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COVERING METRIC SPACES WITH CLOSED SETS

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1. This paper concerns a theorem [**3**] about continua originally due to W. Sierpinski. (For more discussion consult also [**2**, p. 173] and [**1**, p. 440].)

Theorem. *If a connected compact Hausdorff space has a countable cover $\{X_i\}$ by pairwise disjoint closed sets, at most one of the sets X_i is nonvoid.*

We will show (Corollary 1) that if X is a complete, connected, locally connected metric space and if X is covered by countably many proper closed sets E_n, then some two E_n must meet. Sierpinski has shown [**4**] that "locally connected" cannot be deleted here. We increase slightly the hypothesis on X (Theorem 1) and find that some two E_n must meet in at least continuum many points. This last result applies (Corollaries 2, 3 and 4) to several spaces frequently encountered in functional analysis.

2. We prove

Proposition 1. *Let X be a complete, locally connected metric space. Let X be covered by a sequence of proper closed sets (E_n) at least one of which has a nonvoid boundary. Then there exist indices i, j, $i \neq j$, such that $E_i \cap E_j$ is nonvoid.*

Proof. Let $Y = \cup_{n=1}^{\infty}$ (boundary of E_n). By hypothesis, Y is nonvoid, and Y is covered by the closed sets E_n. We claim there is a number $c > 0$ and a $y_0 \in Y$ and an index i such that E_i contains the set $Y \cap S(y_0, c)$ where $S(y_0, c)$ denotes the open ball in X with center y_0

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and radius c. This follows from Baire's theorem if Y is closed; if Y is not closed, choose $y_1 \in$ (Y closure)$\backslash Y$ and observe that y_1 lies in the interior of some E_i, and the conclusion is clear. Because X is locally connected, there is a connected neighborhood U of y_0 such that $y_0 \in U \subset S(y_0, c)$.

Now assume that the conclusion of Proposition 1 is false. Then E_i cannot contain U. For otherwise y_0 would be in the interior of E_i and also in the boundary of some E_j, $j \neq i$. So we assume, without loss of generality, that there is an index $j \neq i$, such that E_j meets U. But $y_0 \notin E_j$, and U is connected. Hence, E_j has a boundary point in U, in $Y \cap S(y_0, c)$ and in E_i. □

When we assume that X is also connected, we need not mention the nonvoid boundary in our hypothesis.

Corollary 1. *Let X be a complete, connected, locally connected metric space. Let X be covered by a sequence of proper closed sets (E_n). Then there exist i, j, $i \neq j$, such that $E_i \cap E_j$ is nonvoid.*

Proof. Because X is connected, the boundary of each E_n is nonvoid. The conclusion follows from Proposition 1. □

Sierpinski [**4**] provided an example of a complete, connected metric space that is the union of a sequence of mutually disjoint proper closed subsets. Thus, local connectedness cannot be omitted in Corollary 1. His example is a closed subspace of Euclidean 3-space, R^3.

3. We say that a topological space X is *strongly locally connected* if for each $x \in X$ and each neighborhood V of x, there exists a neighborhood U of x such that $U \subset V$ and either $U = \{x\}$ or the difference $U \backslash \{x\}$ is connected. Thus, for example, Euclidean n-space is strongly locally connected for $n > 1$, but the real line is not strongly locally connected.

It is easy to see that if X is strongly locally connected, then X is locally connected. For suppose, in the preceding paragraph, x is not an isolated point of X. Then x is in the closure of $U\backslash\{x\}$ and the union $\{x\} \cup (U\backslash\{x\}) = U$ is also a connected set. We begin with two lemmas.

Lemma 1. *Let E be a closed subset of a strongly locally connected metric space X. Then the boundary of the interior of E is a perfect set.*

Proof. We assume, without loss of generality, that this boundary is nonvoid. Let B denote the boundary of the interior of E, and let $y \in B$. Choose any $c > 0$. Because y is in the boundary of a set, y is not an isolated point of X. Hence, there is a neighborhood U of y such that $U \subset S(y,c)$, where $S(y,c)$ is the open ball with center y and radius c, and $U\backslash\{y\}$ is a connected set. Moreover, $(U\backslash\{y\}) \cap (\text{interior}\, E)$ is nonvoid because U is a neighborhood of a boundary point of interior E. But interior E cannot contain the set $U\backslash\{y\}$, for otherwise the closed set E would contain U and interior E would contain y. Because $U\backslash\{y\}$ is connected $U\backslash\{y\}$ must contain a boundary point x of interior E.

It follows that $x \in S(y,c) \cap B$. Thus the set B is dense in itself. And B is clearly closed, so B is a perfect set. □

In Lemma 2, X need not be a strongly locally connected space. For example, X could be the real line.

Lemma 2. *Let X be a complete, connected, locally connected metric space. Let X be covered by a sequence of proper closed sets (E_n) such that the boundary of the interior of each E_n is a perfect set. Then there exist indices i, j, $i \neq j$, such that $E_i \cap E_j$ contains at least continuum many points.*

Proof. By Baire's theorem, the interior of some E_n is nonvoid, and by connectedness, the boundary of this interior is nonvoid. Let $Y = \cup_{n=1}^{\infty}$ (boundary of interior E_n). Then Y is nonvoid. Again, by Baire's theorem, there is an index i, a point $y \in Y$, and a positive number c such that $E_i \supset Y \cap S(y,c)$. Now y is not an isolated point of X because y is on the boundary of a set, so there is a neighborhood U of y such that $U \subset S(y,c)$ and $U\backslash\{y\}$ is connected.

We assume, without loss of generality, that y is on the boundary of interior E_i. For otherwise there is a k ($k \neq i$) such that y is on the boundary of interior E_k and (because X is complete and this boundary is a perfect set) $S(y,c)$ contains continuum many points in the boundary of interior E_k, and all these points lie in $Y \cap S(y,c)$ and in E_i. But E_i cannot contain the set $U \backslash \{y\}$ for otherwise y would be in interior E_i.

There is a nonvoid open set $Z \subset U \backslash \{y\}$ such that $Z \cap E_i$ is void. By Baire's theorem, there is an index j ($j \neq i$) such that $Z \cap$ (interior E_j) is nonvoid. Thus, $(U \backslash \{y\}) \cap$ (interior E_j) is also nonvoid. We may assume, without loss of generality, that E_j does not contain $U \backslash \{y\}$. For if it did, interior E_j would contain continuum many of the points in the perfect set (boundary of interior E_i).

Because $U \backslash \{y\}$ is connected, $U \backslash \{y\}$ contains a boundary point of interior E_j. It follows that U contains continuum many points of the boundary of interior E_j and so do $Y \cap S(y,c)$ and E_i. □

We turn again to strongly locally connected metric spaces.

Theorem 1. *Let X be a complete, connected, strongly locally connected metric space. Let X be covered by a sequence of proper closed sets (E_n). Then there exist indices i, j ($i \neq j$) such that $E_i \cap E_j$ contains at least continuum many points.*

Proof. By Lemma 1, the boundary of the interior of each E_n is a perfect set. Moreover, X is locally connected because X is strongly locally connected. The conclusion follows from Lemma 2. □

Certain metric spaces encountered in functional analysis are strongly locally connected.

Lemma 3. *Let X be the unit sphere of a normed linear space not of dimension* 1 *or* 2. *Then X is strongly locally connected.*

Proof. By the segment joining vectors x and y in X for which $x+y \neq 0$ we mean the set of vectors $\{(tx+(1-t)y)/||tx+(1-t)y|| : 0 \leq t \leq 1\}$.

Now let $u = tx + (1-t)y$ for some t, $0 \leq t \leq 1$. Clearly, $||u|| \leq 1$. Also, $x - u/||u|| = (x-u) + u(||u||-1)/||u||$ and $||x - u/||u|| \, || \leq ||x-u|| + 1 - ||u||$. Moreover, $||u|| \geq ||x|| - ||x-u|| \geq 1 - ||x-y||$. It follows that $||x - u/||u|| \, || \leq 2||x-y||$.

Choose any number $c > 0$. Let $U = \{y \in X : ||x-v|| < c$ for all v in the segment joining x and $y\}$. It follows that any $y \in X$ for which $||x-y|| < (1/2)c$ lies in U, so U is a neighborhood of x containing the open ball $S(x,(1/2)c)$.

Now let y and z be vectors in U. Because the vector space does not have dimension 1 or 2, it follows that y and z can be joined by a connected path in U that avoids x in the ball $S(x,(1/2)c)$. Clearly, $U \backslash \{x\}$ is connected and X is strongly locally connected. □

By a *convex body* in a normed linear space we mean a convex set with nonvoid interior.

Lemma 4. *Let X be a convex body in a normed linear space not of dimension* 1. *Then X is strongly locally connected.*

Proof. The argument is much like the proof of Lemma 3 only it is easier because we do not normalize the vector u. So we leave it. □

In Corollaries 2, 3 and 4 we need not assume that X is complete.

Corollary 2. *Let X be the unit sphere of a normed linear space not of dimension* 1 *or* 2. *Let X be covered by a sequence of proper closed sets (E_n). Then there exist indices i, j $(i \neq j)$ such that $E_i \cap E_j$ contains at least continuum many points.*

Proof. Choose vectors x and y in X and let T denote the segment joining x and y as in the proof of Lemma 3. Let $z \in X$ such that $x - y$, $z - y$ are linearly independent. Let N be an index such that $T \cap E_N$ is uncountable. But E_N is closed so $T \cap E_N$ contains continuum many points. Choose $w \in X \backslash E_N$. Let X_0 be the unit sphere of the finite dimensional subspace generated by the vectors x, y, z, w. We assume, without loss of generality, that no E_i contains X_0; for if it did, then

$E_i \cap E_N$ contains continuum many points. Finally, X_0 is compact and hence complete, and the rest follows from Lemma 3 and Theorem 1. □

Corollary 3. *Let X be a convex body in a normed linear space not of dimension 1. Let X be covered by a sequence of proper closed sets (E_n). Then there exist indices i, j $(i \neq j)$ such that $E_i \cap E_j$ contains at least continuum many points.*

Proof. Choose vectors x and y in X and let T denote the set $\{tx + (1-t)y : 0 \leq t \leq 1\}$. Let $z \in X$ such that $x - y$, $z - y$ are linearly independent. Let N be an index such that $T \cap E_N$ is uncountable. But E_N is closed so $T \cap E_N$ contains continuum many points. Choose $w \in X \backslash E_N$. Let X_0 be the smallest convex set containing the vectors x, y, z, w. Then X_0 is compact and is homeomorphic to a convex body in some finite dimensional normed linear space not of dimension 1. We assume, without loss of generality, that no E_i contains X_0; for if it did, then $E_i \cap E_N$ contains continuum many points. The rest follows from Lemma 4 and Theorem 1. □

We turn now to a special kind of normed linear space, the inner product spaces.

Corollary 4. *Let X be a convex body in an inner product space not of dimension 1. Let X be covered by a sequence of closed balls (B_n). Then there exist indices i, j $(i \neq j)$ such that (interior B_i) $\cap$ (interior B_j) is nonvoid.*

It follows easily from the geometry of an inner product space, that if the balls B_i and B_j have more than one common point, then the distance between the centers of B_i and B_j is less than the sum of the radii of B_i and B_j and there interiors must meet. The proof then is an easy consequence of Corollary 3, so we leave it.

It is well to note that we cannot state Corollary 4 for general normed linear spaces. Consider the plane with the norm of (u, v) equal to $|u| + |v|$. Then the balls are just squares in the plane, and an appropriate tiling of the plane provides our counterexample.

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