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SPHERICALLY SYMMETRIC SOLUTIONS OF AN ELLIPTIC-PARABOLIC NEUMANN PROBLEM

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1. Introduction. In [**4**] we gave an existence theorem for bounded weak solutions of the following problem:

$$(1.1) \qquad (c(u))_t = \Delta u \quad \text{in } Q_T = \Omega \times (0,T]$$

$$(1.2) \qquad (\mathbf{N}) \quad \frac{\partial u}{\partial \nu} = f \geq 0 \quad \text{on } \partial\Omega \times (0,T]$$

$$(1.3) \qquad c(u(x,0)) = v_0(x), \qquad x \in \Omega.$$

Here Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $u = u(x,t)$ is the unknown function to be found and f and v_0 are given boundary values. Note that we prescribe the outward normal derivative $\partial u/\partial \nu$ at the lateral boundary of Q_T.

The function $c : \mathbf{R} \to \mathbf{R}$ is also given and it is assumed to be increasing on $\mathbf{R}^-$ and identically equal to one on $\mathbf{R}^+$, see Figure 1. Leaving smoothness assumptions aside for the moment, we recall that (1.1) reduces to

$$(1.4) \qquad \Delta u = 0 \text{ (elliptic)} \quad \text{for } u > 0$$

whereas

$$(1.5) \qquad u_t = \left\{ \frac{1}{c'(u)} \right\} \Delta u \text{ (parabolic)} \quad \text{for } u < 0.$$

The physical background of (1.1) lies in the theory of partially saturated flows in porous media. In that context u stands for the hydrostatic potential due to capillary suction and $c(u)$ for the moisture content or saturation. The part of Ω where u is negative is the unsaturated region, and that where u is positive the saturated region. The set where $u = 0$ is usually referred to as the interface or free

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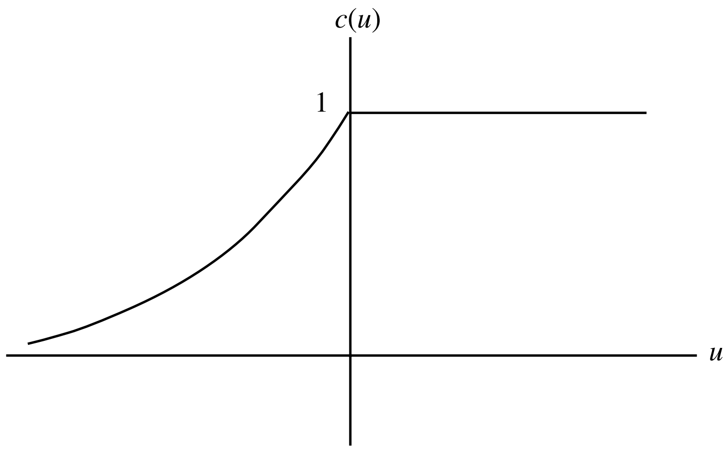

FIGURE 1. The function c.

boundary. In one space dimension there is now an extensive literature on this free boundary, see, e.g., [**1,3,7,8**], but in more dimensions no results have been obtained so far.

The purpose of this paper is, after extending some of the results in one space dimension to the spherically symmetric case, to take a closer look at what happens if the medium saturates completely in a finite time T_s. We recall that DiBenedetto and Gariepy showed that for bounded weak solutions u of (1.1) the saturation $c(u)$ is continuous [**2**] and that we know from [**4**] that Problem (N) has a bounded solution if $T < T_s$. We shall see, however, that for $n \geq 2$ solutions u generally blow up near T_s and hence may be discontinuous. *Our main result is a discontinuity criterion for $c(u)$ as t tends to T_s if $n \geq 3$. In addition, we describe the behavior of the interface.*

The saturation time T_s depends explicitly on the boundary conditions v_0 and f. To see this, observe that integrating (1.1) and using (1.2) and (1.3) yields a conservation law in terms of the initial concentration v_0 and the flux f at the boundary:

$$\int_\Omega c(u(x,t))\,dx = \int_\Omega v_0(x)\,dx + \int_0^t \int_{\delta\Omega} f(x,s)\,dx\,ds. \tag{1.6}$$

Since $c(u) \le c(0) = 1$, a necessary compatibility condition for v_0 and f is that the right-hand side of (1.6) does not exceed the value $|\Omega| = \int_\Omega dx$. T_s is precisely the first time t for which the right-hand side of (1.6) equals $|\Omega|$, if such a time exists.

To state our results, consider Problem (N) on the unit ball B in $\mathbf{R}^n$, with spherically symmetric initial data, i.e., $v_0 = v_0(r)$, where $r = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$, and, for simplicity, $f \equiv 1$. Thus we arrive at:

$$
\begin{aligned}
&(1.7) && (c(u))_t = r^{1-n}(r^{n-1}u_r)_r, && 0 < r < 1,\ 0 < t \le T\\
&(1.8)\quad (\mathbf{NS}) && u_r(1,t) = 1, && 0 < t \le T\\
&(1.9) && c(u(r,0)) = v_0(r), && 0 \le r \le 1.
\end{aligned}
$$

Note that we always have that $u_r(0,t) = 0$ since we are working with spherically symmetric solutions. The boundary condition at $r = 1$ states that there is a constant fluid flux at the boundary into the medium, and (1.6) can be rewritten as

$$\int_0^1 r^{n-1} c(u(r,t))\, dr = \int_0^1 r^{n-1} v_0(r)\, dr + t. \tag{1.10}$$

Because of the Jacobian r^{n-1}, T_s now is the first time for which the right-hand side of (1.10) equals $1/n$. We introduce four hypotheses:

H1. $c : \mathbf{R} \to \mathbf{R}$ is uniformly Lipschitz continuous, $c \equiv 1$ on $\mathbf{R}^+$, $c \in C^{1,\beta}((-\infty, 0])$ with $0 < \beta \le 1$, and c' is positive on $(-\infty, 0]$;

H2. There exists a bounded function $u_0 \in W^{1,\infty}{}_{\mathrm{loc}}(0,1)$ with $|r^{n-1}u_0'(r)|$ bounded by a constant L, such that $v_0 = c(u_0)$ is not identically equal to one.

H1*. c satisfies H1 and in addition c is concave.

H2*. v_0 satisfies H2 and in addition $u_0 \in C^{2,\beta}([0,1])$ with $0 < \beta \le 1$, $u_0'(0) = 0$, $u_0'(1) = 1$, $u_0' \ge 0$ on $[0,1]$, and

$$r^{1-n}(r^{n-1}u_0'(r))' \ge -Kc'(u_0(r)), \qquad 0 < r \le 1 \tag{1.11}$$

for some constant $K \ge 0$ whenever $u_0(r) \ne 0$.

Theorem 1. *Let* H1–2 *be satisfied.*

(i) *For every* $0 < T < T_s$ *Problem* NS *has a unique weak solution in the sense of* [**4**] (*see the next section*) *which satisfies* (1.7). *In addition,* $r^{n-1}u_0'(r) \in L^\infty((0,1) \times (0,T_s))$ *and* $r^{n-1}c(u(r,t))$ *is uniformly continuous on* $[0,1] \times [0,T_s)$.

(ii) *There exists a continuous function* $\zeta : [0,T_s] \to [0,1]$, *called the interface, with* $\zeta(T_s) = 0$, *such that for* $0 < t < T_s$,

$$u(r,t) < 0, \qquad 0 \le r < \zeta(t) \tag{1.12}$$

$$u(r,t) > 0 \text{ and } r^{n-1}u_r(r,t) = 1, \qquad \zeta(t) < r \le 1 \tag{1.13}$$

$$u(\zeta(t),t) = 0, \qquad \text{if } \zeta(t) < 1 \tag{1.14}$$

and with

$$\zeta(0) = \sup\{0 \le r \le 1 : v_0(r) < 1\}. \tag{1.15}$$

Remarks 1. The uniform parabolicity of (1.1) in the unsaturated region near $u = 0$ (i.e. c' being positive on $(-\infty, 0]$) is only needed to prove $\zeta(T_s) = 0$ in (ii). If $u_0' \ge 0$ we can drop this assumption. Note that $\zeta(T_s) = 0$ and (1.13) imply the unboundedness of u^+ on $[0,1] \times (0,T_s)$.

2. The weak solution is classical in the unsaturated region.

3. The level curve method used in [**1**] to obtain more regularity of the interface cannot be applied here because of the appearance of quadratic terms in the equation for $-u_t/u_r$.

Theorem 2. (i) *Let* H1–2 *be satisfied. Then*

$$\liminf_{t \uparrow T_s} \zeta(t)(T_s - t)^{-1/n} > 0 \tag{1.16}$$

if $n \ge 3$, *then*

$$\limsup_{t \uparrow T_s} u(0,t) < 0 \tag{1.17}$$

and $c(u)$ *is discontinuous in the point* $(0, T_s)$.

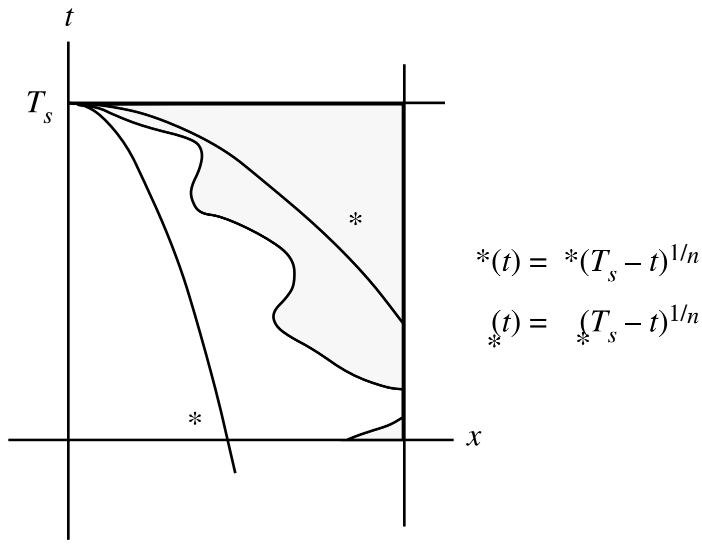

FIGURE 2. The interface ζ for $n \geq 3$; γ^* and γ^* are positive constants with $\gamma^* < \gamma^*$.

(ii) *Let* H1*-2* *be satisfied. Then*

$$\limsup_{t \uparrow T_s} \zeta(t)(T_s - t)^{-1/n} < \infty. \tag{1.18}$$

Remark 4. The uniform parabolicity is needed to prove (1.17) and (1.18), but not for (1.16). If $n = 1$, (1.17) is false, for $n = 2$ we do not know the answer. For $n \geq 3$ (1.16) and (1.18) combined state that

$$\zeta(t) \sim (T_s - t)^{1/n} \quad \text{as } t \uparrow T_s. \tag{1.19}$$

2. Discussion of Theorem 1. In this section we briefly discuss the proof of Theorem 1. We begin by adjusting the definition of a weak solution given in [**4**] to the spherically symmetric case.

Definition. A weak solution of Problem (NS) is a spherically symmetric function $u \in L^2(0, T; H^1(B))$ such that

(i) $c(u) \in C((0, T];\, L^2(B))$;

(ii) for all spherically symmetric test functions $\phi \in H^1(B \times (0, T])$

with $\phi(\cdot, T) = 0$ a.e. in B, the following integral equality holds:

$$\tag{2.1} \int_0^T \int_0^1 \{r^{n-1}\phi_r u_r - r^{n-1}\phi_t c(u)\}\, dx\, dt = \int_0^1 r^{n-1}\phi(r,0)v_0(r)\, dr + \int_0^T \phi(1,t)\, dt.$$

By [**4**], there exists a unique weak solution to Problem (N) for $\Omega = B$, $f \equiv 1$ and $v_0 = v_0(r)$. The uniqueness of this solution immediately implies that it is spherically symmetric. However, it is more efficient to construct this weak solution along the lines of the one-dimensional argument as given in [**8**] or [**5**]. First of all, this will give more regularity of u, and, moreover, the a priori estimates used to show convergence of the classical solution of the parabolic regularization of (NS) are easier to establish. Whereas in [**8**] and [**5**] the main tool was an equation for u_x, we now have an equation for $p = r^{n-1}u_r$:

$$\tag{2.2} p_t = \frac{1}{c'(u)}\left\{p_{rr} - (n-1)\frac{p_r}{r} - \frac{c''(u)}{c'(u)}pp_r\right\}.$$

Using the same parabolic regularization as in [**8**] or [**5**], i.e., approximating c with a sequence of smooth functions c_k satisfying $c_k' \geq 1/k$, $k = 1, 2, \ldots$, we obtain a sequence of smooth classical solutions u_k for which (2.2) makes sense. This yields a uniform bound (L) on $r^{n-1}u_{kr}$ on $[0,1] \times [0, T_s)$. By a variant of Proposition 4 in [**8**] it follows that $r^{n-1}c_k(u_k)$ is uniformly bounded in $C^{0+1}([0,1] \times [0, T_s])$. These two estimates lead to the existence of a weak solution u satisfying the same two estimates by taking the limit of an appropriate subsequence of u_k.

As for the second part of Theorem 1, we refer to [**3**], since the proof is almost identical. Existence of the interface ζ follows from the strong maximum principle (if $c'(0^-) = 0$, one needs a generalization of Sabinina's results [**6**] to the spherically symmetric case). Continuity of the interface ζ follows from a straightforward variant of the subsolution argument in the proof of Lemma 2 in [**3**]. We observe, however, that Lemma 7 in [**3**], which states that for t close to T_s the function u is monotone with respect to the space variable if u_0 is piecewise monotone, cannot be proved for $n > 1$ by the same argument as in [**3**] since the proof in [**3**] explicitly uses the continuity of $c(u)$ at $t = T_s$. This explains the nature of Remark 1 in the previous section.

3. Proof of Theorem 2. The main tool in the proof of Theorem 2 is the following formal coordinate transformation:

$$\rho = \frac{r}{\zeta(t)}, \qquad \tau = \int_0^t \zeta(s)^{-2}\, ds \tag{3.1}$$

which maps the region $\{(r,t) : 0 \le r \le \zeta(t),\ 0 \le t < T_s\}$ into the region $\{(\rho,\tau) : 0 \le \rho \le 1,\ 0 \le \tau < T_s^*\}$, where

$$T_s^* = \int_0^{T_s} \zeta(s)^{-2}\, ds \in (0,\infty] \tag{3.2}$$

and transforms equation (1.7) into

$$w_t = \frac{1}{c'(w)}\{\rho^{1-n}(\rho^{n-1}w_\rho)_\rho + \zeta(t(\tau))\zeta'(t(\tau))\rho w_\rho\}. \tag{3.3}$$

Here we write $w(\rho,\tau) = u(x,t)$. Formally we have $w < 0$ on $[0,1) \times (0,T_s)$. Unfortunately, ζ lacks sufficient regularity for (3.3) to make sense. What we can do, however, is replace ζ in (3.3) by a smooth function χ with $0 < \chi \le \zeta$ on $[0,T_s)$. Our first choice for χ is given by the following lemma.

Lemma 1. *There exists a $\gamma^* > 0$ such that the function χ defined by*

$$\chi(t) = \gamma^*(T_s - t)^{1/n} \tag{3.4}$$

satisfies $0 < \chi \le \zeta$ on $[0,T_s)$.

Note that Lemma 1 proves (1.16) in Theorem 2(i).

Proof. We first observe that by the maximum principle the solution u of Problem (NS) satisfies $u \ge \operatorname{ess\,inf} u_0$ on $(0,T_s)$. Consequently,

$0 \leq 1 - c(u)$ is bounded by a constant $\gamma > 0$. Rewriting (1.10) we obtain

$$T_s - t = \int_0^1 r^{n-1}(1 - c(u(r,t)))\,dr \leq \frac{\gamma(\zeta(t))^n}{n} \tag{3.5}$$

since $c(u) = 1$ for $r > \zeta(t)$. Thus, Lemma 1 holds with $\gamma^* = (n/\gamma)^{1/n}$.

From now on, we assume that $n \geq 3$. In particular,

$$T_s^* < \infty. \tag{3.6}$$

Proof of (1.17). Let χ be as in (3.4). Without loss of generality we may assume that $u_0 < 0$ on $[0, \chi(0))$ and that we can choose a smooth function u_0^* with

$$u_0^{*'} > 0 \quad \text{on } (0, \chi(0)); \qquad u_0^*(\chi(0)) = 0 \tag{3.7}$$

and

$$u_0^* \geq u_0 \quad \text{on } [0, \chi(0)]. \tag{3.8}$$

Now let u^* be the unique classical solution of

$$\text{(P)}\quad \begin{aligned} &u_t = \frac{1}{c'(u)}\{r^{1-n}(r^{n-1}u_r)_r\}, && 0 < r < \chi(t),\ 0<t<T_s && (3.9)\\ &u(\chi(t),t) = 0, && 0 < t < T_s && (3.10)\\ &u(r,0) = u_0^*(0), && 0 \leq r \leq \chi(0). && (3.11) \end{aligned}$$

By (2.2), (3.7) and the maximum principle

$$u_r^* \geq 0,\ 0 \leq r \leq \chi(t), \qquad 0 \leq t < T_s, \tag{3.12}$$

whereas by (3.8) and the comparison principle

$$u \leq u^* \leq 0,\ 0 \leq r \leq \chi(t), \qquad 0 \leq t < T_s. \tag{3.13}$$

What we have to show is that $u^*(0,t)$ is bounded away from zero.

Writing $w^*(\rho,\tau) = u^*(r,t)$ and using (3.12) we see that w^* satisfies

$$w^*_\tau \le \frac{1}{c'(w^*)}\{(\rho^{1-n}(\rho^{n-1}w^*_\rho)_\rho\}, \qquad 0<\rho<1,\ 0<\tau<T^*_s \tag{3.14}$$

$$w^*(1,\tau) = 0, \qquad 0 \le \tau < T^*_s \tag{3.15}$$

$$w^*(\rho,0) = u^*_0(\rho\chi(0)) < 0, \qquad 0 \le \rho < \chi(0) \tag{3.16}$$

where $T^*_s = \int_0^{T_s} \chi(s)^{-2}\,ds < \infty$ because $n \ge 3$. Since c' is bounded away from zero on $[\inf w^*, 0]$, we can apply the strong maximum principle to conclude from (3.14–3.16) that

$$\limsup_{t\uparrow T_s} u^*(0,t) = \limsup_{\tau\uparrow T^*_s} w^*(0,\tau) < 0. \tag{3.17}$$

This completes the proof of (1.17) and thereby of Theorem 2(i). For part (ii) we first need some estimates.

Lemma 2. *Let* H1*–2* *be satisfied. Then*

(i) $u_r \ge 0$ *a.e. on* $(0,1)\times(0,T_s)$;

(ii) $u_t \ge -K$ *in the sense of distributions on* $(0,1)\times(0,T_s)$*, where* K *is the constant in* H2*.

Proof. Both (i) and (ii) are first proved for the classical solution u_k of the parabolic regularization of (NS). Then (i) is immediate from the maximum principle and (2.2), whereas for (ii) we follow [**8**] and differentiate (1.7) with respect to τ to obtain

$$q_t \ge \frac{1}{c'_k(u_k)}\{r^{1-n}(r^{n-1}q_r)_r\} \tag{3.18}$$

for $q = u_{kt}$, implying, again by the maximum principle, that $q \ge -K$. Letting $k \to \infty$ completes the proof.

Proof of (1.18). By Lemma 2(ii) and (1.13) the interface ζ satisfies $\zeta' \le -K$ in the sense of distributions, or, more precisely, $\zeta(t) - Kt$ is nonincreasing on $[0,T_s)$. Thus we can approximate ζ by a sequence of smooth functions χ_m such that

$$0 < \chi_m \le \zeta \quad \text{on } [0,T_s) \tag{3.19}$$

$$\chi'_m \le K \quad \text{on } [0,T_s) \tag{3.20}$$

$$\chi_m \to \zeta \quad \text{in } C([0,T_s]). \tag{3.21}$$

We now apply the coordinate transformation (3.1) with ζ replaced by χ_m. Writing $w(\rho,\tau)=u(r,t)$ and using Lemma 2(i) and (3.20) we arrive at

$$w_\tau \le \frac{1}{c'(w)}\{(\rho^{1-n}(\rho^{n-1}w_\rho)_\rho\} + K\rho w_\rho, \quad 0<\rho<1,\ 0<\tau<T_m^*; \tag{3.22}$$

$$w(1,\tau)\le 0, \qquad 0\le\tau<T_m^*; \tag{3.23}$$

$$w(\rho,0)\le u_0(\rho\zeta(0))<0, \qquad 0\le\rho<1 \tag{3.24}$$

where

$$T_m^* = \int_0^{T_s}(\chi_m(s))^{-2}\,ds \uparrow T_s. \tag{3.25}$$

Note that ρ, τ and w depend on m. To be precise,

$$u(r,t) = w_m\left(\frac{r}{\chi_m(t)}, \int_0^t (\chi_m(s))^{-2}\,ds\right). \tag{3.26}$$

Let w^* be the unique (classical) solution of (3.22–3.24) with all inequalities replaced by equalities and T_m^* by T_s^*. Then

$$w_m \le w^* < 0 \quad \text{on } [0,1)\times[0,T_m^*], \tag{3.27}$$

w^* being independent of m. Combining (3.26) and (3.27) and letting $m\to\infty$, we obtain

$$u(r,t)\le w^*\left(\frac{r}{\zeta(t)}, \int_0^t(\zeta(s))^{-2}\,ds\right)<0 \quad 0\le r<\zeta(t),\ 0\le t<T_s. \tag{3.28}$$

Consequently, there exists a constant $\gamma>0$ such that

$$1-c(u(r,t))\ge\gamma>0, \qquad 0\le r\le\frac{\zeta(t)}{2},\ 0\le t<T_s. \tag{3.29}$$

Thus, using (1.10) again,

$$T_s - t = \int_0^1 r^{n-1}(1-c(u(r,t))\,dr \ge \frac{\gamma}{n}\left(\frac{\zeta(t)}{2}\right)^n \tag{3.30}$$

which completes the proof of Theorem 2. □

Remark. This manuscript was written in 1986 and submitted in 1987. Because of this, the references are no longer complete, and I apologize to anybody who published on this subject since then.

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