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A SURVEY WITH OPEN PROBLEMS ON UNIVALENT FUNCTIONS WHOSE COEFFICIENTS ARE NEGATIVE

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ABSTRACT. Denote by S the family of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk. Of the many subclasses investigated, one of the more manageable has the coefficients of its functions restricted to $a_n \leq 0$ for $n \geq 2$. A characterization of this class T enables us to obtain with relative ease many results that have no simple analog in S. The standard techniques for solving extremal problems in T are explained and applied, with frequent reference to comparable theorems in S or other subclasses whose functions do not have such severe argument limitations. One might, indeed, gain some insight on a hypothesis concerning S by first testing it on the more controllable class T. Not all, however, flows freely in T. Many easily stated problems for T or related classes remain unsolved, which leads to discussions on 19 open problems and conjectures.

1. Preliminaries. A function f is said to be *univalent* in a domain D if, for any two distinct points $z_1, z_2 \in D$, we have $f(z_1) \neq f(z_2)$. A simple consequence of Rouché's Theorem is that an analytic and univalent f must also be *locally univalent*, i.e., $f' \neq 0$ for $z \in D$. That the converse is not true can be illustrated by the function e^z, which is locally univalent but not univalent in the entire plane. Since the univalence of $f(z)$ is not affected by replacing it with $(f(z) - f(0))/f'(0)$, we shall so normalize. We will also restrict our domain to the unit disk $\Delta = \{|z| < 1\}$.

A function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

analytic and univalent in Δ is said to be in the family S. Much of the work on S or its subclasses may be attributed directly or indirectly

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to attempts at resolving the famous and easily stated Bieberbach Conjecture of 1916 that $|a_n| \le n$ for $f \in S$. There has been a rich history of determining bounds on the moduli of the coefficients of f after placing restrictions either on their arguments or on the image domain to which Δ is mapped. For instance, it has long been known that $|a_n| \le n$ if the coefficients of $f \in S$ are real [**15**] or if Δ is mapped onto a star-shaped region [**60**]. In both cases the Koebe function $k(z) = z/(1-z)^2 = \sum_{n=1}^{\infty} nz^n$ is extremal. Though the Bieberbach Conjecture was recently proved by de Branges [**13**], the investigation of S and its many subclasses continues nearly unabated.

The analytic representation for f mapping Δ onto a starlike domain is that $\operatorname{Re}\{zf'/f\} > 0$, $z \in \Delta$, and, for f mapping Δ onto a convex domain is that $\operatorname{Re}\{1+zf''/f'\} > 0$, $z \in \Delta$. Thus f is a convex function (maps Δ onto a convex domain) if and only if zf' is a starlike function (maps Δ onto a starlike domain). In particular, $|a_n| \le 1$ if f is convex.

A function cannot be in S or various subclasses if it has too large a coefficient. One can ask if a function is assured of being in S if its coefficients are sufficiently small. This has a simple answer. For $z_1 \neq z_2$, we have

$$f(z_2) - f(z_1) = (z_2 - z_1)\left(1 + \sum_{n=2}^{\infty} a_n (z_2^{n-1} + z_2^{n-2} z_1 + \cdots + z_1^{n-1})\right) \neq 0$$

as long as $1 - \sum_{n=2}^{\infty} n|a_n| r^{n-1} > 0$ for $|z_1| \le r$, $|z_2| \le r$. Hence the condition

$$\sum_{n=2}^{\infty} n|a_n| \le 1 \tag{2}$$

is sufficient for f, defined by (1), to be in S. The bound in (2) cannot be increased, since $z + (1+\varepsilon)z^n/n \notin S$ for any $\varepsilon > 0$. One can say even more for f satisfying (2). Since $\operatorname{Re} f'(z) \ge 1 - \sum_{n=2}^{\infty} n|a_n| r^{n-1}$, $|z| \le r$, it follows that $\operatorname{Re} f' > 0$, $z \in \Delta$, a condition shown by Noshiro [**61**] and Warschawski [**106**] to be sufficient for univalence. Goodman [**20**] showed that condition (2) is also sufficient for starlikeness. In fact, the values of zf'/f, $z \in \Delta$, will then not only lie in the right half plane but must be further restricted to a disk of radius one. This follows because

$$(3) \qquad \left|\frac{zf'}{f}-1\right| = \left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{1-\sum_{n=2}^{\infty}|a_n|} \leq 1,$$

the last inequality being equivalent to (2).

Our primary interest in this survey is to discuss results and open problems in a family for which condition (2) is necessary as well as sufficient for univalence. In Section 2 we introduce the class consisting of functions in S whose coefficients from the second on are real and nonpositive. In Section 3 we give the extreme and support points for this family T. In Section 4 the subfamily of T consisting of convex functions is discussed, while in Section 5 a class of functions whose coefficients have different arguments are shown to inherit many of the properties of T. Section 6 examines functions in T whose derivatives are univalent as well as coefficient bounds for inverses of functions in T. Functions of positive order are explored in Section 7, with emphasis on properties that do not carry over from the family T. Finally, meromorphic functions are discussed in Section 8, where it is shown that those with positive coefficients behave much like functions in S with negative coefficients.

Probably the nicest quality of the family T is a coefficient characterization that makes many of the computations quite manageable, whereas comparable results for the full family S can be very difficult, very messy, or very false. In any case, insight may be gained into a problem involving S or some other subclass by an argument restriction that reduces the problem to one in T. We should not, however, be deceived by the relative simplicity of the family T. As was the case with the Bieberbach Conjecture, there are several easily stated questions related to the class T that appear quite difficult to solve. Overall, 19 open problems and conjectures are given.

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2. Negative coefficients. Denote by T the subfamily of S consisting of functions of the form

$$(4) \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0.$$

If condition (2) is violated for f of the form (4), then $f'(r) = 1 - \sum_{n=2}^{\infty} n a_n r^{n-1} < 0$ for r sufficiently close to 1. Since $f'(0) = 1$, there must be an r_0, $0 < r_0 < 1$, for which $f'(r_0) = 0$. Thus, f cannot even be locally univalent in Δ. Moreover, inequality (2) is a *necessary* as well as a sufficient condition for inequality (3) when f is of the form (4) because $\lim_{z\to 1^-} |(zf'(z)/f(z)) - 1| = \sum_{n=2}^{\infty}(n-1)a_n/(1 - \sum_{n=2}^{\infty} a_n)$, from which we conclude that (2) is a consequence of (3).

Note that (3) does not imply (2) in the general class S as can be seen by the example $f(z) = ze^z = z + \sum_{n=2}^{\infty} z^n/(n-1)!$, which satisfies $|(zf'(z)/f(z)) - 1| = |z| \le 1$.

Denoting by T^* and T_1^* the families consisting of functions in T that are, respectively, starlike and satisfy $|(zf'/f) - 1| \le 1$, $z \in \Delta$, we can summarize the preceding observations as follows:

Theorem 1. *For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \ge 0$, the following are equivalent*: (i) $\sum_{n=2}^{\infty} n a_n \le 1$, (ii) $f \in T$, (iii) $f \in T^*$, (iv) $f \in T_1^*$, (v) $f' \ne 0$, $z \in \Delta$, and (vi) $\operatorname{Re} f' > 0$, $z \in \Delta$.

In the family S, a function that maximizes the modulus of a single coefficient must do likewise for all the coefficients, the only extremal functions being $z/(1-xz)^2$, $|x| = 1$. In contrast, a function in T that maximizes a single coefficient must *minimize* the remaining. In view of Theorem 1, the extremal functions for coefficients in T are $z - z^n/n$, $n = 2, 3, \dots$.

One can ask, see [**85**], how similar restrictions on the arguments of coefficients affect the modulus.

Open Problem 1. If $f_\lambda(z) = z + e^{i\lambda} \sum_{n=2}^{\infty} a_n z^n \in S$, $a_n \ge 0$ and λ real, find a function $g(\lambda, n)$ for which $a_n \le g(\lambda, n)$ is sharp for every n.

Here we have restricted the coefficients to a ray whose argument is λ. Note that $g(0, n) = n$, with the Koebe function $z/(1-z)^2$ being extremal; and $g(\pi, n) = 1/n$, with extremal functions $z - z^n/n$. Little is known when $\lambda \ne 0, \pi$. In [**98**] it was shown that if $\sum_{n=2}^{\infty} n a_n > 1$, $a_n \ge 0$, then there exists an $\varepsilon > 0$ such that $z - e^{i\lambda} \sum_{n=2}^{\infty} a_n z^n \notin S$ for $|\lambda| \le \varepsilon$. On the other hand, for each $N > 3$, a finite complex sequence $\{a_n\}$ was constructed with the property that $\sum_{n=2}^N n|a_n| > 1$ and $f_\lambda(z) = z + e^{i\lambda}\sum_{n=2}^\infty a_n z^n \in S$ for every real λ. See also [**42**].

If $f(z) = z + \sum_{n=2}^N a_n z^n - \sum_{n=N+1}^\infty a_n z^n$, $a_n \geq 0$, is in S, then condition (2) need not hold. However, noting that $f'(r) \neq 0$ and then letting $r \to 1^-$, we do obtain $\sum_{n=N+1}^\infty n a_n \leq 1 + \sum_{n=2}^N n a_n$. Consequently, it is still true that $a_n = o(1/n)$. Functions of this type in S need not be starlike, as illustrated by $f(z) = z + 3\cos(2\pi/5)z^2/2 - \cos^2(\pi/5)z^3 - z^4/4$. See [**104**].

Open Problem 2. Find $\max a_2$ for functions in S of the form

$$z + a_2 z^2 - \sum_{n=3}^\infty a_n z^n, a_n \geq 0.$$

We know that $\max a_2 \geq 4/5$ because $f(z) = z + 4z^2/5 - 2z^4/5 - z^5/5 \in S$. See [**87**].

Open Problem 3. If $z + a_2 z^2 - \sum_{n=3}^\infty a_n z^n \in S$, $a_n \geq 0$ for $n \geq 3$ and a_2 arbitrary, is it still true that $a_n = o(1/n)$?

3. Extreme and support points. A function f in a family G is an *extreme point* of G if f cannot be expressed as a proper convex combination of two distinct functions in G. For any compact subfamily G of S, the maximum or minimum value of the real part of any continuous linear functional on G defined over the set of analytic functions occurs at one of the extreme points of the closed convex hull of G ($\operatorname{cl}\operatorname{co} G$). Consequently, the determination of the extreme points enables us to solve many extremal problems. See, for example, [**7**] and [**9**].

Most of the subfamilies G of S that have been studied are rotationally invariant, that is, $\bar{x}f(xz) \in G$, $|x| = 1$, whenever $f \in G$. For such families, the functions $\bar{x}f(xz)$ are all extreme points whenever $f(z)$ is an extreme point because $\bar{x}f(xz) = \lambda f_1(z) + (1-\lambda)f_2(z)$ implies $f(z) = \lambda x f_1(\bar{x}z) + (1-\lambda)x f_2(\bar{x}z)$. Therefore, any rotationally invariant compact subfamily $G(\not\equiv z)$ of S must have uncountably many extreme points. We will prove that the family T, which is not rotationally invariant, has countably many extreme points. The necessary and sufficient coefficient bounds of Theorem 1 show that T is a *convex family* ($f_1, f_2 \in T$ and $0 < \lambda < 1$ implies $\lambda f_1 + (1-\lambda) f_2 \in T$). Thus, the closed convex hull of T is T itself.

Theorem 2. [**85**]. *Set* $f_1(z) = z$ *and* $f_n(z) = z - z^n/n$, $n = 2, 3, \dots$. *Then* $f \in T$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, *where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. If $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \lambda_n z^n/n$, then $\sum_{n=2}^{\infty} n(\lambda_n/n) = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1$ and $f \in T$. Conversely, if $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$, we may set $\lambda_n = n a_n$, for $n = 2, 3, \dots$, and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$. Then $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$. □

Consequently, the extreme points of T are the functions $f_n(z)$, $n = 1, 2, \dots$. By examining the extreme points of T, the following distortion result is obtained.

Corollary. *If* $f \in T$, *then*

$$r - r^2/2 \leq |f| \leq r + r^2/2, \qquad |z| \leq r,$$
$$1 - r \leq |f'| \leq 1 + r, \qquad |z| \leq r,$$

with equality for $f_2(z) = z - z^2/2$, $z = \pm r$.

A function f is said to be a *support point* of a compact family F if there exists a continuous linear functional J on A, the set of functions analytic in Δ, such that $\operatorname{Re} J(f) \geq \operatorname{Re} J(g)$ for all $g \in F$, with $\operatorname{Re} J$ nonconstant on F. In [**105**] it is shown that J is a continuous linear functional on A if and only if there exists a sequence $\{b_n\}$, $\limsup_{n\to\infty} |b_n|^{1/n} < 1$, such that $J(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n$. When $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$, $\operatorname{Re} a_n$ is maximized by f_n, $n \geq 2$, whereas $\operatorname{Re}(-a_n)$ is maximized (though not uniquely) by f_1. Thus, the extreme points of T are also support points.

In attempting to solve linear extremal problems, it is the support points that we really want to know. However, for many families the extreme points have been easier to determine and the support points too numerous to give much practical information. In any case, the

Krein-Milman Theorem shows that every linear extremal problem has a solution within the extreme points. A family having identical extreme and support points may be found in [**107**]; families where they differ are in [**30, 34,** and **79**]. H. Hamilton as well as Duren and Leung [**17**] constructed extreme points of $\mathrm{cl}\,\mathrm{co}\,S$ that are not support points, while Brickman and Leung [**8**] showed that if there exists a support point of S that is not an extreme point of $\mathrm{cl}\,\mathrm{co}\,S$ then it must be of a very special type. Properties of extreme and support points of S are discussed in [**16, 48,** and **79**].

The following result, due to Deeb [**14**], shows that there are considerably more support points of T than there are extreme points.

Theorem 3. *With the notation of Theorem* 2, *the function* $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ *is a support point of* T *if and only if* $\lambda_k = 0$ *for some* $k \geq 2$.

Proof. For $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} a_n z^n$, set $J_k(f) = -a_k$ and note that $J_k(f) = -\lambda_k/k$ is maximized when $\lambda_k = 0$. Hence f is a support point if $\lambda_k = 0$ for some $k \geq 2$.

Conversely, suppose $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ is a support point of T with associated functional J and sequence $\{b_n\}$. Now $\mathrm{Re}\, J(f_n)$ cannot equal $\mathrm{Re}\, J(f)$ for every $n \geq 1$ because J is not constant on T. Thus, there exists a $k(\geq 1)$ for which $\mathrm{Re}\, J(f_k) < \mathrm{Re}\, J(f)$. But λ_k must then vanish to avoid the contradiction

$$\begin{aligned} \mathrm{Re}\, J(f) = \sum_{n=1}^{\infty} \lambda_n \mathrm{Re}\, J(f_n) &\leq \sum_{n \neq k} \lambda_n \mathrm{Re}\, J(f) + \lambda_k \mathrm{Re}\, J(f_k) \\ &< \sum_{n=1}^{\infty} \lambda_n \mathrm{Re}\, J(f) = \mathrm{Re}\, J(f). \end{aligned}$$

We conclude our proof by showing that $\lambda_k = 0$ for some $k \geq 2$ even when $\lambda_1 = 0$. For if $\lambda_n > 0$, $n \geq 2$, then $\mathrm{Re}\, J(f_n) = \mathrm{Re}\, J(f) = \mathrm{Re}\, J(z - z^n/n) = b_1 - b_n/n$, $n \geq 2$. Setting $\mathrm{Re}\, J(f_n) = c$, c a constant, we get $b_n = (b_1 - c)n$. Now $b_1 \neq c$ because J is not constant on T. Hence, $|b_n|^{1/n} \to 1$, contradicting the fact that $\limsup_{n\to\infty} |b_n|^{1/n} < 1$. Therefore, $\lambda_k = 0$ for some $k \geq 2$, and the proof is complete. $\square$

Remark . The actual proof given by Deeb in [**14**] neglected to treat explicitly the case $\lambda_1 = 0(\sum_{n=2}^{\infty} na_n = 1)$ and $\lambda_n \neq 0$ for $n \geq 2$. Our proof shows that this special case does not affect the statement of Theorem 3.

In [**14**] Deeb also considered the family M of functions of the form (1) for which $\{na_n\}$ is a positive nondecreasing sequence bounded by 2. He showed that there are countably many extreme points of M and that the relationship of the support points to the extreme points of M is similar to that for T found in Theorem 3. See [**30**] and [**79**] for classes containing uncountably many extreme points in which not every one is a support point.

Open Problem 4. In a compact family with countably many extreme points, must all the extreme points be support points?

4. Convex functions. Since the operator L defined by $Lf(z) = \int_0^z (f(t)/t)\,dt$ is an isomorphism from $T = T^*$ to C, the subfamily of T consisting of convex functions, a consequence of Theorem 2 is that the extreme points of C are z and $z - z^n/n^2$, $n = 2, 3, \dots$. Hence, a necessary and sufficient condition for f of the form (4) to be in C is that $\sum_{n=2}^{\infty} n^2 a_n \leq 1$. This also follows from the relationship $f \in C$ if and only if $zf' \in T$. Note that f in T is convex in the disk $|z| < r_0$ if $|zf''/f'| < 1$ or, equivalently, $\sum_{n=2}^{\infty} n^2 a_n r_0^{n-1} \leq 1$. The radius of convexity of f in T is the largest disk $|z| < r_0 \leq 1$ that is mapped onto a convex domain. For $f \notin C$, this is the $r_0 (< 1)$ for which $\sum_{n=2}^{\infty} n^2 a_n r_0^{n-1} = 1$. The radius of convexity of T is thus the largest r_0 for which $\sum_{n=2}^{\infty} na_n \leq 1$ implies $\sum_{n=2}^{\infty} n^2 a_n r_0^{n-1} \leq 1$ whenever $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$. This will hold if $n^2 r_0^{n-1} \leq n$ for $n \geq 2$ or, equivalently, $r_0 \leq \min_n (1/n)^{1/(n-1)} = 1/2$. Therefore, the radius of convexity of the family T is $1/2$, with extremal function $z - z^2/2$. See [**85**].

Schild [**77**] investigated the family consisting of polynomials $p_N(z) = z - \sum_{n=2}^{N} a_n z^n$, $a_n \geq 0$, for which $\sum_{n=2}^{N} na_n = 1$. Most of his interesting results are readily generalizable to all of T. See [**39**] and [**85**]. We mention the following problem studied by Schild, but state it for the more general class T.

Let $d^* = \min_\theta |f(e^{i\theta})|$ and $d_0 = \min_\theta |f(r_0 e^{i\theta})|$, where r_0 is the radius of convexity of f. Schild [**77**] showed that $d_0/d^* \geq 2/3$ for $f \in T$ and conjectured that the sharp lower bound was $3/4$. Lewandowski [**40**] proved the Schild Conjecture. We give a simpler proof due to Gray and Schild [**24**].

Theorem 4. *If $f \in T$, then $d_0/d^* \geq 3/4$. The result is sharp, with extremal function $f(z) = z - z^2/2$.*

Proof. Since $|f(re^{i\theta})| \geq f(r)$, we must show that $d_0/d^* = (r_0 - \sum_{n=2}^\infty a_n r_0^n)/(1 - \sum_{n=2}^\infty a_n) \geq 3/4$, where r_0 is the radius of convexity of f. But this is equivalent to

$$(r_0 - 3/4) + \sum_{n=2}^{\infty} a_n (3/4 - r_0^n) \geq 0. \tag{5}$$

Since $\sum_{n=2}^\infty n^2 a_n r_0^{n-1} = 1$, we may write (5) as $\sum_{n=2}^\infty a_n [n^2 r_0^n - (3/4) n^2 r_0^{n-1} + 3/4 - r_0^n] = \sum_{n=2}^\infty a_n b(r_0, n) \geq 0$. The proof is completed by observing that $b(r_0, 2) = 3(r_0 - 1/2)^2 \geq 0$ and then showing that $b(r_0, n+1) - b(r_0, n) \geq 0$ for $n \geq 2$. □

In [**76**] Schild conjectured that $d_0/d^* \geq 2/3$ for all $f \in S^*$, the subfamily of S consisting of starlike functions. This seemed like a natural conjecture because $d_0/d^* = 2/3$ for the Koebe function $k(z) = z/(1-z)^2$, which is extremal for so many problems. Here $r_0 = 2 - \sqrt{3}$, the radius of convexity of S, $d_0 = |k(-r_0)| = 1/6$ and $d^* = |k(-1)| = 1/4$. Schild found a lower bound for d_0/d^*, which was later improved by McCarty and Tepper [**50**]. Surprisingly, however, Barnard and Lewis [**4**] constructed a counterexample to the 2/3 conjecture.

Open Problem 5. Suppose r_0 is the radius of convexity of f, $d_0 = \min_\theta |f(r_0 e^{i\theta})|$, and $d^* = \min_\theta |f(e^{i\theta})|$. Find the largest value of m for which $d_0/d^* \geq m$ for all $f \in S^*$.

We turn next to a different normalization. Montel [**58**] examined functions f analytic and univalent in Δ and satisfying $f(0) = 0$

and $f(z_0) = z_0 \neq 0$. Lewandowski [**41**] showed that all such f may be expressed as $f(z) = z_0 g(z)/g(z_0)$, where $g \in S$. Pilat [**66**] applied this result to modify the class studied by Schild and obtained information when $f(z_0) = z_0$ for $0 < z_0 < 1$. In [**86**], functions of the form $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, were investigated when either $f(z_0) = z_0$ or $f'(z_0) = 1$, $-1 < z_0 < 1$. In the former case $a_1 = 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1}$ and in the latter $a_1 = 1 + \sum_{n=2}^{\infty} n a_n z_0^{n-1}$. The results found reduce to those of T when $z_0 = 0$. Denoting by $G(z_\gamma)$ the subfamily of starlike or convex functions with either $f(z_\gamma) = z_\gamma$ or $f'(z_\gamma) = 1$, it was further shown, for B a subset of the real interval $(0, 1)$, that $\cup_{z_\gamma \in B} G(z_\gamma)$ is a convex family if and only if B is connected. For additional papers on negative coefficients with either of these normalizations, see [**37, 62, 69, 74,** and **10**].

Open Problem 6. What can be proved for either normalization when z_0 is not real?

5. Varying arguments. We should pause, at this point, to ask just what is so special about negative coefficients anyway? The necessary and sufficient coefficient condition affords us the opportunity to obtain many results that are not readily accessible in S or other subfamilies. Is T the only class for which condition (2) is necessary and sufficient for univalence? Not at all. The same holds for functions of the form $f(z) = z + \sum_{n=2}^{\infty} (-1)^n a_n z^n$, $a_n \geq 0$. Since $f'(-r) = 1 - \sum_{n=2}^{\infty} n a_n r^{n-1} \neq 0$, $0 < r < 1$, we must have $\sum_{n=2}^{\infty} n a_n \leq 1$ if $f \in S$. Note here that we chose values of z along the negative real axis instead of the positive real axis, as we did with the family T. We now characterize all functions for which condition (2) is necessary as well as sufficient for univalence. See [**87**].

A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be in $V(\theta_n)$ if $f \in S$ and $\arg a_n = \theta_n$ for all n (when $a_n = 0$, θ_n may be chosen arbitrarily). If, further, there exists a real number β such that

$$\theta_n + (n-1)\beta \equiv \pi \pmod{2\pi}, \tag{6}$$

then f is said to be in $V(\theta_n; \beta)$. The union of $V(\theta_n; \beta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers β is denoted by V.

Theorem 5. [**87**]. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V$, *then* $\sum_{n=2}^{\infty} n|a_n| \leq 1$.

Proof. If $f \in V(\theta_n; \beta)$ with $f(z) = z + \sum_{n=2}^{\infty} |a_n| e^{i\theta_n} z^n$, then $f'(\mathrm{r}e^{i\beta}) = 1 - \sum_{n=2}^{\infty} n|a_n| r^{n-1}$. Since $f' \neq 0$ for $z \in \Delta$, the result follows upon letting $r \to 1^-$. □

There is a kind of converse to Theorem 5 in that if $\{\theta_n\}$ is a sequence of real numbers for which there does not exist a real number β that satisfies (6), then there exists a function in $V(\theta_n)$ for which $\sum_{n=2}^{\infty} n|a_n| > 1$.

The family V, unlike T, is not convex. Even though $f(z) = z + z^2/2$ and $g(z) = z + z^3/3$ are in V, the function $(f(z) + g(z))/2$ is not. The closed convex hull of V is $\{f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\sum_{n=2}^{\infty} n|a_n| \leq 1\}$, and the extreme points of the hull are $\{z + xz^n/n, |x| = 1\}$. See [**87**].

We now return to T and its subfamilies, keeping in mind that the techniques used to obtain results about T often work in the family V.

6. Further properties of T. Denote by T' the subfamily of T consisting of functions f for which f' is also univalent in Δ. Since the second coefficient of a function in T' cannot vanish, the only extreme point of T that is also a member of T' is $f_2(z) = z - z^2/2$. Although the bound on the second coefficient for T' is the same as that for T, the bounds on the remaining coefficients appear difficult to obtain because there is no simple coefficient characterization of T'. We do have

Theorem 6. [**90**]. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$, $a_2 > 0$, *then a sufficient condition for* f *to be in* T' *is that* $\sum_{n=3}^{\infty} n(n-1)a_n \leq 2a_2$.

Proof. The function $f'(z)$ is univalent in Δ if and only if

$$g(z) = (1 - f'(z))/2a_2 = z + \sum_{n=2}^{\infty} (n+1)a_{n+1} z^n / 2a_2 = z + \sum_{n=2}^{\infty} b_n z^n$$

is in S. The result follows upon noting that $\sum_{n=2}^{\infty} nb_n \leq 1$ is sufficient for the univalence of g. □

If $f(z) = z - \sum_{n=2}^{N} a_n z^n \in T'$, then $\max a_3 = 1/9$ if $N = 3$ and $\max a_3 = (\sqrt{2}-1)/3$ if $N = 4$. See [**90**]. The general case is unknown.

Open Problem 7. Find $\beta = \sup\{a_3 : f \in T'\}$. (We know [**90**] that $(\sqrt{2}-1)/3 \le \beta \le 1/6$.)

Shah and others have investigated extensively the subfamily of S consisting of functions having univalent derivatives of all orders. See [**10, 81, 82,** and **83**]. A conjecture by Shah and Trimble [**82**] on the upper bound for the second coefficient was disproved by Lachance [**38**]. Denote by T_∞ the subfamily of T consisting of functions having univalent derivatives of all orders. It was shown in [**90**] that

$$F_\varepsilon(z) = z - (1-\varepsilon)z^2/2 - \varepsilon \sum_{n=3}^{\infty} z^n/2^{n-3}n! \in T_\infty$$

for $0 < \varepsilon \le \varepsilon_0 = (2e^{1/2}-1)^{-1}$, so that $\sup\{a_2 : f \in T_\infty\} = 1/2$. The lack of an extremal function demonstrates that T_∞ is not a compact family. In fact, the sequence $\{f_k(z)\}$ defined by

$$f_k(z) = z - \left(\frac{1-1/k}{2}\right) z^2 - \frac{1}{k} \sum_{n=3}^{\infty} \frac{z^n}{2^{n-3}n!}$$

is in T_∞ for every integer $k \ge 3$, yet $\{f_k(z)\}$ converges uniformly on compact subsets of Δ to $z - z^2/2 \notin T_\infty$.

Open Problem 8. Find $\gamma = \sup\{a_3 : f \in T_\infty\}$. (It is known that $\varepsilon_0/6 \le \gamma \le 1/6$.)

In 1923 Löewner [**45**] verified the Bieberbach Conjecture for the third coefficient and also found sharp bounds on all the coefficients for the family consisting of inverses of functions in S. On the other hand, in many subfamilies of S for which sharp coefficient bounds have long been known, the coefficient problem for inverse functions remains unresolved. See, for example, [**35, 43,** and **80**]. We now state a result for inverses in a family that includes T.

Theorem 7. [**93**]. *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *with* $\sum_{n=2}^{\infty} n|a_n| \le 1$, *and set* $f^{-1}(w) = g(w) = w + \sum_{n=2}^{\infty} b_n w^n$. *Then* $|b_n| \le B_n = \binom{2n-3}{n-2}/n2^{n-2}$. *Equality holds for rotations of* $F(z) = z - z^2/2$, *where* $F^{-1}(w) = G(w) = 1 - (1-2w)^{1/2} = w + \sum_{n=2}^{\infty} B_n w^n$.

This theorem is proved by applying the identity, see [**22** or **43**],

$$b_n = \frac{1}{2\pi i} \int_{|z|=r} \left(\frac{1}{f(z)}\right)^n dz$$

and then showing that the modulus of b_n is maximized when $f(z) = F(z)$. Note that if $\sum_{n=2}^{\infty} n|a_n| \le 1$, then $\sum_{n=2}^{\infty} |a_n| \le 1/2$ and $|f(z)| \ge |z| - \sum_{n=2}^{\infty} |a_n| z^n \ge r - r^2/2$. Hence f maps Δ onto a domain that contains the disk $|w| < 1/2$, which means that the series expansion for $f^{-1}(w)$ has radius of convergence at least $1/2$. Equality holds for $G(w) = w + \sum_{n=2}^{\infty} B_n w^n$.

Another classical problem is that of determining integral means. Using his star-function, Baernstein [**3**] proved for $f \in S$ and $k(z) = z/(1-z)^2$ that $\int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\theta \le \int_0^{2\pi} |k(re^{i\theta})|^\lambda \, d\theta$ for all $r < 1$ and $\lambda > 0$. Since $f_2(z) = z - z^2/2$ often serves as an extremal function in T as $k(z)$ does in S, we ask the following.

Open Problem 9. For $f \in T$, is it true that $\int_0^{2\pi} |f(\mathrm{re}^{i\theta})|^\lambda \, d\theta \le \int_0^{2\pi} |f_2(\mathrm{re}^{i\theta})|^\lambda \, d\theta$ for all $r < 1$ and $\lambda > 0$?

If H is a family consisting of functions analytic in Δ, we define the Koebe domain $K(H)$ by $K(H) = \bigcap_{f \in H} f(\Delta)$, that is, the set of points in the w plane covered by all $f \in H$. For rotationally invariant families, the Koebe domain will be a disk $|w| < R$. The "1/4 Theorem" for $f \in S$ says that $K(S)$ is $\{|w| < 1/4\}$. The Koebe domain for some families that are not rotationally invariant may be found in [**21, 46, 52,** and **67**]. For an extensive discussion of Koebe domains, see [**23**].

Open Problem 10. Find the Koebe domain of T.

From Theorem 2, $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ for $f \in T$. Thus, $f(1) = \sum_{n=1}^{\infty} \lambda_n f_n(1) \geq f_2(1) = 1/2$ and $f(-1) = \sum_{n=1}^{\infty} \lambda_n f_n(-1) \leq f_3(-1) = -2/3$. Therefore, $K(T)$ contains the real interval $(-2/3, 1/2)$.

Open Problem 11. Does $K(T) = \cap_{n=1}^{\infty} f_n(\Delta)$, where $\{f_n\}$ is the set of extreme points of T?

7. Positive order. A function f in S is said to be *starlike of order* α, $0 \leq \alpha \leq 1$, if $\operatorname{Re}\{zf'/f\} \geq \alpha$, $z \in \Delta$, and is said to be *convex of order* α if $\operatorname{Re}\{1 + zf''/f'\} \geq \alpha$, $z \in \Delta$. Sharp coefficient bounds for these families were found by Robertson [**71**]. Extreme points were determined in [**7**]. Denote by $T^*(\alpha)$ and $C(\alpha)$ the subclasses of T that are, respectively, starlike of order α and convex of order α, and note that $T^*(0) = T$. A necessary and sufficient condition for f of the form (4) to be in $T^*(\alpha)(C(\alpha))$ is

$$(7) \qquad \sum_{n=2}^{\infty} (n-\alpha)a_n \leq 1-\alpha, \qquad \sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha.$$

See [**51**] or [**85**]. As in Theorem 2, we can show that the extreme points of $T^*(\alpha)$ are $f_1(z) = z$ and $f_n(z) = z - (1-\alpha)z^n/(n-\alpha)$, $n = 2, 3, \ldots$, and the extreme points of $C(\alpha)$ are $g_1(z) = z$ and $g_n(z) = z - (1-\alpha)z^n/n(n-\alpha)$, $n = 2, 3, \ldots$.

The necessary and sufficient coefficient condition (7) enables us to make only minor modifications on most of the results previously given for T to obtain comparable information for the more general family $T^*(\alpha)$. We state two exceptions.

Open Problem 12. Suppose r_0 is the radius of convexity of f, $d_0 = \min_\theta |f(r_0 e^{i\theta})|$, and $d^* = \min_\theta |f(e^{i\theta})|$. Find the largest value of $m = m(\alpha)$ for which $d_0/d^* \geq m(\alpha)$ for all $f \in T^*(\alpha)$.

We saw from Theorem 4 that $m(0) = 3/4$. In [**85**] it was shown that f in $T^*(\alpha)$ is convex for $|z| < r(\alpha) = \inf_n((n-\alpha)/n^2(1-\alpha))^{1/(n-1)}$, $n = 2, 3, \ldots$, with an extreme point of $T^*(\alpha)$ being extremal for each

α. We also believe $m(\alpha)$ to arise from an extreme point of $T^*(\alpha)$ for each α, and thus conjecture that

$$m(\alpha) = \inf_n \left[\left(\frac{n-\alpha}{n^2(1-\alpha)} \right)^{1/(n-1)} \left(\frac{(n+1)(n-\alpha)}{n^2} \right) \right].$$

In Theorem 7 we gave coefficient bounds for inverses of functions in T. For $f \in T^*(\alpha)$, sharp coefficient bounds for inverses were found [**93**] for $n = 2, 3$, and 4. In the cases $n = 3$ and 4, the degree of the extremal polynomial function depended on α. We state the following conjecture.

Open Problem 13. There exists a sequence $\{\alpha_n\}$, $0 < \alpha_n < 1$, such that the maximum of the n^{th} coefficient for the inverses of functions in $T^*(\alpha)$ is

$$B_n(\alpha) = \begin{cases} \dfrac{2}{n} \dbinom{2n-3}{n-2} \left(\dfrac{1-\alpha}{2-\alpha} \right)^{n-1}, & 0 \le \alpha \le \alpha_n, \\ (1-\alpha)/(n-\alpha), & \alpha_n \le \alpha \le 1. \end{cases}$$

Equality holds for $f \in T^*(\alpha)$ when $f(z) = z - (1-\alpha)z^2/(2-\alpha)$, $0 \le \alpha \le \alpha_n$, and $f(z) = z - (1-\alpha)z^n/(n-\alpha)$, $\alpha_n \le \alpha \le 1$.

The values for α_3 and α_4 are given in [**93**] where it is also shown that there exist positive sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ for which $B_n(\alpha)$ is extremal in the intervals $0 \le \alpha \le \varepsilon_n$ and $1 - \delta_n \le \alpha \le 1$. To prove the conjecture, it is necessary to show that $\varepsilon_n + \delta_n = 1$ for all n.

Marx [**49**] and Strohhäcker [**103**] showed that convex functions must be starlike of order 1/2. MacGregor [**47**] found the order of starlikeness for functions convex of order α. See also [**84**] and [**96**]. We now determine the order of starlikeness for $C(\alpha)$.

Theorem 8. [**85**]. *$C(\alpha) \subset T^*(2/(3-\alpha))$, with extremal function $f(z) = z - (1-\alpha)z^2/2(2-a)$.*

Proof. In view of (7), we must prove, for $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in C(\alpha)$, that $\sum_{n=2}^{\infty}(n - 2/(3-\alpha))/(1 - 2/(3-\alpha))a_n \le 1$ whenever $\sum_{n=2}^{\infty} n(n-\alpha)a_n/(1-\alpha) \le 1$. It suffices to show that $n(n-\alpha)/(1-\alpha) \ge$

$(n-2/(3-\alpha))/(1-2/(3-\alpha))$ for $n=2,3,\dots$, which is equivalent to $n^2-3n+2\ge 0$. This completes the proof. $\square$

By a criterion of Kaplan [**32**], a function of the form (1) is in S if $\int_{\theta_1}^{\theta_2}\operatorname{Re}\{1+zf''/f'\}\,d\theta>-\pi$ for all z in Δ and for all θ_1,θ_2 satisfying $-\pi\le\theta_1<\theta_2\le\pi$. Hence, a function convex of order α, $\alpha\ge -1/2$, must be in S. Since $\int_0^z dt/(1-t)^2(1-\alpha)=((1-z)^{2\alpha-1}-1)/(1-2\alpha)\notin S$ for $\alpha<-1/2$, this bound cannot be lowered.

On the other hand, there is no requirement in the proof of Theorem 8 that α be nonnegative. Thus $C(\alpha)\subset T$ for all real α, but not conversely. A necessary and sufficient condition for f in T to be in $C(\alpha)$ for some real α is that its coefficients satisfy the inequalities

$$\sum_{n=2}^{\infty} na_n<1 \quad\text{and}\quad \sum_{n=2}^{\infty} n^2a_n<\infty. \tag{8}$$

This follows because

$$\begin{aligned}\operatorname{Re}\left\{1+\frac{zf'(z)}{f(z)}\right\}&=\operatorname{Re}\left\{1-\frac{\sum_{n=2}^{\infty} n(n-1)a_nz^{n-1}}{1-\sum_{n=2}^{\infty} na_nz^{n-1}}\right\}\\ &\ge 1-\frac{\sum_{n=2}^{\infty} n(n-1)a_n}{1-\sum_{n=2}^{\infty} na_n}=\frac{1-\sum_{n=2}^{\infty} n^2a_n}{1-\sum_{n=2}^{\infty} na_n}=1+\frac{f''(1)}{f'(1)}.\end{aligned}$$

Note that, in T, the functions $z-z^n/n$ satisfy the latter but not the former condition of (8), whereas the functions $z-c\sum_{n=2}^{\infty} z^n/n^3$, $0<c<6/\pi^2$, satisfy the former but not the latter. Now (8) also plays a role in another subfamily of T.

A function $f(z)=z+\cdots$, analytic in Δ, is said to be α-*convex* if

$$\operatorname{Re}\{(1-\alpha)zf'/f+\alpha(1+zf''/f')\}>0$$

for all $z\in\Delta$. We denote this family by $M(\alpha)$ and observe that $M(0)$ and $M(1)$ are, respectively, the families of starlike and convex functions. In [**54**] it was shown that $M(\alpha)$ contains only starlike functions for every real α and that $M(\beta)\subset M(\alpha)$ for $0\le\alpha\le\beta$. Denote by $N(\alpha)$ the subfamily of $M(\alpha)$ consisting of functions of the form (4). It was shown in [**91**] that if (8) holds and $0\le\alpha\le 1$, then a necessary and sufficient condition for f to be in $N(\alpha)$ is that

$$(1-\alpha)\frac{\sum_{n=2}^{\infty}(n-1)a_n}{1-\sum_{n=2}^{\infty} a_n}+\alpha\frac{\sum_{n=2}^{\infty}(n^2-n)a_n}{1-\sum_{n=2}^{\infty} na_n}\le 1. \tag{9}$$

This condition is still necessary when $\alpha > 1$. Inequality (9) was then used to find the (positive) order of starlikeness and sharp coefficient bounds for $f \in N(\alpha)$, $\alpha > 0$. The order of starlikeness of the general class $M(\alpha)$, $\alpha \geq 1$, was determined by Miller, Mocanu, and Reade [**55**]. Eenigenburg and Nelson [**18**] have constructed functions in $M(\alpha)$, $0 < \alpha < 1$, that are not starlike of any positive order. The coefficient bounds for $M(\alpha)$ remain open. See [**53**].

Open Problem 14. Find the extreme points of $N(\alpha)$.

Functions of the form $F_n(z) = z - A_n z^n$ for which equality holds in (9) uniquely maximize the n^{th} coefficient, so that $\{F_n(z)\}$ are extreme points. We have not been able to determine if there are additional extreme points.

The convexity of the families $T^*(\alpha)$ and $C(\alpha)$ are immediate consequences of (7). The more complicated condition (9) does not readily lend itself to showing convexity for $N(\alpha)$.

Open Problem 15. Is $N(\alpha)$ a convex family?

Another family for which the extreme points have not been characterized is

$$H(\alpha) = \{p : p(z) = zf'(z)/f(z), \quad f \in T^*(\alpha)\}.$$

In [**99**] it was shown that if $p(z) = 1 - \sum_{n=1}^{\infty} b_n z^n \in H(\alpha)$, then $b_n \leq n(1-\alpha)/(n+1-\alpha)$. For each n, equality holds only for a function p whose related f is an extreme point of $T^*(\alpha)$. Though these must be extreme points of $H(\alpha)$, the existence of other extreme points was proved. It was also shown that $H(\alpha)$ is not a convex family. Mullins and Ziegler [**59**] actually exhibited additional extreme points for $H(\alpha)$. In fact, they showed that if $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$ and $a_{n-1}a_{2n-1} = 0$ for all $n > 2$, then $p = zf'/f$ is an extreme point of $H(\alpha)$. However, there are still additional extreme points.

Open Problem 16. Find the extreme points of $H(\alpha)$.

We next mention a family for which the extreme points are known for some values. A function f in T is said to be in $T^*(a,b)$, a real and $b \geq 0$, if $|(zf'/f)-a| \leq b$ for all z in Δ. Note that $T^*(1,1-\alpha) = T^*(\alpha)$. In [**6**] the extreme points of $T^*(a,b)$ were found when $a \geq 1$. For $a < 1$, sharp coefficient bounds were determined as well as the extreme points for some b when f is of the form $f(z) = z - \sum_{n=1}^{\infty} a_{2n} z^{2n}$.

Open Problem 17. Find the extreme points of $T^*(a,b)$, $a < 1$.

If $f \in S$ and $w \notin f(\Delta)$, then $F(z) = wf(z)/(w - f(z))$ is also in S. The historical importance of this function $F(z) = z + (a_2 + 1/w)z^2 + \cdots$ with associated $f(z) = z + a_2 z^2 + \cdots$ was in applying the bound on the second coefficient to obtain the well-known $1/4$ covering theorem for functions in S. We have $|a_2 + 1/w| \leq 2$ so that $1/|w| \leq 2 + |a_2| \leq 4$, or $|w| \geq 1/4$. Hall [**29**] proved that $F(z)$ has bounded coefficients if f is convex by showing that $|z^2 f'(z)/f^2(z)| > 4/\pi^2$. He essentially showed that $F(z)$ will have bounded coefficients if $z^2 f'(z)/f^2(z)$ is bounded away from zero for $z \in \Delta$. As a consequence (see [**88**]) for a bounded function f that is starlike of order α, $\alpha > 0$, the corresponding F will have bounded coefficients. This follows for $|f| \leq M$ from the identity

$$\left|\frac{z^2 f'(z)}{f^2(z)}\right| = \left|\frac{z}{f(z)}\right| \left|\frac{zf'(z)}{f(z)}\right| \geq \frac{\alpha}{M} > 0.$$

The restriction $\alpha > 0$ cannot be eliminated, not even in T. If we take $f(z) = z - z^2/2$ and $w = 1/2$, then

$$F(z) = z + \sum_{n=2}^{\infty} (n+1) z^n / 2. \tag{10}$$

The problem of obtaining *sharp* coefficient bounds on the class investigated by Hall appears to be very difficult. Barnard and Schober [**5**] applied variational methods to find a sharp bound on the second coefficient.

It was shown, in [**88**], that if f of the form (4) is in $T^*(\alpha)$, then, for any $w \notin f(\Delta)$, the function $F(z) = wf(z)/(w - f(z)) = z + \sum_{n=2}^{\infty} b_n z^n$ will satisfy $|b_2| \leq (3-\alpha)/2$. Equality holds for $f(z) = z - (1-\alpha)z^3/(3-\alpha)$

and $w = 2/(3-\alpha)$. When $\alpha = 0$, equality also holds in (10). For $\alpha = 0$, we believe (10) to be extremal for all n.

Open Problem 18. If $f \in T$, $w \notin f(\Delta)$ and $F(z) = wf(z)/(w - f(z)) = z + \sum_{n=2}^{\infty} b_n z^n$, then $|b_n| \leq (n+1)/2$ with equality for $F(z)$ defined by (10).

An argument involving (11) in the next theorem may be used to verify the conjecture when f is an extreme point of T.

By choosing w on the boundary of $f(\Delta)$, the corresponding F is unbounded. This leads to a result on the lower bound of the coefficients of F.

Theorem 9. *If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$, then there exists a $w \notin f(\Delta)$ such that the n^{th} coefficient, b_n, for $F(z) = wf(z)/(w-f(z))$ satisfies $|b_n| \geq 1$ for all n. Equality holds for $f(z) = z$ and $w = e^{i\alpha}$, α real.*

Proof. For any $w \notin f(\Delta)$, we have $b_2 = 1/w - a_2$, $b_3 = (b_2 - a_2)/w - a_3$, and

$$b_n = \frac{b_{n-1} - a_{n-1} - \sum_{k=2}^{n-2} a_k b_{n-k}}{w} - a_n, \qquad n \geq 4. \tag{11}$$

Setting $w = 1 - \sum_{n=2}^{\infty} a_n$ and noting that $b_2 \geq 1/(1-a_2) - a_2 \geq 1$, it suffices to show inductively that $\{b_n\}$ is a nondecreasing function of n. Now $b_3 = (b_2 - a_2)/w - a_3 \geq b_2$ is equivalent to

$$b_2 \left(\frac{1}{w} - 1 \right) = \frac{b_2 \sum_{n=2}^{\infty} a_n}{1 - \sum_{n=2}^{\infty} a_n} \geq \frac{a_2 + a_3 - a_3 \sum_{n=2}^{\infty} a_n}{1 - \sum_{n=2}^{\infty} a_n},$$

which is true because $b_2 \geq 1$. If $b_k \geq b_{k-1}$ for $2 \leq k \leq n-1$, we will show that $b_n \geq b_{n-1}$ or, equivalently, from (11),

$$b_{n-1} \left(\frac{1}{w} - 1 \right) \geq \frac{a_n w + a_{n-1} + \sum_{k=2}^{n-2} a_k b_{n-k}}{w}.$$

But this holds if

$$b_{n-1} \sum_{k=2}^{\infty} a_k \geq a_n + a_{n-1} + \sum_{k=2}^{n-2} a_k b_{n-k}. \tag{12}$$

From the inductive hypothesis, the right side of (12) is bounded above by

$$a_n + a_{n-1} + b_{n-2}\sum_{k=2}^{n-2} a_k \le b_{n-1}\sum_{k=2}^{n} a_k,$$

and the proof is complete. □

We conjecture that this result for T is true for the family S as well.

Open Problem 19. If $f \in S$, does there exist a $w \notin f(\Delta)$ such that the n^{th} coefficient, b_n, for $F(z) = wf(z)/(w - f(z))$ satisfies $|b_n| \ge 1$ for all n?

8. Convolutions and meromorphic functions. The *convolution* or *Hadamard product* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. The Pólya-Schoenberg conjecture that the convolution of two convex functions is itself a convex function was proved by Ruscheweyh and Sheil-Small [**73**]. Convolution results for subclasses of T may be found in [**2, 33, 56, 64, 65, 78, 94**].

Certainly the convolution of two functions in S need not be in S, as can easily be seen by taking $f(z) = g(z) = z/(1-z)^2 = \sum_{n=1}^{\infty} nz^n$. Mandelbrojt and Schiffer conjectured that f and g in S implies $z + \sum_{n=2}^{\infty} a_n b_n z^n/n$ is in S. Had this conjecture been true, it would have furnished us with a simple proof of the Bieberbach Conjecture (de Branges' Theorem). For, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ and $g(z) = z + z^n/n$, then $z + a_n z^n/n^2$ would be in S if and only if $|a_n| \le n$. The conjecture was disproved by Loewner and Netanyahu [**44**]. Krzyz and Lewandowski [**36**] gave an example that showed the conjecture to be false even if we take g to be the convex function $g(z) = z/(1-z)$. They proved that $f(z) = z/(1-iz)^{1-i} = z + \sum_{n=2}^{\infty} a_n z^n \in S$, but that $z + \sum_{n=2}^{\infty} a_n z^n/n \notin S$. Thus, the convolution of $f \in S$ with a convex function need not be in S, since $g(z) = -\ln(1-z) = z + \sum_{n=2}^{\infty} z^n/n$ is convex.

On the other hand, Robertson [**72**] showed that the convolution of two meromorphic functions is better behaved. Denote by Σ the family consisting of functions of the form $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$ that are analytic and univalent in $|z| > 1$. Results for Σ have often been used to obtain results for S. The Area Theorem, $\sum_{n=1}^{\infty} n|a_n|^2 \leq 1$ for functions in Σ, enabled Bieberbach to find a sharp bound on the second coefficient for functions in S. As was the case with S, a function f in Σ is starlike (maps $|z| > 1$ onto a domain whose complement is starlike with respect to the origin) if and only if $\operatorname{Re}\{zf'/f\} > 0$, $|z| > 1$. The same sufficient condition (2) for starlikeness of $f \in S$ works for Σ. Setting

$$\frac{zf'(z)}{f(z)} = \frac{1 - \sum_{n=1}^{\infty} na_n z^{-(n+1)}}{1 + \sum_{n=1}^{\infty} a_n z^{-(n+1)}} = \frac{1 - b(z)}{1 + b(z)},$$

we need to show that $|b(z)| < 1$ for $|z| > 1$ whenever $\sum_{n=1}^{\infty} n|a_n| \leq 1$. Since $b(z) = \sum_{n=1}^{\infty}(n+1)a_n z^{-(n+1)}/(2 - \sum_{n=1}^{\infty}(n-1)a_n z^{-(n+1)})$, the result follows from the inequalities

$$|b(e^{i\theta})| \leq \frac{\sum_{n=1}^{\infty}(n+1)|a_n|}{2 - \sum_{n=1}^{\infty}(n-1)|a_n|} = \frac{\sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} |a_n|}{(2 - \sum_{n=1}^{\infty} n|a_n|) + \sum_{n=1}^{\infty} |a_n|} \leq 1.$$

Theorem 10. [**72**]. *If $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \in \Sigma$ and $g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \in \Sigma$, then $(f * g)(z) = z + \sum_{n=1}^{\infty} a_n b_n z^{-n}$ is univalent and starlike for $|z| > 1$.*

Proof. It suffices to prove that $\sum_{n=1}^{\infty} n|a_n b_n| \leq 1$. From the Cauchy-Schwarz inequality,

$$\sum_{n=1}^{\infty} n|a_n b_n| = \sum_{n=1}^{\infty} \sqrt{n}|a_n|\sqrt{n}|b_n| \leq \left(\sum_{n=1}^{\infty} n|a_n|^2 \sum_{n=1}^{\infty} n|b_n|^2\right)^{1/2}.$$

An application of the Area Theorem shows this last expression to be bounded above by 1, and the proof is complete. $\square$

Since $f'(z) = 1 - \sum_{n=1}^{\infty} na_n z^{-(n+1)}$ for $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$ in Σ, we see that the positive coefficients for $f \in \Sigma$ act as the negative coefficients for $f \in S$. In particular, $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$, $a_n \geq 0$, is univalent and starlike in $|z| > 1$ if and only if $\sum_{n=1}^{\infty} na_n \leq 1$.

Mogra, Reddy, and Juneja [**57**] investigated meromorphic functions of the form $g(z) = 1/z + \sum_{n=1}^{\infty} a_n z^n$ that are analytic and univalent in $0 < |z| < 1$. Note that g is analytic and univalent in $0 < |z| < 1$ if and only if $g(1/z)$ is in Σ. The condition for starlikeness of g is that $\mathrm{Re}\{-zf'/f\} > 0$, $0 < |z| < 1$. They showed that a necessary and sufficient condition for g to be starlike of order α when $a_n \geq 0$ is that $\sum_{n=1}^{\infty}(n+\alpha)a_n \leq 1-\alpha$. Extreme points, distortion theorems, and convolution results for various subfamilies of meromorphic functions with positive coefficients were found and shown to be comparable to companion subclasses of T. Analogous results to those for T have also been obtained for p-valent functions of the form $f(z) = z^p - \sum_{n=k+1}^{\infty} a_n z^n$, $n \geq 2$. See, for example, [**11, 12, 19, 33, 69,** and **70**].

Many additional subfamilies of T have been investigated. Some involve rational expressions [**68, 92, 97, 100**] or have fixed and missing coefficients [**1, 31, 37, 95**]. Others involve integral operators or can be put in the form $|H(f,f')| < \beta$ [**25, 26, 27, 28, 64, 102**]. For properties of other subfamilies of T not previously mentioned in this survey, see [**63, 75, 89**].

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