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MATRIX SUMMABILITY OF CLASSES OF GEOMETRIC SEQUENCES

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ABSTRACT. Recently Fricke and Fridy [**2**] introduced the set G of complex number sequences that are dominated by a convergent geometric sequence. In this paper we define a set G_t, for any fixed t satisfying $0 < t < 1$, as the set of all the sequences which are dominated by a constant multiple of any sequence $\{s^n\}$ with $s < t$. We study the matrices which map the set G_t into another similar set G_w as well as mapping into the set G. The characterizations of such matrices are established in terms of their rows and columns. Also, several classes of well-known summability methods are investigated as mappings on G_t or into G_t.

1. Introduction. If u is a complex number sequence and $A = [a_{n,k}]$ is an infinite matrix, then Au is the sequence whose n-th term is given by

$$(Au)_n = \sum_{k=0}^{\infty} a_{nk} u_k.$$

The matrix A is called an $X - Y$ matrix if Au is in the set Y whenever u is in X. Also, if

$$\sum_{n=0}^{\infty} (Au)_n = \sum_{k=0}^{\infty} u_k$$

for each u in X, then we say that A is a sum-preserving matrix over X. In [**2**] Fricke and Fridy introduced the set G as the set of complex number sequences that are dominated by a convergent geometric sequence, and they gave characterizations of $G - l$ and $G - G$ matrices. In the present study we consider the set G_t for any fixed t satisfying $0 < t < 1$ as the set of complex number sequences of geometrical domination of order less than t, i.e.,

$$G_t = \{u : u_n = O(r^n) \text{ for some } r \in (0, t)\}.$$

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Thus, we have

$$G = \bigcup_{0<t<1} G_t.$$

Macphail [**7**, Theorem 2] established the necessary and sufficient conditions for the matrix A in order that it should be a $G_t - l$ matrix or a sum-preserving $G_t - l$ matrix. These results are listed below:

Theorem 1.1. *The matrix A is a $G_t - l$ matrix if and only if*

$$\sum_{n=0}^{\infty} |a_{nk}| = M_k < \infty, \qquad \text{for } k = 0, 1, 2, \dots , \tag{1}$$

and

$$\limsup_{k} M_k^{1/k} \le \frac{1}{t}. \tag{2}$$

Theorem 1.2. *The $G_t - l$ matrix A is sum-preserving over G_t if and only if*

$$\sum_{n=0}^{\infty} a_{nk} = 1, \qquad \text{for } k = 0, 1, 2, \dots . \tag{3}$$

In Section 2 we investigate $G_t - G$, $G_t - G_w$, and $G - G_t$ matrices, where $0 < t$, $w < 1$. Then we prove results concerning the preservation of the sums of the sequences in G_t. The third section examines $G_t - l$, $G_t - G$, $G_t - G_w$, and $G - G_t$ mapping properties of the classical summability methods of Euler-Knopp, Taylor, the extended forms of these methods, classes of Nörlund, Abel, and Borel matrices.

2. Matrix mappings of G_t into various other sets. It will be useful to have an alternative form of the definition of G_t.

The following proposition, which is easily proved, gives such a characterization.

Proposition 2.1. *A sequence u is in G_t if and only if*

$$\limsup_k |u_k|^{1/k} < t. \tag{4}$$

In order to prove a characterization of $G_t - G$ matrices, we need the following preliminary result.

Lemma 2.1. *If A is a $G_t - G$ matrix, then there is a number $r \in (0,1)$ and a positive number sequence $\{\beta_k\}$ such that for all n and k, $|a_{nk}| \leq \beta_k r^n$.*

Proof. The basis sequences are in G_t, and therefore for each k, there is an $r_k \in (0,1)$ such that

$$|a_{nk}| \leq r_k^n \qquad \text{for sufficiently large } n. \tag{5}$$

Now suppose the conclusion of the lemma is false. This implies that there is no $r \in (0,1)$ such that

$$\limsup_n |a_{nk}|^{1/n} \leq r, \qquad \text{for all } k.$$

Then $\limsup_k r_k = 1$ and for any $s \in (0,1)$, there exists an arbitrarily large k such that

$$\limsup_n |a_{nk}|^{1/n} > s. \tag{6}$$

We now choose sequences $\{s_i\}, \{k(i)\}$, and $\{n(i)\}$ as follows: Let $s_1 \in (1/2, 1)$ and choose $k(1)$ and $n(1)$ so that

$$|a_{n(1),k(1)}| > s_1^{n(1)}.$$

After selecting s_p, $k(p)$ and $n(p)$ for all $p < i$, we choose s_i, $k(i)$ and $n(i)$ as follows: Choose $s_i \in (s_{i-1}, 1)$ satisfying $s_i > r_j$ for $j \leq k(i-1)$. Next we choose $k(i) > k(i-1)$ so that

$$\limsup_n |a_{n,k(i)}|^{1/n} > s_i \tag{7}$$

and

$$\sum_{k\geq k(i)} |a_{n(i-1),k}| \left(\frac{t}{2}\right)^k \leq \frac{t}{4}\left(\frac{t}{2}\right)^{k(i-1)} |a_{n(i-1),k(i-1)}|. \tag{8}$$

This is possible because (7) follows from (6) and the hypothesis that G_t is in the domain of A implies that for each n, the power series $\sum_{k=0}^{\infty} a_{nk}z^k$ has radius of convergence at least t. Next choose $n(i) > n(i-1)$ satisfying

$$\begin{aligned} n(i) &> [k(i)]^2, \\ p^{n(i)} &\leq \frac{t^{k(i)+1}(1-t)s_i^{n(i)}}{(4)\cdot 2^{k(i)}}, \end{aligned} \tag{9}$$

where $p = \max_{j<i} r_{k(j)} < s_i$ and $|a_{n(i),k(i)}| > s_i^{n(i)}$, using (7).

Now for any $j < i$, we have

$$|a_{n(i),k(j)}| \leq [r_{k(j)}]^{n(i)} \leq p^{n(i)} \leq \frac{t^{k(i)+1}(1-t)}{(4)2^{k(i)}}|a_{n(i),k(i)}|.$$

Thus,

$$\sum_{j<i} |a_{n(i),k(j)}| \left(\frac{t}{2}\right)^{k(j)} < \frac{t^{k(i)+1}}{(4)2^{k(i)}}|a_{n(i),k(i)}|. \tag{10}$$

Now consider the sequence x given by

$$x_k = \begin{cases} \left(\frac{t}{2}\right)^{k(i)}, & \text{if } k = k(i) \text{ for } i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $x \in G_t$. If Ax were in G we would have $|(Ax)_n| \leq Hv^n$ for $v \in (0,1)$ and we could choose an $R > 1$ such that $v < 1/R < 1$ which implies that $\sum_{n=0}^{\infty} |(Ax)_n|R^n < \infty$, whence $\lim_n[|(Ax)_n|R^n] =$

0. But we shall show that this last limit is not true. Consider

$$\begin{aligned}|(Ax)_{n(i)}| &= \left|\sum_{j=1}^{\infty} a_{n(i),k(j)} \left(\frac{t}{2}\right)^{k(j)}\right| \\ &\geq \Bigg[|a_{n(i),k(i)}| \left(\frac{t}{2}\right)^{k(i)} - \sum_{j<i} |a_{n(i),k(j)}| \left(\frac{t}{2}\right)^{k(j)} \\ &\qquad - \sum_{j>i} |a_{n(i),k(j)}| \left(\frac{t}{2}\right)^{k(j)}\Bigg] \\ &> \Bigg[|a_{n(i),k(i)}| t \left(\frac{t}{2}\right)^{k(i)} - \frac{t^{k(i)+1}}{(4)2^{k(i)}} |a_{n(i),k(i)}| \\ &\qquad - \frac{t}{4}\left(\frac{t}{2}\right)^{k(i)} |a_{n(i),k(i)}|\Bigg],\end{aligned}$$

using (10) and (8)

$$> s_i^{n(i)} \left(\frac{t}{2}\right)^{k(i)} \frac{t}{2}.$$

Since $\lim_i s_i = 1$ and $R > 1$, there exists a number N such that $s_i R \geq L > 1$ for $i > N$. Thus for $i > N$,

$$\begin{aligned}|(Ax)_{n(i)}| R^{n(i)} &> (Rs_i)^{n(i)} \left(\frac{t}{2}\right)^{k(i)+1} \\ &> L^{[k(i)]^2} \left(\frac{t}{2}\right)^{k(i)+1}, \qquad \text{using (9)} \\ &= \frac{t}{2}\left[L^{k(i)} \frac{t}{2}\right]^{k(i)} \\ &> 1, \qquad \text{for sufficiently large } i.\end{aligned}$$

Hence, Ax is not in G, so A is not a $G_t - G$ matrix. □

Theorem 2.1. *The matrix A is a $G_t - G$ matrix if and only if for any $\varepsilon > 0$, there exists a constant $B = B(\varepsilon)$ and an $r \in (0,1)$, such that*

$$|a_{nk}| \leq Br^n \left(\frac{1}{t} + \varepsilon\right)^k, \qquad \text{for all } n \text{ and } k.$$

Proof. Assume A satisfies the given property. Let $u \in G_t$, say $|u_k| \leq Ms^k$, where $s \in (0,t)$. Choosing ε such that $0 < \varepsilon < (1/s) - (1/t)$, we have

$$\begin{aligned}|(Au)_n| &\leq \sum_{k=0}^{\infty} Br^n \left(\frac{1}{t}+\varepsilon\right)^k Ms^k \\ &= \frac{BM}{1-s(\frac{1}{t}+\varepsilon)} r^n,\end{aligned}$$

where $r \in (0,1)$. Hence, $Au \in G$.

Conversely, assume A is a $G_t - G$ matrix. By Lemma 2.1, there exists an $s \in (0,1)$ and a sequence $\{\beta_k\}$ satisfying

$$|a_{nk}| \leq \beta_k s^n, \qquad \text{for all } n \text{ and } k. \tag{11}$$

We may assume that $1 < \beta_k < \beta_{k+1}$ for all k. Also, for each n,

$$\limsup_k |a_{nk}|^{1/k} \leq \frac{1}{t}. \tag{12}$$

Suppose A does not satisfy the property asserted in the theorem. Then there exists an $\varepsilon > 0$ such that for every $r \in (0,1)$ and for every $B > 0$, there exist $n = n(B,r)$ and $k = k(B,r)$ satisfying

$$|a_{nk}| > Br^n \left(\frac{1}{t}+\varepsilon\right)^k.$$

Now we choose a sequence $\{r_i\}$ as follows:

$$r_1 = \frac{1+s}{2} \quad \text{and} \quad r_{i+1} = \frac{1+r_i}{2}, \qquad \text{for } i \geq 1.$$

Thus, $r_i \in (s,1)$ for all i and r_i increases to 1. For each of these r_is, we get $n(i)$ and $k(i)$ such that

$$|a_{n(i),k(i)}| > r_i^{n(i)} \left(\frac{1}{t}+\varepsilon\right)^{k(i)}.$$

We assert that $\lim_i n(i) = \infty$ and $\lim_i k(i) = \infty$. For, if not, there would be a subsequence $\{i_m\}$ of $\{i\}$ such that either $n(i_m) = c$ or $k(i_m) = d$ for all m. Then either

$$|a_{c,k(i_m)}| > r_{i_m} \left(\frac{1}{t}+\varepsilon\right)^{k(i_m)} \quad \text{or} \quad |a_{n(i_m),d}| \geq r_{i_m}^{n(i_m)} \left(\frac{1}{t}+\varepsilon\right)^{d},$$

which would contradict (12) or (11).

Now let $u = 2t/(2+\varepsilon t)$; then $\{u^k\} \in G_t$. Select $\{i_p\}$ as follows: $i_1 = 1$ and for $m \geq 1$, choose $i_{m+1} > i_m$ satisfying

$$\sum_{k=k(i_{m+1})}^{\infty} |a_{n(i_m),k}| u^k \leq s^{n(i_m)}$$

and

$$\left[\frac{2+2\varepsilon t}{2+\varepsilon t}\right]^{k(i_{m+1})} > 2(1+i_{m+1})\beta_{k(i_m)}.$$

This selection is possible because

$$\frac{2+2\varepsilon t}{2+\varepsilon t} > 1 \quad \text{and} \quad k(i_{m+1}) \geq i_{m+1}.$$

Now consider the sequence x given by

$$x_k = \begin{cases} u^k, & \text{if } k = k(i_m), \text{ for } m = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $x \in G_t$, but we shall show that Ax is not in G. Consider

$$\begin{aligned} |(Ax)_{n(i_m)}| &= \left| \sum_{j=1}^{\infty} a_{n(i_m),k(i_j)} u^{k(i_j)} \right| \\ &\geq |a_{n(i_m),k(i_m)}| u^{k(i_m)} - \sum_{j<m} |a_{n(i_m),k(i_j)}| u^{k(i_j)} \\ &\quad - \sum_{j>m} |a_{n(i_m),k(i_j)}| u^{k(i_j)} \\ &> r_{i_m}^{n(i_m)} \left[\left(\frac{1}{t}+\varepsilon\right) u\right]^{k(i_m)} - \sum_{j<m} \beta_{k(i_j)} s^{n(i_m)} u^{k(i_j)} - s^{n(i_m)} \\ &> r_{i_m}^{n(i_m)} \left[\left(\frac{2\varepsilon t+2}{2+\varepsilon t}\right)^{k(i_m)} - m\beta_{k(i_{m-1})} - 1\right] \\ &> r_{i_m}^{n(i_m)} [2(1+i_m)\beta_{k(i_{m-1})} - (m+1)\beta_{k(i_{m-1})}] \\ &> r_{i_m}^{n(i_m)}, \end{aligned}$$

because $i_m \geq m$ and $1 < \beta_k$ for all k. Since $\lim_m r_{i_m} = 1$, Ax is not in G [**2**, Proposition 1]. Hence, A is not a $G_t - G$ matrix. □

If $0 < t,\ w < 1$, then it is clear that G_w is a proper subset of G_t if and only if $w < t$. Now we investigate those matrices which, while still preserving geometrical domination, map one G_t set into another G_w set.

Theorem 2.2. *The matrix A maps G_t into G_w if and only if for any $\varepsilon > 0$, there exists a constant $B = B(\varepsilon)$ and an $r \in (0, w)$ such that*

$$|a_{nk}| \leq Br^n \left(\frac{1}{t} + \varepsilon\right)^k \qquad \text{for all } n \text{ and } k.$$

To prove Theorem 2.2 we need only to repeat the proof of Lemma 2.1 and Theorem 2.1 with the obvious changes (namely, replacing 1 by w). It is worthwhile to note that in Theorem 2.2, the value of w is independent of the value of t. Consequently, this result is true when w equals t. For convenience, we shall state the characterization of $G_t - G_t$ matrices.

Theorem 2.3. *The matrix A maps G_t into itself if and only if for any $\varepsilon > 0$, there exists a constant $B = B(\varepsilon)$ and an $r \in (0, t)$ such that*

$$|a_{nk}| \leq Br^n \left(\frac{1}{t} + \varepsilon\right)^k \qquad \text{for all } n \text{ and } k.$$

It now seems natural that we can get a similar result to characterize $G - G_w$ matrices. We state below a theorem without proof, which can be verified easily by slight modifications in the proof of the Lemma and Theorem 4 in [**2**, 573–577].

Theorem 2.4. *The matrix A is a $G - G_w$ matrix if and only if for any $\varepsilon > 0$, there exists a constant $B = B(\varepsilon)$ and an $r \in (0, w)$ such that*

$$|a_{nk}| \leq Br^n (1 + \varepsilon)^k \qquad \text{for all } n \text{ and } k.$$

In [**5**] Jacob derived similar characterizations of above matrix transformations using the topological properties of the spaces G_t and G.

It is clear that a $G_t - G$ matrix or a $G_t - G_w$ matrix is sum-preserving over G_t if and only if (3) holds (see [**2**, Theorem 2]).

3. Well-known summability mappings of G_t. On the following page in Tables 3.1 and 3.2 we have listed the necessary and sufficient conditions for different classes of well-known matrices to be a $G - l$, $G - G$, $G_t - l$, $G_t - G$, $G_t - G_w$, or a $G - G_w$ matrix. We give below outlines of the proofs of these results.

Case of Euler matrix. Theorems 3 and 4 are easily proved using [**2**, Theorem 6] and the fact that $G_t \subset G \subset l^1$. In order to see Theorem 5, let $u \in G_t$ and if $r \in [(1-w)/(1-t), 1]$, then $|(E_r u)_n| \leq M[1-r(1-s)]^n$ implying that $E_r u \in G_w$. Conversely, it suffices to show that E_r is not a $G_t - G_w$ matrix when $r \in (0, (1-w)/(1-t))$. If r lies in this interval, then for a sequence $v \in G_t$ given by $v_k = \rho^k$ where $\rho > 0$ satisfying $(r + w - 1)/r < \rho < t$, we have $|(E_r v)_n| = [1 - r + r\rho]^n > w^n$. Thus the sequence $E_r v$ is not in G_w.

When G_t gets mapped into itself, r cannot lie in the interval $(0, 1)$, for if $0 < r < 1$, then as before we have $(r + t - 1)/r < t$ and therefore repetition of the argument shows that E_r is not a $G_t - G_t$ matrix. This yields Theorem 6. Since the only Euler matrix E_r that maps G_t into itself is the identity matrix and $w < t < 1$ implies that $G_w \subset G_t \subset G$, we get Theorems 7 and 8. In [**3**, p. 116] it is shown that each column sum of the E_r matrix converges absolutely to $1/r$ provided that $r \in (0, 1]$. Thus in Theorems 3, 4, 5, and 6 the matrix rE_r is sum-preserving over G_t.

TABLE 3.1.

THEOREM	Necessary and sufficient condition to be a	Euler Matrix (Lower triangular) $E_r[n,k] = \binom{n}{k} r^k (1-r)^{n-k}$ See [**8**, p. 53]	Extended Euler Matrix (Lower triangular) $E(r_n)[n,k] = \binom{n}{k} r_n^k (1-r_n)^{n-k}$ where $r_n \in (0,1)$ See [**1**, p. 335]	Taylor Matrix (Upper triangular) $T_r[n,k] = \binom{k}{n} r^{k-n} (1-r)^{n+1}$ where r is any real number See [**8**, p. 57]	Extended Taylor Matrix (Upper triangular) $T(r_n)[n,k] = \binom{k}{n} r_n^{k-n} (1-r_n)^{n+1}$ where $r_n \in (0,1)$ See [**6**, p. 25]
1.	$G - l$	* $r \in (0,1]$	Unknown	$r \in [0,1]$	Always
2.	$G - G$	* $r \in (0,1]$	$\liminf_n r_n > 0$	$r \in [0,1]$	Always
3.	$G_t - l$	$r \in (0,1]$	Unknown	$r \in [\frac{1}{2} - \frac{1}{2t}, \frac{1}{2} + \frac{1}{2t}]$	Always
4.	$G_t - G$	$r \in (0,1]$	$\liminf_n r_n > 0$	$r \in [\frac{1}{2} - \frac{1}{2t}, \frac{1}{2} + \frac{1}{2t}]$	Always
5.	$G_t - G_w$ $(w > t)$	$r \in [\frac{1-w}{1-t}, 1]$	$\liminf_n r_n \geq \frac{1-w}{1-t}$	$r \in [\frac{t-w}{t(1+w)}, \frac{t+w}{t(1+w)}]$	Always
6.	$G_t - G_t$	$r = 1$	$\lim_n r_n = 1$	$r \in [0, \frac{2}{1+t}]$	Always
7.	$G_t - G_w$ $(w < t)$	Never	Never	$r \in [\frac{t-w}{t(1-w)}, \frac{t+w}{t(1+w)}]$	$\liminf_n r_n \geq \frac{t-w}{t(1-w)}$
8.	$G - G_w$	Never	Never	$r = 1$	$\lim_n r_n = 1$

* These results are proved in [**2**].

TABLE 3.2.

THEOREM	Necessary and sufficient condition to be a	Nörlund Matrix (Lower triangular) $N_p[n,k] = \frac{p_{n-k}}{P_n}$ where p is a nonnegative sequence, $p_0 > 0, P_n = \sum_{k=0}^n p_k$	Abel Matrix $A_v[n,k] = v_n(1-v_n)^k$ where v is a null sequence in (0,1) See [**4**, p. 86]	Extended Borel Matrix $B_\delta[n,k] = \frac{e^{-n^\delta}(n^\delta)^k}{k!}$ where δ is any real number See [**2**, p. 580]
*3.	$G_t - l$	$p \in l$	$v \in l$	$\delta > 0$
4.	$G_t - G$	$p \in G$	$v \in G$	$\delta \geq 1$
5.	$G_t - G_w$ $(w > t)$	$p \in G_w$	$v \in G_w$	$\delta > 1$, or $\delta = 1$ & $t \leq 1 + \ln w$
6.	$G_t - G_t$	$p \in G_t$	$v \in G_t$	$\delta > 1$
7.	$G_t - G_w$ $(w > t)$	Never	$v \in G_w$	$\delta > 1$
8.	$G - G_w$	Never	$v \in G_w$	$\delta > 1$

* In the cases of these three classes of matrices, Theorems 1 and 2 are proved in [**2**].

Case of extended Euler matrix. If $\liminf_n r_n = 0$, then $\limsup_n$

$|E(r_n)[n,0]|^{1/n} = 1$. Consequently, by Theorem 2.1, $E(r_n)$ is not a $G_t - G$ matrix. If $\liminf_n r_n > 0$, for any $u \in G$, it can be easily shown that $E(r_n)u \in G$ also. This yields Theorems 2 and 4. If $\liminf_n r_n \geq (1-w)/(1-t)$, then for $u \in G_t$, say $|u_k| \leq Ms^k$ for $0 < s < t$, we can find an r satisfying $(1-w)/(1-s) < r < (1-w)/(1-t)$ and $r \leq r_n$ for large n. Thus $E(r_n)u \in G_w$. Conversely, suppose $\liminf_n r_n < (1-w)/(1-t)$. Then there exists an r so that $r_n \leq r < (1-w)/(1-t)$ for infinitely many n. By a simple calculation, we get $(r+w-1)/r < t$. Thus, for the sequence $v \in G_t$ as in the proof of Theorem 5 in the case of Euler matrix, we find that $E(r_n)$ is not a $G_t - G_w$ matrix. Thus, Theorem 5 is proved.

In order to see Theorem 6, suppose $\lim_n r_n = 1$ and let $u \in G_t$. For an ε satisfying $0 < \varepsilon < t - s$, $1 - r_n < \varepsilon$ for $n \geq N$. So, $|(E(r_n)u)_n| < M(s+\varepsilon)^n$ for $n \geq N$. Conversely, if $\liminf_n r_n < 1$, we could repeat the proof of converse of Theorem 5 above with the replacement of w by t, and we could prove that $E(r_n)$ is not a $G_t - G_t$ matrix.

In Theorem 7 it is enough to prove that $E(r_n)$ is not a $G_t - G_w$ matrix when $\lim_n r_n = 1$. If $E(r_n)$ were a $G_t - G_w$ matrix, for an ε satisfying $0 < \varepsilon < (\eta/w) - (1/t)$, where $\eta \in (w/t, 1)$ we would get an $s \in (0, w)$ (Theorem 2.2) such that for all n,

$$|E(r_n)[n,n]| = r_n^n \leq Bs^n \left(\frac{1}{t} + \varepsilon\right)^n .$$

But, by the choice of ε, we have for $n \geq N$,

$$\frac{r_n}{s(\frac{1}{t}+\varepsilon)} \geq \frac{\eta}{w(\frac{1}{t}+\varepsilon)} > 1.$$

Case of Taylor matrix. When $r = 0$, T_r is the identity matrix and when $r = 1$, T_r is the zero matrix. If T_r is a $G - l$ matrix (or a $G_t - l$ matrix) using Theorem 1 in [**2**, p. 569] (or Theorem 1.1), we get for $k = 1, 2, \dots, \limsup_k M_k^{1/k} = |r| + |1-r| \leq 1$ (or $1/t$), which implies that $r \in [0,1]$ (or $r \in [\frac{1}{2} - \frac{1}{2t}, \frac{1}{2} + \frac{1}{2t}]$). Also, for $u \in G$, direct calculation shows that $T_r u \in G$, thus proving Theorems 1 and 2. For $u \in G_t$, say $|u_k| \le Ms^k$ where $s \in (0,t)$, we have

$$|(T_r u)_n| \le M \frac{|1-r|}{1-|r|s} \left[\frac{s|1-r|}{1-|r|s}\right]^n .$$

Considering three cases, namely, $r \in [1/2-(1/2t),0)$, $r \in (0,1)$, and $r \in (1,1/2+(1/2t)]$ it is easy to see that $T_r u \in G$, thus proving Theorems 3 and 4.

If $u \in G_t$, then considering the three intervals in which r can lie, namely, $[(t-w)/t(1+w),0)$, $(0,1)$, and $(1,(t+w)/t(1+w)]$, it is not difficult to get that $\limsup_n |(T_r u)_n|^{1/n} < w$. Hence, T_r is a $G_t - G_w$ matrix. In order to prove the converse of Theorem 5, suppose $r < (t-w)/t(1+w)$. Then we could choose a sequence $v \in G_t$ given by $v_k = \rho^k$ where $\rho > 0$ and $w/[1-r(1+w)] < \rho < t$. Thus $T_r v$ would not be in G_w. Similarly, if we suppose that $r > (t+w)/t(1+w)$, then we could choose a sequence $z \in G_t$ given by $z_k = \sigma^k$ where $\sigma > 0$ and $w/[r(1+w)-1] < \sigma < t$, and $T_r z$ would not be in G_w.

It is easy to verify that the proof of Theorem 5 is valid in Theorem 6 by letting $w = t$. Theorem 7 can be proved in the same method as used in Theorem 5. Also, since $\cup_{0<t<1} G_t = G$ we can get Theorem 8 by considering

$$\bigcap_{t<1} \left[\frac{t-w}{t(1-w)}, \frac{t+w}{t(1+w)}\right] = \{1\}.$$

Case of extended Taylor matrix. If $|u_k| \le Ms^k$, then

$$|T(r_n)u)_n| \le Ms^n \frac{(1-r_n)^{n+1}}{(1-r_n s)^{n+1}}.$$

So, if $u \in G$, then $T(r_n)u \in G$, and if $u \in G_t$, then $T(r_n)u \in G_t$. In order to see Theorem 7, suppose $\liminf_n r_n < (t-w)/t(1-w)$. Then we find a number r such that $0 < r < (t-w)/t(1-w)$ satisfying $r_n \le r$ for infinitely many n. Now we can choose a sequence $y \in G_t$ for which $T(r_n)y \notin G_w$. The sufficiency of the condition can be obtained by direct calculation. Since $(t-w)/t(1-w)$ increases to 1 as t approaches 1, Theorem 7 implies Theorem 8.

Case of Nörlund matrix. Theorems 3 and 4 follow from the proof of Theorem 5 in [**2**, p. 578]. If $p \in G_w$, say $|p_n| \le Br^n$ for some $r \in (0, w)$ and if $u \in G_t$, say $|u_k| \le Ms^k$ for some $s \in (0, t)$, then it is possible to consider that $r > s$. Now using the fact that $s/r < 1$, we can easily get that $N_p u \in G_w$, which implies the sufficiency of the condition in Theorem 5. The necessity follows from the fact that if N_p is a $G_t - G_w$ matrix then its first column is in G_w. In the case where $w < t$ if N_p were a $G_t - G_w$ matrix, then for $\varepsilon = (1/w) - (1/t)$, we would get, by Theorem 2.2, that for each n,

$$\frac{p_0}{P_n} \le Br^n \left(\frac{1}{t} + \varepsilon\right)^n = B\left(\frac{r}{w}\right)^n.$$

Now $r < w$ implies that $p \notin l^1$. Hence, N_p would not be a $G_t - l$ matrix, which would lead us to a contradiction.

Case of Abel matrix. The first two theorems can be obtained from [**2**, Theorem 7] and the fact that if A_v is a $G_t - l$ or a $G_t - G$ matrix, then the first column sequence is in l^1 or G. If $v \in G_w$, say $|v_n| \le Bs^n$ for some $s \in (0, w)$, then $|A_v[n, k]| \le Bs^n$ for all n and k. Now Theorem 2.2 enables us to conclude that A_v is a $G_t - G_w$ matrix. Since no relation between t and w is used here, it is clear that the result holds for all three cases, namely, $w > t$, $w = t$ and $w < t$.

Case of extended Borel matrix. We first notice that if $x \in G_t$ given by $|x_k| \le Ms^k$, then

$$|(B_\delta x)_n| \le Me^{n^\delta (s-1)} = M[e^{(s-1)n^{\delta-1}}]^n.$$

Using Theorem 8 in [**2**, p. 580] we have that if $\delta > 0$, B_δ is a $G_t - l$ matrix and if $\delta \ge 1$, B_δ is a $G_t - G$ matrix. If $\delta < 0$, then considering $y_k = s^k$ we have that $(B_\delta y)_n$ tends to 1 as $n \to \infty$, and if $\delta = 0$, then $(B_\delta y)_n$ is a constant sequence. Thus, $B_\delta y \notin l^1$. If $\delta < 1$, then $\limsup_n |(B_\delta y)_n|^{1/n} = 1$. Thus, $B_\delta y \notin G$. Hence, we get Theorems 3 and 4.

Let $u \in G_t$. If $\delta > 1$, then for large n we have

$$n^{\delta-1} \ge \frac{\ln p}{s-1} > 0,$$

where $0 < p < w < 1$. Thus, for large n, we have $|(B_\delta u)_n| \leq Mp^n$, implying that $B_\delta u \in G_w$. If $\delta = 1$, then for the sequence $z \in G_t$ given by $z_k = \rho^k$ where $\rho > 0$ and $1 + \ln t < \rho < t$, the nth term of the transformed sequence is $(B_\delta z)_n = [e^{\rho-1}]^n > t^n$. Thus, $B_\delta z \notin G_t$. Thus, we have proved Theorems 6, 7, and 8.

In case $w > t$ and $\delta = 1$, if $t > 1 + \ln w$, then we could choose z as before to get $B_\delta z \notin G_w$. Conversely, if $t \leq 1 + \ln w$ then for any $x \in G_t$, we have $s < 1 + \ln w$ and so $e^{s-1} < w$. Thus, $B_\delta x \in G_w$, yielding Theorem 5.

REFERENCES

1. R.P. Agnew, *Euler transformations*, Amer. J. Math. **66** (1944), 313–338.

2. G.H. Fricke and J.A. Fridy, *Matrix summability of geometrically dominated series*, Canad. J. Math. **39** (1987), 568–582.

3. J.A. Fridy, *Absolute summability matrices that are stronger than the identity mapping*, Proc. Amer. Math. Soc. **47** (1975), 112–118.

4. ———, *Tauberian theorems via block dominated matrices*, Pacific J. Math. **81** (1979), 81–91.

5. R.T. Jacob, *Matrix transformations involving simple sequence spaces*, Pacific J. Math. **70** (1977), 179–187.

6. J.P. King, *An extension of the Taylor summability transform*, Proc. Amer. Math. Soc. **16** (1965), 25–29.

7. M.S. Macphail, *Some theorems on absolute summability*, Canad. J. Math. **3** (1951), 386–390.

8. P.E. Powell and S.M. Shah, *Summability theory and applications*, Prentice Hall of India, New Delhi, 1988.

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