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ASYMPTOTIC ANALYSIS OF QUENCHING PROBLEMS

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Introduction. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary and let $\alpha > 0$. Consider the problem

$$(P) \begin{cases} u_t - \Delta u = -u^{-\alpha} & \text{in } (0,T) \times \Omega, \\ u = 1 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Here $0 < u_0(x) \leq 1$ is assumed throughout the paper. It is well known that for sufficiently large domains Ω the solution can approach zero in finite time, see [**10,2**]. This phenomenon is called quenching and throughout this paper we assume that u quenches at time $T < \infty$.

It was furthermore shown that $u_t \to -\infty$ as $u \to 0$, see [**10,5,1,6**].

In the present paper we derive some asymptotic estimates for u near the point $(T, 0)$ in which u is supposed to quench. They will be of the type

$$\min_{x \in \Omega} u(t,x) \leq [(1+\alpha)(T-t)]^{1/(1+\alpha)}, \tag{1}$$

$$u(t,x) \geq C_1 (T-t)^{1/(1+\alpha)}, \tag{2}$$

$$u(T,r) \leq C_2 r^{2/(1+\alpha)} \qquad \text{for } \alpha < 1, \tag{3}$$

$$u(t,r) \geq C_3 r^{2/(1+\gamma)} \qquad \text{for } 0 < \gamma < \alpha. \tag{4}$$

Notice that (2) implies the blow up of u_t at quenching. Therefore, (1) and (2) give us the rate at which $u_t \to -\infty$. Notice further that (3) implies $u_r(T, 0) = 0$ for $0 < \alpha < 1$, see also Remark 2.10. (3) and (4) will be derived only in a radial situation where Ω is a ball and u_0 radially symmetric and for (3) $u_0 \equiv 1$. A consequence of (4) is the fact that for $n \geq 2$ or $\alpha < 3$

$$||1 - u(t, \cdot)||_{H_0^1(\Omega)} \leq C_4 \quad \text{for any } t \in (0,T),$$

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so that u does not blow up in $H^1(\Omega)$ as $t \to T^-$. For $\alpha \in (0,1)$ one can establish this result without using (4) and for general domains Ω, see Theorem 3.1.

Moreover, we are interested in the behavior of $\int_\Omega u^{-\alpha}\,dx$, or even of

$$I_\lambda(t) = \int_\Omega u^{-\lambda}(t,x)\,dx \tag{5}$$

as $t \to T^-$, where $\lambda > 0$. It turns out that there exists a number $\lambda^* = (n/2)(1+\alpha)$ such that $I_\lambda(t)$ blows up for $\lambda > \lambda^*$ and remains bounded in time for $\lambda < \lambda^*$. More precisely, if $\Omega \subset \mathbf{R}^n$ is pseudoconvex, i.e., if the mean curvature of $\partial\Omega$ is nonnegative, and if $\lambda > \lambda^*$, then

$$I_\lambda(t) \to \infty \quad \text{as } t \to T^-. \tag{6}$$

But if $\Omega \subset \mathbf{R}^n$ is a ball and u_0 satisfies certain assumptions (see Theorem 3.1), then for $\lambda < \lambda^*$

$$I_\lambda(t) \le C_5 < \infty \quad \text{as } t \to T^-,$$

while for $\lambda = \lambda^*$, $0 < \alpha < 1$ and $u_0 \equiv 1$

$$I_\lambda(T) = \infty.$$

The proofs of these results are based on the maximum principle and were partly inspired by papers of Friedman and McLeod [**7**] and Bebernes, Bressan and Lacey [**4**] on blow up problems. Since blow up and quenching problems are essentially of the same nature (see [**1, 11**]), it is not surprising to find out that they are amenable to similar techniques. We point out that the estimate (2) is implicitly contained in the work of Deng and Levine [**6**] on quenching.

After this paper had been submitted for publication, H. Levine kindly pointed out the work of Guo [**8,9**] to us. In terms of our notation, Guo obtained the following results: In [**8**] Guo shows for $n = 1$ and $\alpha \ge 3$ and for fairly general initial data u_0 that

$$\lim_{t\to T} u(t,x)(T-t)^{-\gamma} = k \tag{$*$}$$

uniformly on the parabolic domains $|x| \le C\sqrt{T-t}$. Here $\gamma = 1/(1+\alpha)$, $k = k(\gamma, u_0)$ and C is any positive constant. Notice that $(*)$ is stronger than our Theorem 1.2. In [**9**] Guo derives (∗) for balls in $\mathbf{R}^n$, $n \geq 2$, of radius $R^2 > 2\gamma(\gamma + n - 2)$ for $u_0 \equiv 1$ and for $\alpha > 1$. In contrast to Guo's result, we have weaker assumptions in Theorem 1.2 (convex Ω and more general u_0) and we obtain a weaker statement than (∗).

1. Time decay. In this section we establish the estimates (1) and (2). By definition, a point $x \in \Omega$ is called *quenching point* if there exists a sequence $(t_m, x_m) \in (0, T) \times \Omega$ such that $t_m \to T^-$, $x_m \to x$ and $u(t_m, x_m) \to 0$ as $m \to \infty$.

Lemma 1.1. *Suppose that Ω is convex. Then the set of quenching points lies in a compact subset of Ω.*

We refer to [**6**] for a proof.

Theorem 1.2.

a) $\min_{x\in\Omega} u(t,x) \leq [(1+\alpha)(T-t)]^{1/(1+\alpha)}$ *for* $0 \leq t < T$.

b) *Suppose that $\Delta u_0 - u_0^{-\alpha} \leq 0$ in Ω and that Ω is convex. Then there exists a positive constant C depending on u_0 such that*

$$u(t,x) \geq C(T-t)^{1/(1+\alpha)} \quad \textit{for } 0 < t < T \textit{ and } x \in \Omega.$$

Proof. It is easy to see that $\underline{u}(t) := \min_{x\in\Omega} u(t,x)$ is locally Lipschitz and $\underline{u}' \geq -\underline{u}^{-\alpha}$ for a.e. $t \in (0,T)$. Integrating the inequality $(1/(1+\alpha))(\underline{u}^{-\alpha})' \geq -1$ from t to T we obtain statement a) (cf. Theorem 4.5 in [**7**]).

To prove b) we observe that Lemma 1.1 implies the existence of a constant $\eta > 0$ such that $\Omega^\eta := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \eta\}$ contains all quenching points. For δ sufficiently small it can be shown that

$$\tilde{J}(t,x) := u_t + \delta u^{-\alpha} \leq 0 \quad \text{in } (\eta, T) \times \Omega^\eta$$

as in the proofs [**7**, Lemma 4.1, Theorem 4.2]; see [**6**, proof of Theorem 3.1] for details. This completes the proof of Theorem 1.2. □

Remark 1.3. Theorem 1.2a gives a simple lower bound for the quenching time T, namely

$$T \geq \frac{1}{1+\alpha} \min_{x\in\Omega}(u_0(x))^{1+\alpha}.$$

2. Spatial asymptotics at $t = T$. Let us introduce some notation. We denote the spatial gradient $(\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ of u by ∇u and its components by u_i, $i = 1, \dots, n$. Correspondingly, u_{ij} denotes $\partial^2 u/\partial x_i \partial x_j$. Furthermore, we introduce the functions

$$f(u) = -u^{-\alpha}, \qquad \alpha > 0, \tag{2.1}$$

$$\tilde{f}(u) = \begin{cases} \frac{-1}{1-\alpha} u^{1-\alpha} & \text{if } 0 < \alpha < 1, \\ \ln u & \text{if } \alpha = 1, \\ \frac{1}{1-\alpha} u^{1-\alpha} & \text{if } \alpha > 1, \end{cases} \tag{2.2}$$

and

$$P(t,x) = \frac{1}{2}|\nabla u|^2 + \tilde{f}(u(t,x)) \tag{2.3}$$

for $(t,x) \in (0,T) \times \Omega$. If $\alpha \geq 1$, a straightforward calculation shows that

$$P_t - \Delta P = 2\alpha u^{-\alpha-1}|\nabla u|^2 - f^2(u) - \sum_{i,j=1}^{n} u_{ij}^2 \quad \text{in } (0,T) \times \Omega, \tag{2.4}$$

so that we cannot infer anything about P from the parabolic maximum principle. If, however, $\alpha < 1$, we calculate

$$P_t - \Delta P = f^2(u) - \sum_{i,j=1}^{n} u_{ij}^2 \quad \text{in } (0,T) \times \Omega \tag{2.5}$$

and observe that

$$\sum_{i=1}^{n}(P_i - f(u)u_i)^2 = \sum_{i,j=1}^{n} (u_j u_{ij})^2 \leq |\nabla u|^2 \sum_{i,j=1}^{n} u_{ij}^2. \tag{2.6}$$

A combination of (2.5) and (2.6) gives

$$P_t - \Delta P \le \vec{b} \cdot \nabla P,$$

where $\vec{b} = |\nabla u|^{-2}(2f(u)\nabla u - \nabla P)$ is locally bounded in $((0,T) \times \Omega)\backslash\{(t,x)|\nabla u(t,x) = 0\}$. Therefore, P can only attain a maximum in a point where $\nabla u = 0$ or on the parabolic boundary of $(0,T) \times \Omega$. But in points where $\nabla u = 0$ we have $P \le 0$ so that a positive maximum cannot occur in those points.

In summary, we have shown that for $\alpha < 1$ any positive maximum of $P(t,x)$ will have to be attained on the parabolic boundary of $(0,T)\times\Omega$.

Remark 2.1. The usefulness of the P-function for semilinear (and quasilinear) elliptic and parabolic problems is demonstrated in the book of Sperb [**15**] in great detail.

We shall now restrict the set of points in which P attains a positive maximum even further. We need the following definition: A domain $\Omega \subset \mathbf{R}^n$ is called *pseudoconvex* if $\partial\Omega \in C^{3+\gamma}$ for some $\gamma \in (0,1)$ and if the mean curvature $H(x)$ of $\partial\Omega$ is nonnegative.

Lemma 2.2. *If $\alpha < 1$ and if Ω is pseudoconvex, then $P(t,x)$ attains any positive maximum initially. Thus, if $P(0,x) \le 0$ in Ω, then $P(t,x) \le 0$ in $(0,T) \times \Omega$.*

Proof. By contradiction, suppose that there is a positive time t_0 and a point $x_0 \in \partial\Omega$ such that P attains a positive maximum in (t_0, x_0). Then by Hopf's second lemma,

$$\frac{\partial P}{\partial \nu}(t_0, x_0) > 0, \tag{2.7}$$

where ν denotes the exterior normal to $\partial\Omega$. But (2.7) can be rewritten as

$$u_\nu u_{\nu\nu} + f(u)u_\nu = (u_{\nu\nu} + f(u))u_\nu > 0. \tag{2.8}$$

If we write the differential equation for u in curvilinear coordinates on the boundary, we obtain

$$-u_{\nu\nu} - (n-1)H(x_0)u_\nu = f(u). \tag{2.9}$$

But (2.9) implies a contradiction to (2.8), i.e.,

$$(u_{\nu\nu} + f(u))u_\nu = -(n-1)H(x_0)u_\nu^2 \le 0. \qquad \square$$

Remark 2.3. If $P \le 0$ in $(0,T) \times \Omega$, then

$$\frac{\partial}{\partial \nu} u^{(1+\alpha)/2} \le \frac{1+\alpha}{\sqrt{2(1-\alpha)}}, \tag{2.10}$$

a statement which can be used to get upper bounds on u at the quenching time. This will be done in the following theorem.

Theorem 2.4. *Let $\Omega = B_R(0)$, $0 < \alpha < 1$ and $u_0 \equiv 1$. then*

$$u(T,r) \le C_\alpha r^{2/(1+\alpha)}, \tag{2.11}$$

where

$$C_\alpha = (1+\alpha)^{2/(1+\alpha)}[2(1-\alpha)]^{-1/(1+\alpha)}. \tag{2.12}$$

Moreover,

$$u_r(T,0) = 0. \tag{2.13}$$

Proof. Since $\Delta u_0 - u_0^{-\alpha} \le 0$ in Ω, the solution u is nonincreasing in t; hence, there is a pointwise limit $u(T,\cdot)$. From Lemma 2.2 we have $P(t,x) \le 0$ in $(0,T) \times \Omega$. Now (2.11) follows from an integration of (2.10) and from the fact that u quenches in $(T,0)$, because $u_r \ge 0$ in $(0,T) \times \Omega$. Finally, (2.13) follows from (2.11) because

$$\frac{1}{r}\{u(T,r) - u(T,0)\} \le C_\alpha r^{(1-\alpha)/(1+\alpha)}. \qquad \square$$

Remark 2.5. Theorem 2.4 holds also for slightly more general initial data. We used the assumptions

$$u_0 = u_0(r), \qquad \frac{\partial}{\partial r} u_0(r) \ge 0 \quad \text{in } \Omega, \tag{2.14}$$

$$\frac{\partial^2}{\partial r^2} u_0 + \frac{n-1}{r}\frac{\partial}{\partial r} u_0 - u_0^{-\alpha} \le 0 \quad \text{in } \Omega, \tag{2.15}$$

and $P(0,r) \leq 0$. These assumptions are satisfied, e.g., for

$$u_0(r) = 1 - \delta R^2 + \delta r^2$$

provided $\delta > 0$ is sufficiently small.

Remark 2.6. Theorem 2.4 makes no statement about the limiting case $\alpha \to 1^-$, since C_α tends to $+\infty$ as $\alpha \to 1^-$.

It is the purpose of the following considerations to derive a lower bound for u which complements the one given by (2.11). To this end, we follow an idea of Friedman and McLeod [**7**]. We set $w = r^{n-1}u_r$ and $J(t,r) = w + c(r)F(u)$ with c and F to be determined later. This Ansatz leads to the differential equation

$$J_t + \frac{n-1}{r}J_r - J_{rr} = B \quad \text{in } (0,T) \times \Omega \tag{2.16}$$

with

$$\begin{aligned} B = {} & f'(u)w + cF'f + \frac{2(n-1)}{r}cF'u_r \\ & + \frac{n-1}{r}c'F - cF''u_r^2 - 2c'F'u_r - c''F. \end{aligned} \tag{2.17}$$

We use $u_r = wr^{1-n}$ and $w = -cF + J$ to find out that

$$\begin{aligned} B = {} & bJ - c(f'F - fF') - \frac{c^3}{r^{2n-2}}F'F^2 + \frac{2cc'}{r^{n-1}}F'F \\ & + \frac{n-1}{r}c'F - 2\frac{n-1}{r^n}c^2F'F - c''F, \end{aligned}$$

where b is bounded for $0 < r < R$.

We intend to show that $B - bJ \geq 0$, since then there is hope to derive $J \geq 0$ via the maximum principle as well. This is the reason why we pick now $c(r) = -\varepsilon r^n$ and $F(u) = u^{-\gamma}$ with $\gamma > 0$. Then $B - bJ \geq 0$ can be established, provided that

$$(\gamma - \alpha)u^{\gamma-\alpha} \leq -2\varepsilon n\gamma \tag{2.18}$$

holds. Notice that the other terms in $B - bJ$ remain nonnegative because the signs of c, F and their derivatives are under control. Property (2.18) can be satisfied only if $\gamma < \alpha$ and, since $u \leq 1$, if

$$\varepsilon \leq \frac{\alpha - \gamma}{2n\gamma}. \tag{2.19}$$

Under assumption (2.19), the function J can only attain a negative minimum for $t = 0$ or $r = 0$ or $r = R$. But $J(t, 0) = 0$ for $t \in (0, T)$. We also calculate

$$J_r(t, r) = r^{n-1}(u_t - f(u)) - \varepsilon n r^{n-1} u^{-\gamma} + \varepsilon \gamma r^n u^{-\gamma-1}$$

so that $J_r(t, R) \geq 0$ as long as

$$\varepsilon \leq \frac{1}{n}. \tag{2.20}$$

So, under conditions (2.19) and (2.20), the function J can only attain a negative minimum initially.

Theorem 2.7. *Let $\Omega = B_R(0)$, and suppose that the initial data u_0 satisfy $J(0, r) \geq 0$, i.e.,*

$$u_0 = u_0(r) \quad \textit{and} \quad \frac{\partial}{\partial r} u_0 \geq \varepsilon r u_0^{-\gamma} \quad \textit{in } \Omega, \tag{2.21}$$

where $\gamma < \alpha$ and $\varepsilon \leq \min\{(\alpha - \gamma)/2n\gamma, 1/n\}$. Then

$$u(t, r) \geq C_{\gamma,\varepsilon} r^{2/(1+\gamma)} \quad \textit{in } (0, T) \times \Omega, \tag{2.22}$$

and

$$C_{\gamma,\varepsilon} = \left[\frac{(\gamma + 1)\varepsilon}{2}\right]^{1/(1+\gamma)}. \tag{2.23}$$

Proof. By the maximum principle $J(t, x) \geq 0$ in $(0, T) \times \Omega$, i.e., $u_r \geq \varepsilon r u^{-\gamma}$ in $(0, T) \times \Omega$. A simple integration gives $(1/(\gamma + 1))u^{\gamma+1}(t, r) \geq \varepsilon(r^2/2)$ and the proof is complete. □

Remark 2.8. Assumption (2.21) can be verified for initial data of the type

$$u_0(r) = 1 - \delta R^2 + \delta r^2 \tag{2.24}$$

provided $\delta \geq \varepsilon/2$, cf. Remark 2.5.

Remark 2.9. Notice that in (2.22) the limit $\gamma \to \alpha^-$ does not give any new information since ε, and thus $C_{\gamma,\varepsilon} \to 0$ as $(\alpha - \gamma) \to 0$.

Remark 2.10. If $\alpha > 1$ and if $u \in C([0,T] \times \Omega)$, then (2.22) implies that $u_r(T, 0) = +\infty$.

3. Integral estimates. In the present section we shall investigate integrals like

$$\int_\Omega |\nabla u|^2 \, dx \quad \text{or} \quad \int_\Omega u^{-\lambda} \, dx \quad \text{for } \lambda > 0 \quad \text{as } t \to T^-.$$

Theorem 3.1. *Let u be a solution of* (P) *which quenches at time T, and suppose that* $(1 - u_0) \in H_0^1(\Omega)$.

a) *If $\alpha < 1$, then there exists a constant M depending on u_0 and $|\Omega|$ such that*

$$||u(t, \cdot)||_{H^1(\Omega)} \leq M \quad \textit{for } t \in (0, T). \tag{3.1}$$

b) *If $\alpha \in (0,3)$ and $n = 1$, or if $n \geq 2$ and $\alpha > 0$, if $\Omega = B_R(0)$ and if u_0 satisfies the assumptions of Theorem* 2.7, *then there exists a constant M depending on u_0, R and $C_{\gamma,\varepsilon}$, such that* (3.1) *holds.*

c) *If $\alpha > 3$, $n = 1$ $(\Omega = (-R, R))$ and if u_0 satisfies the assumptions of Theorem* 2.7, *then $\int_\Omega |u_x|^2 \, dx \to \infty$ as $t \to T^-$.*

Proof. We recall that the Liapunov functional

$$V(t) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \left(\int_1^u f(w) \, dw \right) dx$$

is nonincreasing in time, since $V'(t) = -\int_\Omega u_t^2 \, dx \leq 0$. Therefore,

$$\int_\Omega |\nabla u|^2 \, dx \leq 2V(0) + 2 \int_\Omega \left(\int_1^u f(w) \, dw \right) dx. \tag{3.2}$$

But if $\alpha \neq 1$, we have

$$\int_\Omega \left(\int_1^u f(w)\,dw \right) dx = \frac{-1}{1-\alpha} \int_\Omega (u^{1-\alpha} - 1)\,dx, \tag{3.3}$$

and for $\alpha < 1$, this expression is always bounded by $1/(1-\alpha)$. This proves statement a) for $\alpha < 1$.

To prove part b) for $\alpha \neq 1$, we use (2.22) to conclude that (3.3) is bounded as long as $2(\alpha - 1) < n(1+\gamma)$, i.e., for $n \geq 2$ and $\alpha > 0$ or for $n = 1$ and $0 < \alpha < 3$.

For $\alpha = 1$, one sees that

$$\int_\Omega \left(\int_1^u f(w)\,dw \right) dx = \int_\Omega - \ln u\,dx \leq - \ln C_{\gamma,\varepsilon} - \frac{2}{\gamma+1} \int_\Omega \ln r\,dr \leq M,$$

which concludes the proof of b).

Finally, we prove c) by contradiction. Assume that there is a sequence $t_n \to T^-$ such that $u(t_n, \cdot)$ is bounded in $H^1(\Omega)$. Then, after passing to a subsequence, $u(t_n, \cdot)$ converges weakly in $H^1(\Omega)$ to some function v. By Theorem 2.7, $v \geq C_{\gamma,\varepsilon} x^{2/(1+\gamma)}$ in $(-R, R)$ for any $\gamma < \alpha$. This is a contradiction, since v is in $H^1(\Omega)$ which is embedded in $C^\beta(\Omega)$ if $\beta < 1/2$. □

In the remainder of this section we shall consider

$$I_\lambda(t) = \int_\Omega u^{-\lambda}(t,x)\,dx.$$

Under the assumptions of Theorem 2.7, we know that $u(t,x) \geq C_{\gamma,\varepsilon} r^{2/(1+\gamma)}$, so that

$$u^{-\lambda}(t,x) \leq C_{\gamma,\varepsilon}^{-\lambda} r^{-2\lambda/(1+\gamma)}. \tag{3.4}$$

But the right-hand side of (3.4) is integrable near zero as long as $n > 2\lambda/(1+\gamma)$ or $\lambda < (n/2)(1+\gamma)$. This proves (3.5) below while (3.6) is a simple consequence of (2.11).

Corollary 3.2. *Under the assumptions of Theorem* 2.7*,*

$$\overline{\lim_{t \to T^-}}\, I_\lambda(t) < \infty \tag{3.5}$$

for $\lambda < (n/2)(1+\alpha)$, *and under the assumptions of Theorem* 2.4,

$$I_\lambda(T) = \infty \tag{3.6}$$

for $\lambda \geq (n/2)(1+\alpha)$.

Note that Corollary 3.2 does not apply to Kawarada's original problem in which $\lambda = \alpha = n = 1$. Moreover, Corollary 3.2 is based on radial symmetry assumptions. In the following theorem we shall treat more general, namely pseudoconvex domains Ω. For this purpose, we introduce the functions

$$g(u) = \int_\mu^1 2f(s)\, ds \tag{3.7}$$

and

$$\underline{u}(t_0) = \min_{x\in\Omega} u(t_0, x). \tag{3.8}$$

Lemma 3.3. *Let* $\Omega \subset \mathbf{R}^n$ *be pseudoconvex. Suppose that for some* $t_0 \in (0, T)$

$$-g(\underline{u}) \geq -g(u_0) + \frac{1}{2}|\nabla u_0|^2 \quad \text{in } \Omega \tag{3.9}$$

holds. Then

$$\frac{1}{2}|\nabla u|^2 \leq -g(\underline{u}) + g(u) \quad \text{in } (0, t_0) \times \Omega. \tag{3.10}$$

Proof. For the proof, which is a modification of the proof of Theorem 3.1 in [**7**], we fix t_0 and treat $\underline{u}(t_0)$ as a fixed constant. Then the function $w = u - \underline{u}$ satisfies

$$\begin{aligned} &w_t - \Delta w = f(w + \underline{u}) =: h(w) && \text{in } (0, t_0) \times \Omega,\\ &w = u - \underline{u} && \text{on } (0, t_0) \times \partial\Omega,\\ &w(0, x) = u_0(x) - \underline{u} && \text{in } \Omega. \end{aligned}$$

We set $H(w) = \int_0^w h(s)\,ds$ and introduce

$$\tilde{P} = \frac{1}{2}|\nabla w|^2 + H(w).$$

Then, as in the proof of Lemma 2.1, we obtain

$$\tilde{P}_t - \nabla\tilde{P} \le b \cdot \nabla\tilde{P} \quad \text{in } (0, t_0) \times \Omega. \tag{3.11}$$

Here b is bounded where ∇u is positive. So $\tilde{P}$ takes a positive maximum either where $\nabla u = 0$, but $\tilde{P} \le 0$, or on the parabolic boundary of $(0, t_0) \times \Omega$. But initially, $\tilde{P} \le 0$ by (3.9) since

$$H(u_0 - \underline{u}) = g(\underline{u}) - g(u_0). \tag{3.12}$$

And, on the later boundary, we have, using curvilinear coordinates

$$\frac{\partial\tilde{P}}{\partial\nu} = w_\nu h(w) - (n-1)H(x)w_\nu^2 \le u_\nu h(w) \le 0.$$

Thus, by the maximum principle, it follows that $\tilde{P} \le 0$ in $(0, t_0) \times \Omega$. Replacing u_0 in (3.12) by u, one sees that

$$\tilde{P} = \frac{1}{2}|\nabla w|^2 + H(u - \underline{u}) = \frac{1}{2}|\nabla u|^2 + g(\underline{u}) - g(u),$$

so that (3.10) follows and the proof is complete. □

What about assumption (3.9)? For $\alpha \ge 1$ and t_0 close to the quenching time T, we can verify (3.9) because

i) for $\alpha = 1$, we have $g(u) = \ln u \to -\infty$, and

ii) for $\alpha > 1$, we have $g(u) = 1/(\alpha - 1)(1 - u^{1-\alpha}) \to -\infty$, as $u \to 0$.

So we can establish (3.10) for $\alpha \ge 1$ and proceed further from there.

In the sequel we distinguish the cases $\alpha = 1$ and $\alpha > 1$. If $\alpha = 1$, then (3.10) reads

$$\frac{1}{2}|\nabla u|^2 \le -\ln\underline{u} + \ln u, \tag{3.13}$$

and by convexity of the function $-\ln u$, we have

$$\frac{1}{2}|\nabla u|^2 \le -\frac{1}{u}(\underline{u} - u) = \frac{1}{u}(u - \underline{u}). \tag{3.14}$$

Let $\underline{u} = u(t_0, x_0) = \min_{x\in\Omega} u(t_0, x)$ and introduce polar coordinates (r, θ) about x_0. In any direction θ there is a smallest value of r, $r = r_0(\theta)$ say, such that $u(r, \theta) = 2\underline{u}$. Because of (3.14) and the definition of $\underline{u}$, we know that $u_r^2 \le (2/\underline{u})(u - \underline{u})$ or

$$\frac{u_r}{\sqrt{u-\underline{u}}} \le \sqrt{\frac{2}{\underline{u}}}. \tag{3.15}$$

By integration, $2\sqrt{u-\underline{u}} \le \sqrt{(2/\underline{u})}r$, and taking $r = r_0(\theta)$, we obtain

$$r_0(\theta) \ge \sqrt{2\underline{u}}. \tag{3.16}$$

Therefore,

$$\begin{aligned}\int_\Omega u^{-\lambda}(t,x)\,dx &\ge \int_\theta ds_\theta \int_{\{r<r_0(\theta)\}} u^{-\lambda} r^{n-1}\,dr \\ &\ge \int_\theta ds_\theta \int_{\{r<r_0(\theta)\}} (2\underline{u})^{-\lambda} r^{n-1}\,dr \\ &= \int_\theta ds_\theta (2\underline{u})^{-\lambda}\cdot\frac{1}{n}r_0^n \ge 2^{(n/2)-\lambda}\cdot n^{-1}\int_\theta ds_\theta\cdot \underline{u}^{n-\lambda},\end{aligned}$$

and as $\underline{u} \to 0$, the last factor blows up provided $\lambda > n$. But $\underline{u} \to 0$ as $t \to T^-$ according to Theorem 1.2a. This settles the case $\alpha = 1$.

If $\alpha > 1$, then (3.13) is replaced by

$$\frac{1}{2}|\nabla u|^2 \le \frac{1}{\alpha-1}(\underline{u}^{1-\alpha} - u^{1-\alpha}), \tag{3.17}$$

and by convexity of the function $(1/(\alpha-1))u^{1-\alpha}$, we have

$$\frac{1}{2}|\nabla u|^2 \le \underline{u}^{-\alpha}(u - \underline{u}). \tag{3.18}$$

Proceeding as in the case $\alpha = 1$, we arrive at

$$\frac{u_r}{\sqrt{u-\underline{u}}} \le \sqrt{2}\underline{u}^{-\alpha/2} \tag{3.19}$$

and after integration at $2\sqrt{u-\underline{u}} \le 2^{1/2} u^{-\alpha/2} r$, so that analogously to (3.16) we obtain

$$r_0(\theta) \ge \sqrt{2}\underline{u}^{(1+\alpha)/2}. \tag{3.20}$$

Consequently,

$$\int_\Omega u^{-\lambda}(t,x)\,dx \ge 2^{-\lambda+n/2} \cdot n^{-1} \int_\theta ds_\theta \cdot \underline{u}^{-\lambda+n(1+\alpha)/2},$$

and the last factor goes to $+\infty$ as $u \to 0$ for $\lambda > (n/2)(1+\alpha)$. Again, recall that $\underline{u}(t) \to 0$ as $t \to T^-$ (Theorem 1.2a). In summary, we have shown the following

Theorem 3.4. *Let $\Omega \subset \mathbf{R}^n$ be pseudoconvex and $\alpha \ge 1$. Then for $\lambda > (n/2)(1+\alpha)$, we have*

$$\lim_{t \to T^-} I_\lambda(t) = +\infty.$$

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