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STUDYING SINGULAR SOLUTIONS OF A SEMILINEAR HEAT EQUATION BY A DILATION RESCALING NUMERICAL METHOD

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ABSTRACT. A method of dynamic rescaling of variables is used to investigate numerically the nature of the point singularities of the cubic and quadratic nonlinear heat equations in one and two dimensions, with evenness and radial symmetry, respectively. This has allowed solutions to be computed until the amplitude of the spike has grown by a factor of 10^9 or more.

This high numerical resolution is used first to corroborate the occurrence of singularities of the predicted form for several choices of initial data and then to test a conjecture of Galaktionov and Posashkov [**4**] concerning the spatial scale and shape of the solution near the singularity.

1. Background. The equation

$$\phi_t=\Delta\phi+|\phi|^{p-1}\phi, \quad \phi:\mathbf{R}^+\times\Omega\to\mathbf{R}, \quad \Omega\subseteq\mathbf{R}^d, \quad p>1 \quad \phi(0,x)=\phi_0(x) \tag{1.1}$$

is known to develop point singularities in finite time (Fujita [**2,3**]). It also has the trivial singular solutions

$$\phi(t,x)=\kappa(t^*-t)^{-\beta}, \qquad \beta=1/(p-1), \qquad \kappa=\beta^\beta. \tag{1.2}$$

It is conjectured that the growth rate of all singular solutions is the same as for the ODE solutions (1.2). The "ODE growth rate" can be shown to be a lower bound using local in time existence theory. It has been shown to be an upper bound also under numerous combinations of extra hypotheses by Weissler [**10**], Friedman and McLeod [**1**], and Giga and Kohn [**5,6**].

The latter result is related to a rescaling to $u(\tau,\xi)$

$$u=\lambda^{-2\beta}(t)u, \qquad \xi=x/\lambda(t), \qquad d\tau/dt=\lambda^{-2}(t) \tag{1.3a}$$

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$$\lambda(t) = (t^* - t)^{1/2}. \tag{1.3b}$$

This gives

$$u_\tau = \Delta u + |u|^{p-1} u - \beta u - \frac{1}{2}\xi \cdot \nabla u, \tag{1.4}$$

with a time-dependent boundary condition unless $\Omega = \mathbf{R}^d$. I will restrict attention to the latter case from now on.

The only nontrivial, τ-independent solutions are the obvious ones $u(\tau, \xi) = \pm\kappa$, which correspond to the solutions in (1.2). Further, all bounded global solutions converge to one of these or to 0, uniformly on bounded ξ intervals [**6**].

However, this result completely loses track of the behavior for large x and gives no indication of the decay as $\xi \to \infty$ of u for large but finite τ. To study this, another scaling can be used in which the spatial scale is not enlarged so quickly:

$$u = \lambda^{-2\beta}(t) u, \qquad \xi = x/\mu(t), \qquad d\tau/dt = \lambda^{-2}(t) \tag{1.5a}$$

$$1 >> \mu(t) >> \lambda(t) \quad \text{as } t \to t^*. \tag{1.5b}$$

This gives
(1.6)

$$\begin{gathered} u_\tau = \delta(\tau)\Delta u + |u|^{p-1} u - \beta a(\tau) u - b(\tau)\xi \cdot \nabla u, \\ a(\tau) := d(\ln(1/\lambda))/d\tau, \quad b(\tau) := d(\ln(1/\mu))/d\tau, \quad \delta(\tau) := (\lambda/\mu)^2, \\ \to 0 \text{ as } \tau \to \infty \end{gathered}$$

The explicit time dependence of λ and μ has not been specified as it will instead be computed dynamically in the numerical scheme below. For λ, this allows checking of the rate asserted above, without knowing t^* in advance. The choice of μ will be made in a way that keeps u in the middle ground between flattening out (as for $\mu = \lambda$) and forming a narrow spike (as for $\mu = 1$). For example, assuming that ϕ decays monotonically in x, we could require that $u(\tau, 1) = u(\tau, 0)/2$. It is necessary to leave μ unspecified as we have no theorems yet on the form that will give this behavior.

There is, however, a conjecture of Galaktionov and Posashkov [**4**] that the desired form in one dimension is

$$\mu(t) = \lambda(t)(\ln(1/(t^* - t)))^\eta. \tag{1.7}$$

If this is so (and for higher dimensions also), discarding the u_τ and $\delta \cdot \Delta u$ terms in the limit $\tau \to \infty$ suggests that

$$u(\tau, \xi) \to f(\xi) = \kappa(1 + |\xi|^2/c)^{-\beta} \tag{1.8}$$

for κ as above and some positive c, and

$$\delta(\tau) \approx -\tilde{c}\tau^{-2\eta} \tag{1.9}$$

for some positive $\tilde{c}$. Galaktionov and Posashkov further argue that

$$2\eta = 1 \tag{1.10a}$$
$$c = 4p/(p-1)^2. \tag{1.10b}$$

As the numerically computed dilation factors will at best be asymptotic to constant multiples of the conjectured forms, the conjectured form for λ and the limiting value κ of $u(\tau, 0)$ are most conveniently tested by the equivalent statements:

$$a(\tau) := d(\ln(1/\lambda))/d\tau \to \text{ a positive constant} \tag{1.11}$$
$$\rho(\tau) := u(\tau, 0))^{p-1}/a(\tau) \to 1. \tag{1.12}$$

2. The numerical method. Numerical study of the singular solutions described above is not straightforward. The solution forms a spike near which space and time derivatives grow without bound as the singularity time is approached. Correspondingly, the fineness of grid point spacing and smallness of time step size required to maintain numerical accuracy grows without bound, so fixed grids are inadequate.

Periodically refining the spatial grid near the singularity and reducing the time step on the finer grid appropriately has been more successful (R. Kohn and M. Berger, personal communication), but I propose here a method that uses a modified version of the continuous rescalings of (1.3) and (1.5) so that one computes on a fixed grid in the variables u, τ and ξ. It is based on a method introduced for study of the nonlinear Schrödinger equation in LeMesurier [**7**] and in [**8,9**]. It has so far been applied to radially symmetric solutions in two dimensions and even solutions in one dimension, fixing the singularity at the origin.

As mentioned before, the form (1.3b) cannot be used numerically without knowing t^* in advance. Instead one could determine λ dynamically during the computation by requiring $u(\tau, 0) = k$, for example. Setting $u_\tau(\tau, 0) = 0$ in a variant of (1.6) with $\mu = \lambda$ would give an equation determining $a(\tau) = (\ln(1/\lambda))_\tau$ in terms of the current value of $u(\tau, \cdot)$, which can be integrated to determine λ. However, this method is subject to numerical instabilities due to its dependence on data at the single point $\xi = 0$. This is overcome by instead holding constant some ξ-integral functional of $u(\tau, \cdot)$, giving a in terms of a ξ-integral. Specifically, I have used the condition

$$G(u(\tau, \cdot)) := \int |\nabla u|^2 \, d\xi = G_0, \text{ a prescribed constant.} \tag{2.1}$$

This choice is determined more by computational convenience than theoretical considerations and is justified simply by the fact that for the cases studied it gives $u(\tau, 0) \to$ a constant, as desired.

This method gives satisfactory but predictable confirmation of the growth rate and limiting form of u via checking of (1.11) and (1.12). Of more interest is a modification following equations (1.5) and (1.6). For this, two side conditions must be imposed to determine λ and μ, or rather their logarithmic τ derivatives a and b. Guided by the expected form $f(\xi)$ in equation (1.8), and issues of computational expediency, the conditions used are almost but not quite (2.1) and

$$M(u(\tau, \cdot)) := \int u^2 \, d\xi = M_0, \text{ a prescribed constant.} \tag{2.2}$$

These norms are finite for f so long as

$$(p-1)d < 4 \tag{2.3}$$

and this indeed gives the observed limit on p for the cases $d = 1, 2$ studied. Higher p values could no doubt be used by using norms based on integrals of higher powers of u and its gradient.

The reason why (2.1) and (2.2) are not exactly the conditions used is that the presence of δ in the resulting expressions for the logarithmic derivatives of λ and μ would give a coupled pair of nonlinear ODEs to solve. Instead, the approximation of discarding the $\delta \cdot \Delta u$ term from these equations gives one an invertible pair of linear algebraic equations for the logarithmic derivatives. These should give $G(u(\tau,\cdot)) \to G_0$, $M(u(\tau,\cdot)) \to M_0$ due to $\delta \to 0$ so long as (2.3) holds. This is confirmed by the numerical results.

3. Numerical results. Three principal cases were studied: $p = 3$ and $p = 2$ in one space dimension with even data, and $p = 2$ in two space dimensions with radially symmetric data. In addition, several runs were performed with various values of p near 5 in one dimension and near 3 in two dimensions, which confirm that the useful range of the current method is as given in (2.3).

In each case, solution with the dynamic similarity rescaling fashioned after (1.3), (1.4) gave the expected results: u converged to a constant and (1.11) and (1.12) held, confirming the growth rate of (1.3b). More specifically, $1 < \rho(\tau) < 1.01$ at the end of the various computational runs and it was very slowly decreasing towards 1 in each case.

The more interesting results are derived with the two parameter rescaling modeled on (1.5), (1.6). These again confirm the growth rate by having u converge to a constant, bounded profile with (1.11) and (1.12) holding. The final values of $\rho(\tau)$ are in the range 1.004–1.007 and slowly decreasing. The convergence of u is shown in Figures 1–3.

The profiles at the end of each run were then tested against the form f of the conjecture in (1.8). As the numerical rescalings can at best give the expected form up to a horizontal and a vertical dilation, the parameters κ and c in the expression for f were chosen in order to have it meet the u profile at the origin and one other ξ value. The resulting fits are excellent as shown by Figures 4 and 5.

The most interesting question, and the one requiring computations that reach very close to the singularity, is the horizontal scaling rate μ. The conjecture of (1.9) and (1.10) was tested for the $p = 3$, $d = 1$ case by plotting $\delta(\tau) \cdot \tau$ against τ in Figure 6. This indicates approximate convergence to a constant as the conjecture predicts, but to more fairly consider other possible values for η, $\log(\delta(\tau))$ was plotted against $\log(\tau)$ for all three cases in Figures 7–9. The conjecture now implies convergence to a straight line with slope -2η (-1 according to (1.10)).

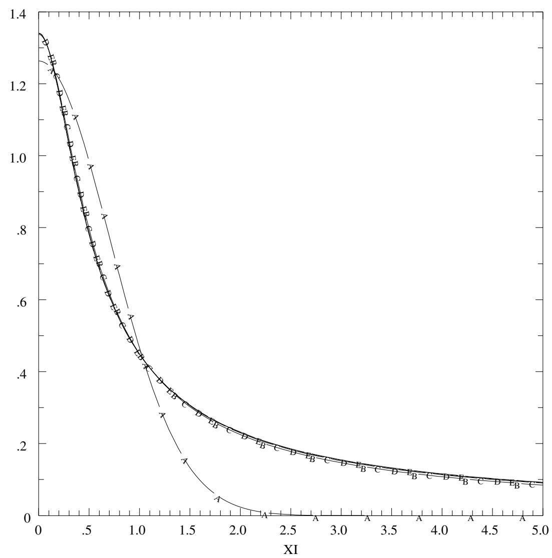

FIGURE 1a. $u(\tau, \xi)$ vs. ξ for $d = 1$, $p = 3$ at $\tau = 0, 5, 10, 15, 20$.

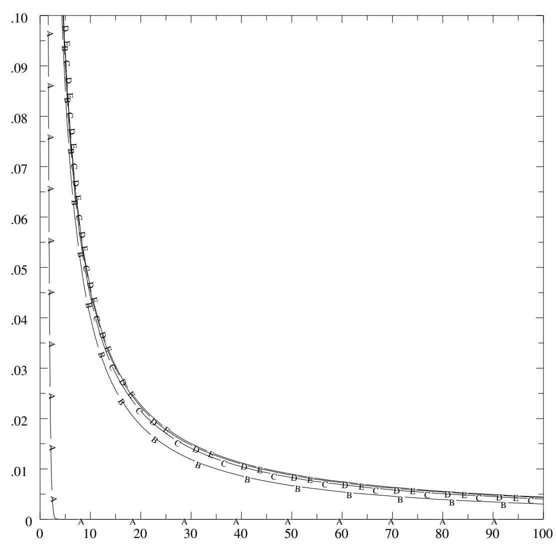

FIGURE 1b. The same, further from the origin.

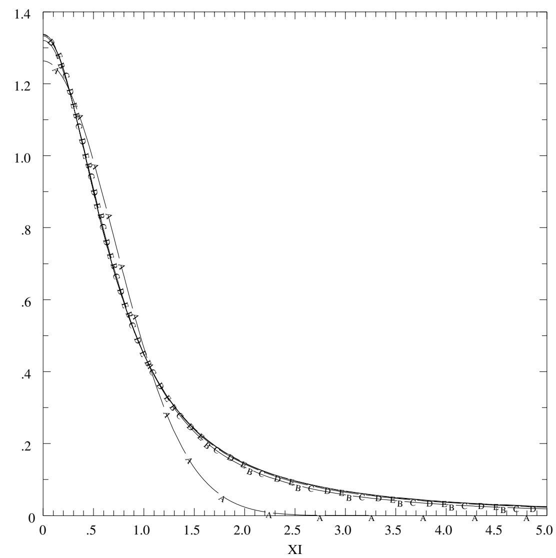

FIGURE 2a. $u(\tau, \xi)$ vs. ξ for $d = 1$, $p = 2$ at $\tau = 0, 10, 20, 30, 40, 50$.

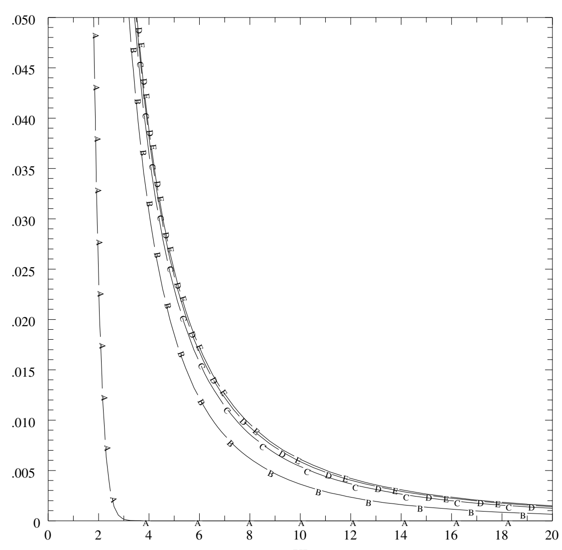

FIGURE 2b. The same, further from the origin.

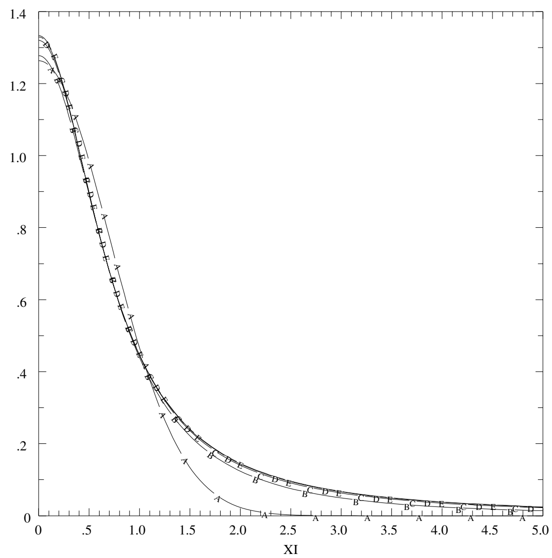

FIGURE 3a. $u(\tau,\xi)$ vs. ξ for $d=2$, $p=2$ at $\tau=0,10,20,30,40$.

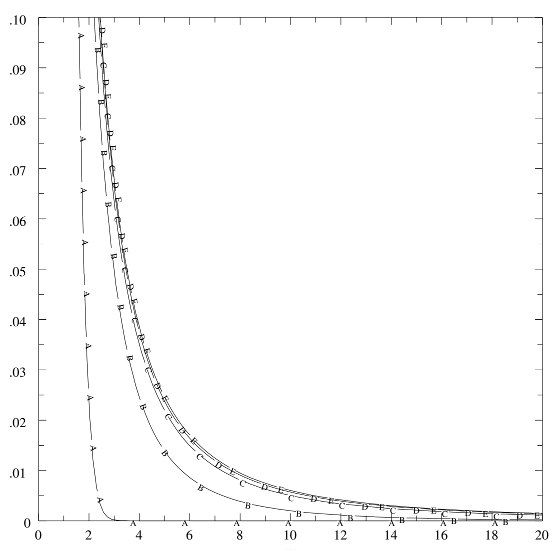

FIGURE 3b. The same, further from the origin.

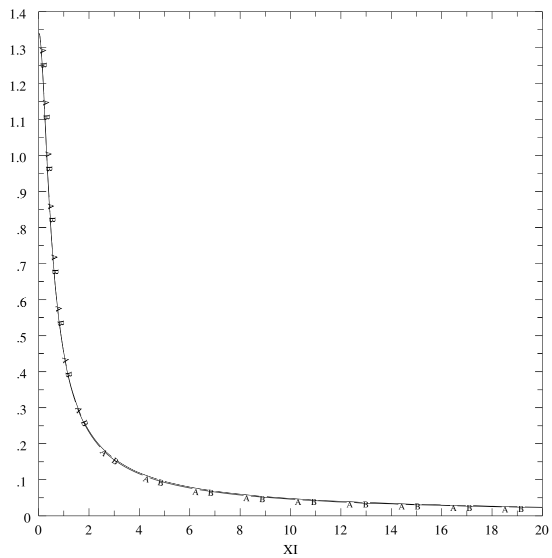

FIGURE 4. Final data $u(20, \xi)$ for $d = 1$, $p = 3$ compared to the conjectured profile.

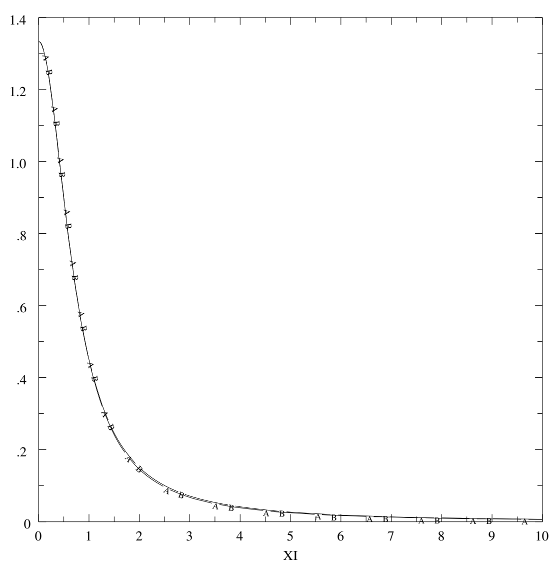

FIGURE 5. Final data $u(40, \xi)$ for $d = 2$, $p = 2$ compared to the conjectured profile.

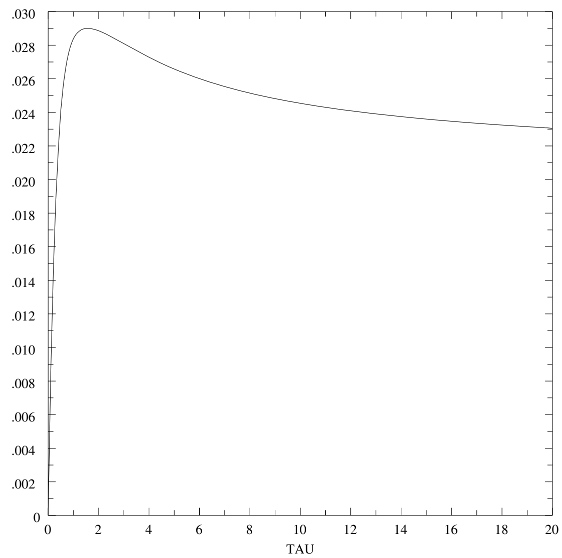

FIGURE 6. $\delta(\tau) \cdot \tau$ vs. τ for $d = 1$, $p = 3$.

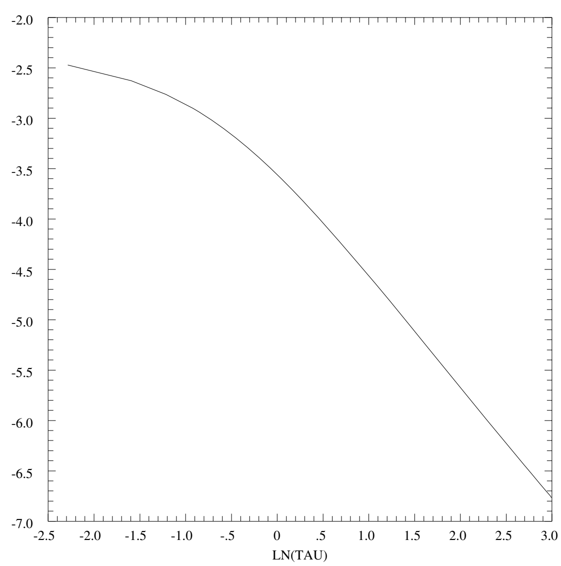

FIGURE 7. Log $\delta(\tau)$ vs. $\log \tau$ for $d = 1$, $p = 3$.

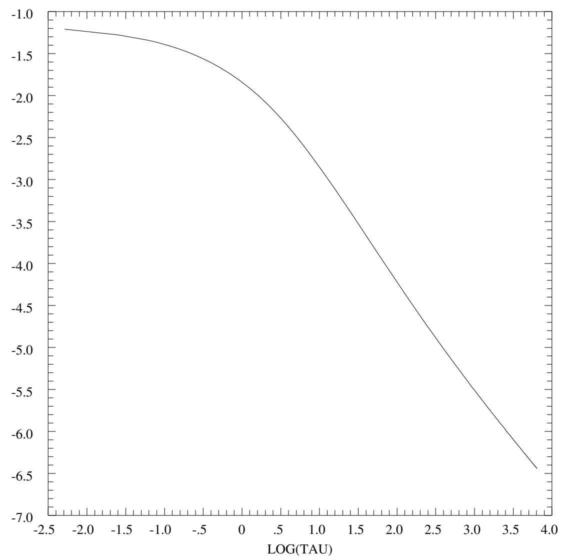

FIGURE 8. Log $\delta(\tau)$ vs. $\log \tau$ for $d = 1$, $p = 2$.

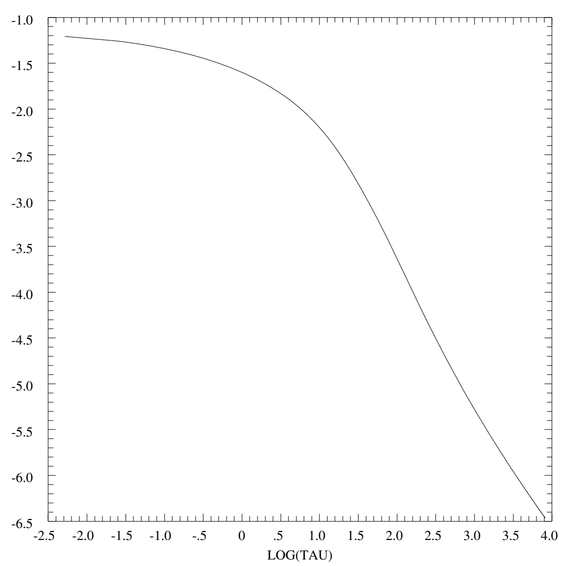

FIGURE 9. Log $\delta(\tau)$ vs. $\log \tau$ for $d = 2$, $p = 2$.

In each one-dimensional case the result is qualitatively confirmed, while in two dimensions the logarithmic form with a different value for η is indicated. The final "2η" values from the graphs are:

$$\text{for each } p \text{ value in one dimension}, 1.1 \text{ and slowly decreasing}$$

$$\text{for } p = 2 \text{ in two dimensions}, 1.4 \text{ with no clear trend up or down.}$$

The convergence to the predicted value here is not nearly as good as for ρ above, as one expects from an attempt to observe a slow logarithmic correction to a power law. Further computations to larger τ values will soon be done to further study this question. Also [**4**] should be extended to radial solutions in several dimensions for checking against the numerical results that are now available. In particular, it is hoped that a different η value will arise.

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