $\alpha(\Omega)$ is bounded, this will insure the boundedness of the sequence of functions.

We begin by assuming that n is large enough to guarantee that (c) is satisfied, and $cr^n(1/1 - \varepsilon) < \delta$, where $K = \varepsilon/r$ for $r < \varepsilon < 1$. Let

$\zeta \in \Omega$. We have

$$
\begin{aligned}
&|f_{n+m}(\alpha)-\alpha| < cr^{n+m} < cr^n(1/1-\varepsilon) < \delta,\\
&|f_{n+m-1}(f_{n+m}(\alpha))-\alpha|\\
&\le |f_{n+m-1}(f_{n+m}(\alpha)) - f_{n+m-1}(\alpha)| + |f_{n+m-1}(\alpha)-\alpha|\\
&< K|f_{n+m}(\alpha)-\alpha| + cr^{n+m-1}\\
&\qquad\qquad\text{(apply } |f_j(z)-f_j(\alpha)| \le \int |f_j'(s)|\,|ds| \text{ and (c))}\\
&< Kcr^{n+m} + cr^{n+m-1}\\
&\qquad = cr^{n+m-1}(1+\varepsilon) < cr^n(1/1-\varepsilon) < \delta,
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
&|f_{n+m-2}(f_{n+m-1}\circ f_{n+m}(\alpha))-\alpha| < \cdots < cr^{n+m-2}(1+\varepsilon+\varepsilon^2)\\
&< cr^n(1/1-\varepsilon) < \delta,\\
&\quad\vdots\\
&|f_n\circ\cdots\circ f_{n+m}(\alpha)-\alpha| < cr^n(1+\varepsilon+\cdots+\varepsilon^m)\\
&\le cr^n(1/1-\varepsilon) < \delta.
\end{aligned}
$$

Consequently, for large fixed $n \ge n(R)$, the sequence $\{F_{n,n+m}(\zeta,\alpha(\zeta))\}$ is uniformly bounded for $\zeta \in$ any compact Ω in R. $\text{Lim}_{m\to\infty} F_{n,n+m}(\zeta,\alpha(\zeta)) = \lambda_n(\zeta)$ for all $\zeta \in \text{Int}\,(R\cap\Delta)$. Hence, $\{F_{n,n+m}(\zeta,\alpha(\zeta))\}$ converges to $\lambda_n(\zeta)$ for all $\zeta \in R\cup\Delta$. □

No attempt will be made to determine whether the extended function $\lambda(\zeta)$ actually coincides with the function one expands in some limit periodic scheme. We begin our analysis with a given expansion.

For the first application of Theorem 2, we consider a power series that is quasi-geometric and, thus allows a limit periodic interpretation.

Example 3. Consider $P(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + \cdots$, where $a_{n+1}/a_n = p_n$ with $|p_n - p| < r^n$ for $0 \le r < 1$ and $|p| < 1$. The ratio test gives $1/|p|$ as the radius of convergence of this series. Then

$$f_n(\zeta,z) \equiv 1 + p_n\zeta z \longrightarrow f(\zeta,z) \equiv 1 + p\zeta z.$$

We shall restrict our ζ and z somewhat to conform to the conditions of Theorem 1. Let $\Delta = \{\zeta : |\zeta| \le 1/|p| - 1/K\}$ and $Z = \{z : |z| \le$

$2K/|p|\}$, where $1 < K < 1/r$ (the coefficient 2 is a somewhat arbitrary convenience). It is clear that $F_n(\zeta, z) = 1 + a_1\zeta + \cdots + a_n\zeta^n z \to \lambda(\zeta) = P(\zeta)$ for $\zeta \in \Delta$ (the radius of convergence of the series is $1/|p|$) and $z \in Z$.

In this example $\alpha(\zeta) = 1/1 - p\zeta$, and it follows that $\zeta \in \Delta \Rightarrow |\alpha(\zeta)| \leq K/|p|$. Set $E(\Delta) = \{\zeta : |\zeta| \leq p < K/|p| \text{ and } |\zeta - 1/p| \geq 1/K\}$. These conditions imply $1/1 + K \leq |\alpha(\zeta)| \leq K/|p|$. Condition (a) is satisfied, and condition (b) takes the form $|f_n(\alpha) - f(\alpha)| = |\zeta|\,|\alpha|\,|p_n - p| \leq (K/|p|)^2 r^n$. Condition (c) is $|f_n'(z)| = |p_n\zeta| < K \Leftrightarrow |\zeta| < K/|p_n|$, which is valid for large n. Since the f_n's are simple entire functions, we see that

$$F_n(\zeta, \alpha(\zeta)) = 1 + a_1\zeta + a_2\zeta^2 + \cdots + a_{n-1}\zeta^{n-1} + a_n(1 - p\zeta)^{-1}\zeta^n \longrightarrow \lambda(\zeta) \text{ for } \zeta \in E(\Delta).$$

This verifies a remarkably simple means of continuation beyond the normal radius of convergence to open sets R whose closures lie in $\operatorname{Int}(E(\Delta))$.

The modification process in this context is Aitken's Δ^2-method in disguised form [**5**]. Thus, we are able to show by fixed point analysis the known fact that Aitken's method analytically continues such power series.

The second application of Theorem 2 is to the analytic theory of continued fractions. Only one of many possible continued fractions will be considered—one whose form is in harmony with the geometrical environment of the continuation theorem. We shall make no attempt to optimize the use of Theorem 2, but, rather, will show how easily it may be applied to produce satisfactory results.

Example 4. Consider the *fixed point* limit periodic continued fraction

$$\frac{\alpha_1(1-\alpha_1)}{1} \underset{-}{} \frac{\alpha_2(1-\alpha_2)}{1} \underset{-}{} \cdots,$$

where $\alpha_n(\zeta) \to \alpha(\zeta) = \zeta$. Let $f_n(\zeta, z) = \alpha_n(1-\alpha_n)/(1-z) \to f(\zeta, z) = \zeta(1-\zeta)/(1-z)$. Here, α_n and $1 - \alpha_n$ are the fixed points of the f_n's. Assume that $|\alpha_n(\zeta) - \zeta| \leq (.001)^n$ if $|\zeta| \leq 15$.

As Example 2 suggests, $F_n(\zeta, 0) \to \lambda(\zeta)$ if $|\zeta| < |1 - \zeta|$, $F_n(\zeta, 0) \to \tau(\zeta)$ if $|\zeta| > |1 - \zeta|$, and $\{F_n(\zeta, 0)\}$ may converge or diverge if $\operatorname{Re}\zeta = 1/2$ [**1**].

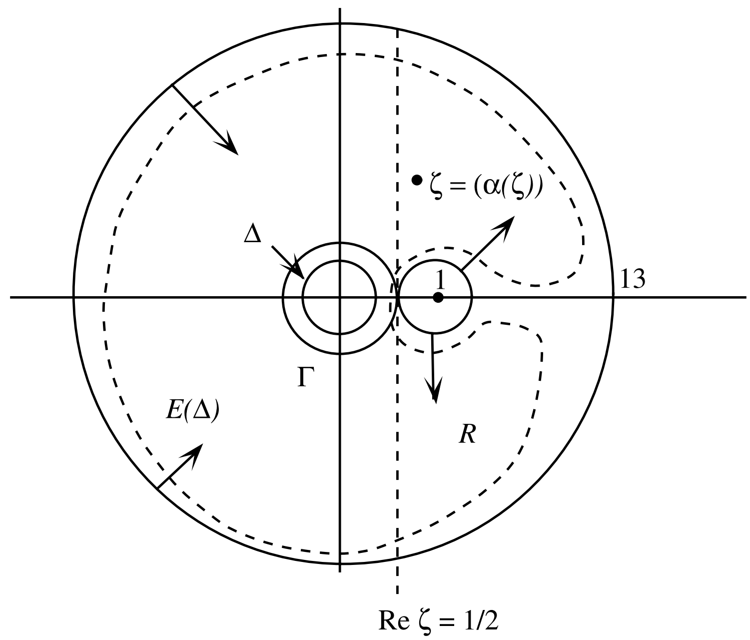

FIGURE 3.

We can effect an analytic continuation of $\lambda(\zeta)$ into a subset of $(|\zeta| > |1-\zeta|)$ by using $\alpha = \zeta$ as a repulsor in the following way: Set $\Delta = \{\zeta : |\zeta| \leq 1/10\}$ and $Z = \{z : |z| \leq 1/2\}$. We shall demonstrate the use of Theorem 1 to verify this nucleus of analyticity of $\{F_n(\zeta, z)\}$, although continued fraction theory is less restrictive.

Let $D_n = \Omega = Z$. It is easily seen that $\operatorname{Int}\Omega \supset f_n(\Delta, \Omega)$ for all n. Also, one quickly finds that $|f(\zeta, z) - \zeta| < |z - \zeta|$ for $\zeta \in \Delta$ and $z \in Z$, i.e., $f_n(\zeta, z) \to f(\zeta, z)$ uniformly on $\Delta \times \Omega$. Therefore, $F_n(\zeta, z) \to \lambda(\zeta)$, analytic on Δ for all $z \in Z$.

For continuation, set $E(\Delta) = \{\zeta : |\zeta - 1| \geq .51 \text{ and } |\zeta| \leq 13\}$. Observe that $\alpha(E(\Delta)) = E(\Delta)$. Let us choose R as shown in Figure 3 with $\operatorname{Inf} d(R, E(\Delta)) > .01$. We see that (a) $\alpha(\zeta) = \zeta$, and (b) $|f_n(\zeta, \alpha(\zeta)) - \zeta| \leq |\alpha_n - \zeta|\ (1 + |\alpha_n| + |\zeta|)/|1-\zeta| \leq 53|\alpha_n - \zeta| \leq 53(.001)^n$. For (c) we proceed directly to the inference stated in the proof of Theorem 2: for $d = .01$, we consider t such that $|t - \alpha(\zeta)| = |t - \zeta| < .01$, $|1-\zeta| > .51$, and $|\alpha_n - \zeta| < .001$. Thus, $|\alpha_n| < |\zeta| + .001 < 13.001$ and

$|1-\alpha_n| < 14.001$. Then

$$|D_t f_n(\zeta,t)| = |\alpha_n(1-\alpha_n)/(1-t)^2| \le |\alpha_n|\,|1-\alpha_n|/(|1-\zeta|-|t-\zeta|)^2$$
$$\le 4(13.001)(14.001) < K \equiv 800 < 1/r = 1,000.$$

Now $n = n(R) = 2$ (from (b) and (1)) so that $F_{2,2+m}(\zeta,\zeta) \to \lambda_2(\zeta)$, analytic on R. Since $f_1(\zeta,\lambda_2(\zeta)) = \alpha_1(1-\alpha_1)/(1-\lambda_2)$, where $|\lambda_2(\zeta)-1| > .5$ for $\zeta \in E(\Delta)$, we conclude that $F_n(\zeta,\zeta) \to \lambda(\zeta)$, analytic on R.

This example demonstrates general consistency with a theorem of Thron and Waadeland [**10**], although their work almost certainly yields more fruitful results in the continued fraction setting. Both perspectives reflect an apparent need for a geometric rate of convergence of coefficients or parameters.

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