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VECTOR-VALUED LOCAL MINIMIZERS OF NONCONVEX VARIATIONAL PROBLEMS

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In recent work with R.V. Kohn [7], a new and general method for obtaining local minimizers of variational problems was established. This technique uses the notion of Γ-convergence of functionals, first introduced by De Giorgi [1] in the 1960's, and yields existence of local minimizers to a Γ-convergent sequence of problems, provided, roughly speaking, that the limit problem possesses a local minimizer which is isolated. In this paper, I apply the method to establish existence of vector-valued local minimizers $u_\varepsilon : \Omega \to \mathbf{R}^2$ of the problem

$$\inf_{u \in H^1(\Omega)} \int_\Omega W(u) + \varepsilon^2 |\nabla u|^2 \, dx, \tag{1}$$

for certain open, bounded sets $\Omega \subset \mathbf{R}^n$ and ε sufficiently small. Here $|\nabla u|^2 = |\nabla u_1|^2 + |\nabla u_2|^2$, $\partial\Omega$ is taken to be Lipschitz-continuous, and W is a nonnegative "double-well" potential vanishing at two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^2$.

In particular, such a minimizer will be a nonconstant solution of the Euler-Lagrange equation (system):

$$2\varepsilon^2 \Delta u = \nabla_u W(u) \quad \text{in } \Omega, \tag{2}$$

with the "natural" Neumann condition

$$\partial_n u = 0 \quad \text{on} \quad \partial\Omega.$$

Variational problems of form (1) arise in the so-called gradient theory of phase transitions [5, 6], as well as in studies of pattern selection [8]. The form of nonconstant local minimizers of (1) was first conjectured in [8].

A full definition of Γ-convergence is given below in (6) and (7), but the essential idea in this setting is to obtain the first term in an asymptotic

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expansion for the energy of a minimizer, u_ε, of (1). In so doing, one characterizes any limit of $\{u_\varepsilon\}$ as being the solution of a new variational problem—the Γ-limit of the original sequence of problems given by (1). One can anticipate the structure of a nonconstant local minimizer of (1)—it should stay close to $\mathbf{a}$ on part of Ω and close to $\mathbf{b}$ on the other part, with a rapid transition layer bridging these two states. Indeed, the construction of a test sequence having this structure leads to energies of order $\mathcal{O}(\varepsilon)$. Hence, we rescale (1) and define

$$F_\varepsilon(u) = \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2 \, dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

At the same time, we define a candidate for the Γ-limit

$$F_0(u) = \begin{cases} 2g(\mathbf{b}) \mathrm{Per}_\Omega \{x : u(x) = \mathbf{a}\} & \text{if } u \in BV(\Omega),\ u(x) \in \{\mathbf{a}, \mathbf{b}\} \text{ a.e.,} \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

Here $BV(\Omega)$ denotes the space of functions of bounded variation in Ω, and $\mathrm{Per}_\Omega \mathbf{A}$ = perimeter of A in Ω = surface area of $\partial A \cap \Omega$ for ∂A "nice" (see [**4**]). The function g is defined through

$$g(u) = \inf_{\substack{\gamma(-1) = \mathbf{a} \\ \gamma(1) = u}} \int_{-1}^{1} \sqrt{W(\gamma(t))} |\gamma'(t)| \, dt \tag{5}$$

among curves γ which are Lipschitz-continuous.

The connection between Γ-convergence and existence of local minimizers is drawn in Theorem 2.1 of [**7**]. We restate it here for convenience. Conditions (6) and (7) below constitute a working definition of Γ-convergence (with respect to the topology $L^1(\Omega)$).

Theorem 1. *Suppose a sequence of functionals* $\{F_\varepsilon\}$ *and a functional* F_0 *satisfy the following conditions*:

(6) *if* $v_\varepsilon \to v_0$ *in* $L^1(\Omega)$ *as* $\varepsilon \to 0$, *then* $\underline{\lim}\, F_\varepsilon(v_\varepsilon) \geq F_0(v_0)$;

(7) *for any* $v_0 \in L^1(\Omega)$ *there is a family* $\{\rho_\varepsilon\}_{\varepsilon > 0}$ *with* $\rho_\varepsilon \to v_0$ *in* $L^1(\Omega)$ *and* $F_\varepsilon(\rho_\varepsilon) \to F_0(v_0)$;

(8) *any family* $\{v_\varepsilon\}_{\varepsilon > 0}$ *such that* $F_\varepsilon(v_\varepsilon) \leq C < \infty$ *for all* $\varepsilon > 0$ *is compact in* $L^1(\Omega)$;

(9) *there exists an isolated* L^1*-local minimizer* u_0 *of* F_0; *that is,* $F_0(u_0) < F_0(v)$ *whenever* $0 < ||u_0 - v||_{L^1(\Omega)} \leq \delta$ *for some* $\delta > 0$.

Then there exists an $\varepsilon_0 > 0$ *and a family* $\{u_\varepsilon\}$ *for* $\varepsilon < \varepsilon_0$ *such that* u_ε *is an* L^1*-local minimizer of* F_ε, *and* $u_\varepsilon \to u_0$ *in* $L^1(\Omega)$.

To establish the conditions of Theorem 1 for the functionals F_ε and F_0 defined by (3) and (4), we take $W : \mathbf{R}^2 \to \mathbf{R}$ to be a C^3, nonnegative function, vanishing only at two points, $\mathbf{a}$ and $\mathbf{b}$. Furthermore, assume

(10) the matrix $\frac{\partial^2 W(u)}{\partial u_i \partial u_j}$ is positive definite at $u = \mathbf{a}, \mathbf{b}$;

(11) there exist positive constants c_1, c_2 and m, and a number $p \geq 2$ such that

$$c_1|u|^p \leq W(u) \leq c_2|u|^p \quad \text{for } |u| \geq m;$$

(12) $V(r,\theta) \stackrel{\text{def}}{=} W(u + r(\cos\theta, \sin\theta)) = r^2 + \mathcal{O}(r^3)$ for r sufficiently small and $u = \mathbf{a}$ or $\mathbf{b}$, where r and θ are local polar coordinates.

The function g defined by (5) plays a crucial role in proving (6)–(8). We summarize its properties in the following lemma, which is proved later.

Lemma. *For every* $u \in \mathbf{R}^2$, *there exists a curve* $\gamma_u : [-1,1] \to \mathbf{R}^2$ *such that* $\gamma(-1) = \mathbf{a}$, $\gamma(1) = u$ *and*

$$g(u) = \int_{-1}^{1} \sqrt{W(\gamma_u(t))}|\gamma_u'(t)|\, dt. \tag{13}$$

The function g *is Lipschitz-continuous and satisfies*

$$|\nabla g(u)| = \sqrt{W(u)} \quad \textit{for a.e. } u. \tag{14}$$

There exists a smooth, increasing function $\beta : (-\infty, \infty) \to (-1, 1)$ *such that the curve* $\zeta(\tau) \stackrel{\text{def}}{=} \gamma_{\mathbf{b}}(\beta(\tau))$ *satisfies*

$$2g(\mathbf{b}) = \int_{-\infty}^{\infty} W(\zeta) + |\zeta'|^2\, d\tau, \tag{15}$$

$\lim_{\tau \to -\infty} \zeta(\tau) = \mathbf{a}$, $\lim_{\tau \to \infty} \zeta(\tau) = \mathbf{b}$, *with these limits being attained at an exponential rate.*

We now prove

Theorem 2. *The sequence* $\{F_\varepsilon\}$ Γ*-converges to* F_0*; that is, conditions* (6) *and* (7) *hold.*

Proof of (6): One need only consider v_0 of the form

$$v_0(x) = \begin{cases} \mathbf{a} & \text{if } x \in A, \\ \mathbf{b} & \text{if } x \in B, \end{cases} \tag{16}$$

for two disjoint sets A and B with $A \cup B = \Omega$. Otherwise, $v_\varepsilon \to v_0$ in $L^1(\Omega)$ implies $F_\varepsilon(v_\varepsilon) \to \infty$.

Now suppose $v_\varepsilon \to v_0$ in $L^1(\Omega)$, and define $h_\varepsilon(x) = g(v_\varepsilon(x))$. It follows from the lemma that

$$|\nabla h_\varepsilon(x)| \leq \sqrt{W(v_\varepsilon(x))}|\nabla v_\varepsilon(x)|. \tag{17}$$

Furthermore,

$$h_\varepsilon \overset{L^1(\Omega)}{\to} g(v_0) = \begin{cases} 0 & x \in A \\ g(\mathbf{b}) & x \in B. \end{cases}$$

Hence, from the Schwartz inequality and the lower-semicontinuity of the BV-norm under L^1-convergence ([**4**]), we have

$$\begin{aligned} \underline{\lim}\, F_\varepsilon(v_\varepsilon) \geq \underline{\lim}\, 2\int_\Omega \sqrt{W(v_\varepsilon)}|\nabla v_\varepsilon|\, dx &\geq \underline{\lim}\, 2\int_\Omega |\nabla h_\varepsilon|\, dx \\ &\geq 2\int_\Omega |\nabla g(v_0)| \\ &= 2g(\mathbf{b})\ \mathrm{Per}_\Omega\{x : v_0(x) = \mathbf{a}\} \\ &= F_0(v_0). \end{aligned}$$

Proof of (7): One may assume $v_0 \in BV(\Omega)$ and again take v_0 of the form (16), for otherwise the trivial construction $\rho_\varepsilon \equiv v_0$ for each ε will suffice. Furthermore, we take $\Gamma \overset{\text{def}}{=} \partial A \cap \partial B$ to be smooth, since one can always approximate a set of finite perimeter by a sequence of sets having smooth boundary ([**4, 9**]).

Now define the distance function $d : \Omega \to \mathbf{R}$ by

$$d(x) = \begin{cases} -\mathrm{dist}\,(x, \Gamma) & \text{if } x \in A, \\ \mathrm{dist}\,(x, \Gamma) & \text{if } x \in B, \end{cases}$$

where "dist" refers to Euclidean distance. Near Γ, d will be smooth and satisfy

$$(18)\qquad |\nabla d(x)| = 1,\ \lim_{s\to 0} \mathcal{H}^{n-1}\{x : d(x) = s\} = \mathcal{H}^{n-1}(\Gamma) = \mathrm{Per}_\Omega \mathbf{A},$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorf measure.

Then define $\{\rho_\varepsilon\}$ through

$$\rho_\varepsilon(x) = \begin{cases} \zeta\left(\frac{-1}{\sqrt{\varepsilon}}\right) & \text{if } d(x) < -\sqrt{\varepsilon}, \\ \zeta\left(\frac{d(x)}{\varepsilon}\right) & \text{if } |d(x)| \le \sqrt{\varepsilon}, \\ \zeta\left(\frac{1}{\sqrt{\varepsilon}}\right) & \text{if } d(x) > \sqrt{\varepsilon}. \end{cases}$$

The L^1-convergence of ρ_ε to v_0 is an immediate consequence of (15). Using (15), (18) and the co-area formula ([**2**]), we calculate

$$\begin{aligned} \overline{\lim}\, F_\varepsilon(\rho_\varepsilon) &= \overline{\lim}\, \frac{1}{\varepsilon} \int_{\{|d(x)|<\sqrt{\varepsilon}\}} W\left(\zeta\left(\frac{d(x)}{\varepsilon}\right)\right) + \left|\zeta'\left(\frac{d(x)}{\varepsilon}\right)\right|^2 dx \\ &= \overline{\lim}\, \frac{1}{\varepsilon} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left[W\left(\zeta\left(\frac{s}{\varepsilon}\right)\right) + \left|\zeta'\left(\frac{s}{\varepsilon}\right)\right|^2\right] \mathcal{H}^{n-1}\{x : d(x) = s\} ds \\ &= \overline{\lim} \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} [W(\zeta(\tau)) + |\zeta'(\tau)|^2] \mathcal{H}^{n-1}\{x : d(x) = \varepsilon\tau\}\, d\tau \\ &\le 2g(\mathbf{b}) \left(\overline{\lim} \max_{|s|\le\sqrt{\varepsilon}} \mathcal{H}^{n-1}\{x : d(x) = s\}\right) = F_0(v_0). \end{aligned}$$

Combining this result with (6) yields (7). □

We now establish the compactness required in Theorem 1.

Proposition. *For W satisfying hypotheses* (10)–(12), *condition* (8) *holds.*

Proof. Suppose $F_\varepsilon(v_\varepsilon) \le C$ for some family $\{v_\varepsilon\}_{\varepsilon>0}$. Let $h_\varepsilon(x) = g(v_\varepsilon(x))$. Then, as in the proof of (6),

$$\int_\Omega |\nabla h_\varepsilon(x)|\, dx < C.$$

Furthermore, from the growth hypothesis (11), it follows that the v_ε are uniformly bounded in L^p, while the h_ε are uniformly bounded in L^1. Hence, we have

$$||h_\varepsilon||_{BV(\Omega)} = \int_\Omega |h_\varepsilon| dx + \int_\Omega |\nabla h_\varepsilon| < C.$$

Since bounded sequences in BV are compact in L^1 ([**4**]), one can extract a subsequence h_ε, converging in L^1 to a limit h_o taking the form

$$h_o(x) = \begin{cases} 0 & x \in A\ , \\ g(\mathbf{b}) & x \in B\ , \end{cases}$$

for sets A and B with $A \cup B = \Omega$. The continuity of g and the fact that $g(u) = 0$ only for $u = \mathbf{a}$ leads to the convergence in measure of $\{u_{\varepsilon j}\}$ (or a subsequence) to $\mathbf{a}$ on A. Then the uniform L^p bound gives L^1 convergence on this set. Switching $\mathbf{b}$ for $\mathbf{a}$ in the definition of g (5) and repeating the argument above leads to L^1-convergence of $\{u_{\varepsilon j}\}$ to $\mathbf{b}$ on a set $B' \subset B$. It then easily follows that, in fact, $B' = B$. □

In order to apply Theorem 1, it remains to establish the existence of an L^1-local minimizer u_o of F_o (condition (9)). This cannot always be done, but for certain nonconvex domains Ω, such a u_o will exist. The partition of Ω associated with a function u_o satisfying (9) has the property that any modification by a set of small measure inevitably increases the area of the interface $\partial\{u_o = \mathbf{a}\} \cap \partial\{u_o = \mathbf{b}\}$. Roughly speaking, a class of domains for which such a partition exists includes those sets Ω which possess a "neck." For example, in [**7**], it is verified that the two-dimensional region pictured in Figure 1 possesses such an L^1-local minimizer. Hence, for such an Ω, one can apply Theorem 1 to obtain nonconstant local minimmizers of (1) for all ε sufficiently small.

We conclude with a proof of the lemma. Since much of the proof involves only a slight modification of the argument found in Section 2, part B of [**9**], we shall only sketch the main ideas.

Proof of (13): One would like to apply a direct method to obtain a minimizing curve γ_u for (3), but a lack of compactness causes difficulty. Instead, first perturb the integrand by $\delta > 0$ and find a minimizer γ^δ

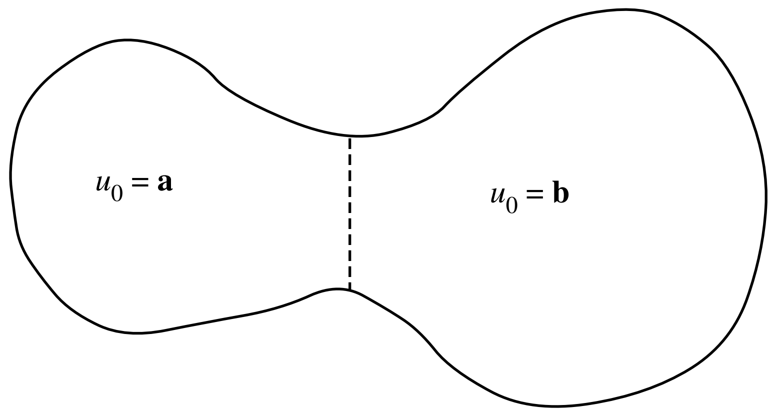

Figure 1. A domain Ω for which F_o possesses an isolated local minimizer u_o.

of the problem

$$\inf_{\substack{\gamma(-1)=\mathbf{a}\\ \gamma(1)=u}} \int_{-1}^{1} (\sqrt{W(\gamma(t))}+\delta)|\gamma'(t)|\,dt \stackrel{\text{def}}{=} g^\delta(u).$$

Such a curve will exist since $g^\delta(u)$ corresponds to distance in a Riemannian metric which is conformal to the Euclidean metric on the plane—a geodesic which minimizes the distance between two points will always exist. If one can establish the compactness of the sequence $\{\gamma^\delta\}$, then the desired curve γ_u is obtained as a subsequential limit; that is, it can be shown that

$$g(u) = \lim_{\delta_j \to 0} g^{\delta_j(u)} = \int_{-1}^{1} \sqrt{W(\gamma(t))}|\gamma_u'(t)|\,dt.$$

Compactness is achieved through an appeal to the Arzela-Ascoli Theorem once a uniform bound on the Euclidean arclength of γ^δ is established. Controlling the length of γ^δ away from $\mathbf{a}$ or $\mathbf{b}$ is easy since there $\sqrt{W(\gamma^\delta)}$ will be bounded away from zero. To bound uniformly the arclength near $\mathbf{a}$, we parametrize γ^δ by arclength and describe γ^δ near $\mathbf{a}$ in local polar coordinates as $(R^\delta(s), \Theta^\delta(s))$. Then introduce

$\lambda^\delta(s)$ defined through

$$\lambda^\delta(s) = R^\delta(s)\frac{\dot{\Theta}^\delta(s)}{\dot{R}^\delta(s)} \cdot \qquad \left(\cdot = \frac{d}{ds}\right).$$

A bound on the arclength of γ^δ near $\mathbf{a}$ will follow from a bound on λ^δ, since arclength is given by

$$ds^2 = dr^2 + r^2 d\theta^2.$$

One can derive a pair of differential inequalities satisfied by λ^δ using the Euler-Lagrange equation for γ^δ. They take the form

$$(19) \quad \dot{\lambda}^\delta \leq \frac{-a_1\dot{R}^\delta}{a_2R^\delta+\delta}\lambda^\delta + a_3 \quad \text{for } \lambda^\delta \text{ restricted to } \{s : 0 \leq \lambda^\delta(s') \leq 1 \text{ for all } s' \in [0,s]\},$$

$$(20) \quad \dot{\lambda}^\delta \geq \frac{-a_1\dot{R}^\delta}{a_2R^\delta+\delta}\lambda^\delta - a_3 \quad \text{for } \lambda^\delta \text{ restricted to } \{s : -1 \leq \lambda^\delta(s') \leq 0 \text{ for all } s' \in [0,s]\},$$

where a_1, a_2 and a_3 are positive constants depending on W, but not δ. It is here that (12) is used. This hypothesis is unessential but greatly simplifies the analysis. Integrating (19) and (20) yields a uniform bound on λ^δ, and so on the arclength of γ^δ near $\mathbf{a}$. (A similar argument also works near $\mathbf{b}$ if needed.) Compactness follows.

Proof of (14): See Lemma 11 of [**9**] for a proof in a similar setting.

Proof of (15): Define β as the solution of

$$\beta' = \frac{\sqrt{W(\gamma_{\mathbf{b}}(\beta))}}{|\gamma'_{\mathbf{b}}(\beta)|}, \qquad \beta(0) = 0.$$

Then $\zeta(\tau) \stackrel{\text{def}}{=} \gamma_{\mathbf{b}}(\beta(\tau))$ satisfies $|\zeta'| = \sqrt{W(\zeta)}$. Since the value of the integral in (3) is invariant under reparametrization, we find

$$2g(\mathbf{b}) = 2\int_{-\infty}^{\infty} \sqrt{W(\zeta(\tau))}|\zeta'(\tau)|\,d\tau = \int_{-\infty}^{\infty} W(\zeta(\tau)) + |\zeta'(\tau)|^2\,d\tau.$$

It follows that ζ also solves the problem

$$\inf_{\substack{\gamma(-\infty)=\mathbf{a}\\ \gamma(\infty)=\mathbf{b}}} \int_{-\infty}^{\infty} W(\gamma(t)) + |\gamma'(t)|^2\, dt,$$

and hence satisfies the equation $2\zeta'' = \nabla W(\zeta)$. Differentiating this equation and using (10), one sees that $|\zeta'|$ must decay to zero at an exponential rate as $\tau \to \pm\infty$. This proves the second half of (15). □

Remarks (1). One could no doubt generalize the lemma to include $W : \mathbf{R}^m \to \mathbf{R}$ for $m > 2$ using, for example, spherical coordinates to prove (13) when $m = 3$. This would immediately lead to the existence of local minimizers u^ε in $\mathbf{R}^m$, $m > 2$, since the proofs of (6)–(9) follow from the lemma.

(2) After completing this work, the author learned of the recent work of Fonseca and Tartar [**3**] who establish Γ-convergence without requiring the existence of a geodesic.

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