ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

NEUTRAL GEOMETRY AND THE GAUSS-BONNET THEOREM FOR TWO-DIMENSIONAL PSEUDO-RIEMANNIAN MANIFOLDS

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1. Introduction. The Gauss-Bonnet theorem was first extended to pseudo-Riemannian manifolds by Avez [**1**] and Chern [**3**]. These authors produced a global Gauss-Bonnet theorem. For example, Chern [**3**] considers oriented pseudo-Riemannian vector bundles of even rank over compact manifolds and interprets the Gauss-Bonnet formula as the assertion that the relevant curvature form (that which appears as the integrand in the Gauss-Bonnet formula) equals the Euler class of the bundle. This is now the standard abstract formulation of the generalized Gauss-Bonnet theorem, though usually stated only for the Riemannian case (cf., e.g., Milnor and Stasheff [**7**]). For the tangent bundle of a compact, oriented, pseudo-Riemannian manifold, this statement reduces to the usual Gauss-Bonnet result.

The obvious elegance of this global Gauss-Bonnet-Chern theorem does not preclude interest in a pseudo-Riemannian version of the classical Gauss-Bonnet formula for a two-dimensional domain D with piecewise smooth boundary Γ:

$$\int_D K\,dV + \int_\Gamma k_g\,ds + \sum \theta_{\text{exterior}} = 2\pi \tag{1.1}$$

where K is the Gaussian curvature of some metric on D, k_g the geodesic curvature, and θ_{exterior} the exterior angle at a nonsmooth point of Γ. It is fairly straightforward to carry over the differential-geometric aspects of a proof of this result to the pseudo-Riemannian context. The only two-dimensional indefinite signature is Lorentzian, and the essential difference that occur between the Riemannian and Lorentzian versions of the Gauss-Bonnet formula arise from the differences between the corresponding orientation-preserving isometry groups $\mathbf{SO}\,(\mathbf{2})$ and $\mathbf{SO}\,(\mathbf{1},\mathbf{1})$. In particular, these groups essentially determine the relevant notion of angle and hence θ_{exterior}.

Received by the editors on January 9, 1990.

In the Lorentzian case, it is necessary to assume that the smooth segments of Γ are nonnull and, hence, either time-like or space-like. Although it is convenient for the differential geometry to work with the bundle of unit time-like vectors, say, this creates no difficulty on the space-like segments of Γ since for each unit space-like vector $\mathbf{x}$ there exists a unique unit time-like vector t orthogonal to x and such that $\{t, x\}$ has a prescribed orientation. The obstruction to obtaining a Gauss-Bonnet formula when Γ contains both time-like and space-like segments arises from the problem of how to measure the exterior angles where a time-like and space-like segment meet. If u and v are unit vectors in the Minkowski plane $\mathbf{R}^{1,1}$, i.e., each has squared norm plus or minus one, then the angle between u and v is to be determined by the geometric quantity $g(u, v)$, where g is the two-dimensional Minkowski metric. If u and v are both time-like or both space-like, then $u = L(v)$ for some unique L in $\mathbf{SO}\,(\mathbf{1}, \mathbf{1})$. Assuming an orientation specified for $\mathbf{R}^{1,1}$ so that oriented angle may be introduced, choose an oriented pseudo-orthonormal basis $\{\mathbf{t}, \mathbf{x}\}$. With respect to this basis, if

$$\underline{\mathrm{L}} = \pm \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \tag{1.2}$$

write L as $\pm L(\beta)$. The real number β is independent of the choice of oriented pseudo-orthonormal basis. For the moment, suppose $L = L(\beta)$. Then $g(u, v) = g(L(\beta)(v), v) = \pm\cosh\beta$, according to whether u and v are both time-like or both space-like. Define the oriented angle from v to u to be β. This approach to defining angle has an obvious geometric appeal. The differences between the Euclidean and Minkowskian cases may then be viewed in terms of the topologies of $\mathbf{SO}\,(\mathbf{2})$ and $\mathbf{SO}\,(\mathbf{1}, \mathbf{1})$. Although the Minkowskian case is more complicated due to the disconnectedness of $\mathbf{SO}\,(\mathbf{1}, \mathbf{1})$ (this does not create a serious difficulty however), from another point of view the Minkowskian case may be regarded as simpler because the identity-connected component of $\mathbf{SO}\,(\mathbf{1}, \mathbf{1})$ is topologically simpler than $\mathbf{SO}\,(\mathbf{2})$. The serious difficulty with the Minkowski case, however, is the fact that isometries cannot carry space-like vectors to time-like vectors or vice versa.

Birman and Nomizu [**2**] appear to be the first to have considered a Lorentzian version of the classical Gauss-Bonnet theorem. They assumed that Γ consisted only of time-like segments and so the above problem did not arise. With a suitable choice of conventions, the result they obtain is the formula of equation (1.1) but with zero on the right-hand side. This is a natural result given the restriction to time-like boundary segments. Dzan [**4**], however, contains a formalism which produces a closer analogue of equation (1.1) and requires only that the segments of Γ be nonnull. This is achieved by mimicking formally what one does in the Riemannian case. Thus, Dzan employs the Euclidean notion of angle as

$$\theta = \cos^{-1}\{g(u,v)/\sqrt{g(u,u)}\sqrt{g(v,v)}\}$$

by allowing the norm $\sqrt{g}$ to be complex valued (on space-like vectors with my conventions). This definition is then combined with Euclidean notions of orientation to yield a concept of oriented angle which allows the derivation of a Gauss-Bonnet result looking formally identical to (1.1). In Dzan's formulation, however, various differential-geometrical quantities on the left-hand side of the equation are complex valued. This is not necessary and Dzan's Gauss-Bonnet result may be restated with the left-hand side having its usual (real) Lorentzian differential-geometric interpretation, only the right-hand side becomes $2\pi i$.

The formalism of Dzan, in imitating closely Euclidean notions in a rather formal way, leaves unclear how this result is related to the intrinsic Lorentzian geometry. To understand how Dzan's result emerges from the geometry of indefinite signature rather than as a successful "Euclideanization" of a piece of Lorentzian geometry, I utilize the fact that a Lorentzian metric in two dimensions has neutral signature. This observation permits the enlargement of $\mathbf{SO}\,(\mathbf{1},\mathbf{1})$ to a new group $\mathbf{SNO}\,(\mathbf{1})$ by including orientation-preserving anti-isometries. In arbitrary dimension, and ignoring orientation, this idea gives rise to a symmetry group $\mathbf{NO}\,(\mathbf{n})$ which I call the neutral orthogonal group. Section two describes the basic structure of $\mathbf{NO}\,(\mathbf{n})$. In Section three, a geometric formulation of a notion of oriented angle between any pair of nonnull (unit) vectors in $\mathbf{R}^{1,1}$ is provided in terms of the action of $\mathbf{SNO}\,(\mathbf{1})$. Elementary properties of this angle are then established and some simple trigonometry presented in Section four. Finally, the Gauss-Bonnet formula is discussed in Section five.

The treatment of the Gauss-Bonnet formula in this paper provides a natural geometric formulation of Dzan's result in the spirit of Birman and Nomizu's paper and, in so doing, indicates that, geometrically, it is more appropriate to view $\mathbf{R}^{1,1}$ as a neutral space rather than merely Lorentzian.

Further applications of the neutral orthogonal group will be presented elsewhere (cf. Law [**5, 6**]).

I end this introduction with some general notation. By $\mathbf{R}^{p,q}$ I shall mean $\mathbf{R}^n$, $n = p + q$, equipped with the inner product

$$g(u, v) = u^1v^1 + \cdots + u^pv^p - \cdots - u^nv^n$$

where (u^i) and (v^i) are components of the vectors u and v with respect to the standard basis of $\mathbf{R}^n$. By n-dimensional Minkowski space, I mean $\mathbf{R}^{1,n-1}$. In any $\mathbf{R}^{p,q}$, a vector u for which the squared norm $g(u, u)$ is positive, negative, or zero is called time-like, space-like, or null, respectively. For nonnull vectors, I shall refer to their time-likeness or space-likeness as their *character* (in phrases such as "of opposite or like character"). A unit vector is a vector of squared norm plus or minus one. A pseudo-orthonormal basis is a basis of unit, mutually orthogonal vectors.

2. The neutral orthogonal group. The approach taken in this paper to the problem of defining a concept of angle between vectors of opposite character is to introduce an appropriate symmetry group. Clearly, an anti-isometry is an example of a linear transformation that does switch the character of a vector.

Remark 2.1. An anti-isometry L on $\mathbf{R}^{p,q}$ is a linear transformation satisfying $g(L(u), L(v)) = -g(u, v)$, for all vectors u and v in $\mathbf{R}^{p,q}$. A linear transformation of $\mathbf{R}^{p,q}$ is an anti-isometry if and only if $L^*L = -1$, where L^* denotes the adjoint of L with respect to g. Consequently, L is invertible and the inverse is an anti-isometry. There exists an anti-isometry on $\mathbf{R}^{p,q}$ if and only if $p = q$. For these facts, consult, for example, Porteous [**9**].

In light of the last stated fact, attention is now focused upon the neutral spaces $\mathbf{R}^{n,n}$. Because of the neutrality of the signature, one can regard the time-like and space-like vectors as being on an equal footing. The pseudo-sphere $\mathbf{S}(\mathbf{R}^{n,n}) := \{u \in \mathbf{R}^{2n} : g(u, u) = 1\}$ is homeomorphic to $\mathbf{S}^{n-1} \times \mathbf{R}^n$. Define the neutral pseudo-sphere

$\mathbf{S}^{n,n} := \{u \in \mathbf{R}^{2n} : g(u,u) = \pm 1\}$. $\mathbf{S}^{n,n}$ has two components for $n \geq 2$ and four components for $n = 1$.

The metric has been defined in a conventional fashion in terms of the standard basis $\{e_1, \ldots, e_{2n}\}$. This choice is in accord with various conventions in relativity theory, but there remains a conflict between the conventional choice of orientation as determined by the standard basis and the orientation conventions of Minkowski diagrams in the two-dimensional case. I, therefore, choose to specify the "preferred" orientation of $\mathbf{R}^{n,n}$ as that determined by the basis $\{e_1, \ldots, e_n, -e_{n+1}, \ldots, -e_{2n}\}$ which, hereafter, I refer to as the "preferred" basis of $\mathbf{R}^{n,n}$. The preferred orientation of $\mathbf{R}^{n,n}$ is therefore opposite to the standard orientation for odd n, and this fact is important to remember when, for example, employing the exterior calculus with its conventions regarding induced orientations.

Definition 2.2. The collection of linear transformations L of $\mathbf{R}^{n,n}$ which are either isometries or anti-isometries, equivalently, such that $L^*L = \pm 1$, form a group called "the neutral orthogonal group" and denoted $\mathbf{NO}\,(\mathbf{n})$.

Lemma 2.3. $\mathbf{O}\,(\mathbf{n},\mathbf{n})$ *is a subgroup of* $\mathbf{NO}\,(\mathbf{n})$ *of index two and hence a normal subgroup.*

Proof. The product of any two anti-isometries is an isometry. ◻

Lemma 2.4. *The linear transformation* $T : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$, *with matrix*

$$\underline{\mathrm{T}} = \begin{pmatrix} \underline{0}_n & \underline{1}_n \\ \underline{1}_n & \underline{0}_n \end{pmatrix}$$

with respect to the preferred basis of $\mathbf{R}^{2n}$ ($\underline{0}_n$ *and* $\underline{1}_n$ *are the* $n \times n$ *zero and identity matrix, respectively*), *is an anti-isometry of* $\mathbf{R}^{n,n}$. *Note that* $T^2 = 1$. *With respect to the basis* $\{e_1, -e_{n+1}, \ldots, e_n, -e_{2n}\}$, $\mathbf{R}^{n,n}$

is a direct sum of n Minkowski planes and T has matrix

$$\underline{T} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

whence one calculates that $\det(T) = (-1)^n$.

Corollary 2.5. (i) $\mathbf{NO}(\mathbf{n}) = \mathbf{O}(\mathbf{n},\mathbf{n}) \amalg T\mathbf{O}(\mathbf{n},\mathbf{n})$.

(ii) *For L in $\mathbf{NO}(\mathbf{n})$,* $\det(L) = \pm 1$.

(iii) $\mathbf{NO}(\mathbf{n})$ *is a Lie group with eight connected components which may be described as follows. Let R and I be the linear transformations whose matrix representations with respect to the preferred basis of $\mathbf{R}^{n,n}$ are*

$$\underline{\mathrm{R}} = \begin{pmatrix} \underline{\mathrm{J}}_n & \underline{0}_n \\ \underline{0}_n & \underline{1}_n \end{pmatrix}, \qquad \underline{\mathrm{I}} = \begin{pmatrix} \underline{\mathrm{J}}_n & \underline{0}_n \\ \underline{0}_n & \underline{\mathrm{J}}_n \end{pmatrix}$$

where $\underline{J}_n$ is the $n \times n$ diagonal matrix $\underline{J}_n := \mathrm{diag}(-1,1,\dots,1)$. If $\mathbf{SO}^+(\mathbf{n},\mathbf{n})$ is the identity-connected component of $\mathbf{SO}(\mathbf{n},\mathbf{n})$, and hence of $\mathbf{NO}(\mathbf{n})$, then $\mathbf{SO}^+(\mathbf{n},\mathbf{n})$ is a normal subgroup of index four in $\mathbf{O}(\mathbf{n},\mathbf{n})$ and of index eight in $\mathbf{NO}(\mathbf{n})$. The other connected components of $\mathbf{NO}(\mathbf{n})$ are the cosets $\mathbf{SO}^-(\mathbf{n},\mathbf{n}) := I\mathbf{SO}^+(\mathbf{n},\mathbf{n})$, $R\mathbf{SO}^+(\mathbf{n},\mathbf{n})$, $RI\mathbf{SO}^+(\mathbf{n},\mathbf{n})$, $T\mathbf{SO}^+(\mathbf{n},\mathbf{n})$, $TI\mathbf{SO}^+(\mathbf{n},\mathbf{n})$, $TR\mathbf{SO}^+(\mathbf{n},\mathbf{n})$, and $TRI\mathbf{SO}^+(\mathbf{n},\mathbf{n})$. The first three, together with $\mathbf{SO}^+(\mathbf{n},\mathbf{n})$, form $\mathbf{O}(\mathbf{n},\mathbf{n})$. Each coset is therefore a connected manifold of dimension $n(2n-1)$.

The following simple lemma singles out certain subgroups of $\mathbf{NO}(\mathbf{n})$ by the group structure.

Lemma 2.6. *Let $\mathbf{N}$ be a normal subgroup of a group $\mathbf{G}$. For g in $\mathbf{G}\backslash\mathbf{N}$, put $\mathbf{N}(g) := \mathbf{N} \amalg g\mathbf{N} = \mathbf{N} \amalg \mathbf{N}g$. Then, $\mathbf{N}(g)$ is a subgroup of $\mathbf{G}$ if and only if g^2 and g^{-1} belong to $\mathbf{N}(g)$.*

Lemma 2.7. *The following relations among I, R and T hold*:

(i) $TIT = I$,

(ii) $TRT = RI$,

(iii) $RIR = I$,

(iv) $RTR = TI$,

(v) $IRI = R$,

(vi) $ITI = T$.

Proposition 2.8. *Let* $\mathbf{N} := \mathbf{SO}^+(\mathbf{n},\mathbf{n})$. $\mathbf{N}(I) = \mathbf{SO}(\mathbf{n},\mathbf{n})$, $\mathbf{N}(R) =: \mathbf{O}_+(\mathbf{n},\mathbf{n})$, $\mathbf{N}(RI) =: \mathbf{O}^+(\mathbf{n},\mathbf{n})$, $\mathbf{N}(T) =: \mathbf{P}^+(\mathbf{n})$, *and* $\mathbf{N}(TI) =: \mathbf{P}^-(\mathbf{n})$ *are subgroups of* $\mathbf{NO}(\mathbf{n})$ *but* $\mathbf{N}(TR)$ *and* $\mathbf{N}(TRI)$ *are not.*

Proof. Straightforward calculations using (2.7) and (2.6). Note that $(TR)^2 = (TRI)^2 = I$ which belongs to neither $\mathbf{N}(TR)$ nor $\mathbf{N}(TRI)$. □

Proposition 2.9. $\mathbf{N} := \mathbf{SO}(\mathbf{n},\mathbf{n})$ *is a normal subgroup of* $\mathbf{NO}(\mathbf{n})$ *of index four*; *the other cosets may be written* $R\mathbf{N}$, $T\mathbf{N}$, *and* $TR\mathbf{N}$. $\mathbf{N}(R) = \mathbf{O}(\mathbf{n},\mathbf{n})$, $\mathbf{N}(T) =: P(\mathbf{n})$, *and* $\mathbf{N}(TR) =: \mathbf{Q}(\mathbf{n})$ *are subgroups of* $\mathbf{NO}(\mathbf{n})$.

Proof. $\mathbf{SO}(\mathbf{n},\mathbf{n})$ is of index two in $\mathbf{O}(\mathbf{n},\mathbf{n})$ and hence a normal subgroup of $\mathbf{O}(\mathbf{n},\mathbf{n})$. By (2.5)(i), and since $T^{-1} = T$, it suffices to show that TAT belongs to $\mathbf{N}$ for any A in $\mathbf{N}$. But TAT must be an orientation-preserving isometry, and so does lie in $\mathbf{N}$. Hence, $\mathbf{N}$ is normal, and the index is readily seen to be four. The remaining assertions follow easily from (2.7) and (2.6). □

The relations between the various cosets and subgroups are summarized in the following diagram, in which $\mathbf{N} := \mathbf{SO}^+(\mathbf{n},\mathbf{n})$.

Corollary 2.10. i) $\mathbf{NO}(\mathbf{n}) = \mathbf{P}(\mathbf{n})(R) = \mathbf{Q}(\mathbf{n})(R)$; $\mathbf{P}(\mathbf{n})$ *and* $\mathbf{Q}(\mathbf{n})$ *are therefore normal subgroups.*

ii) *The "special neutral orthogonal group," denoted* $\mathbf{SNO}(\mathbf{n})$, *of orientation-preserving neutral orthogonal transformations of* $\mathbf{R}^{n,n}$ *is*

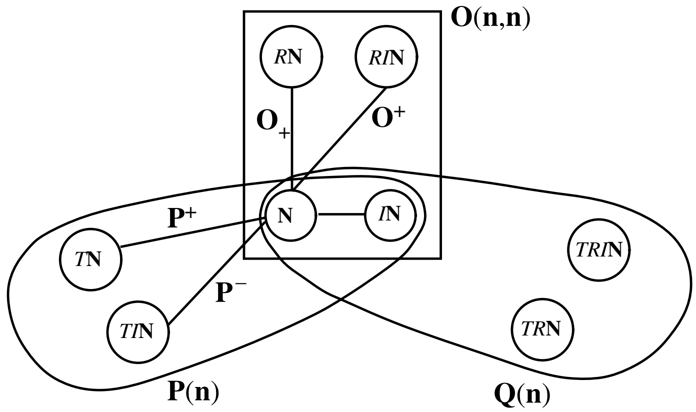

FIGURE 1.

given by

$$\mathbf{SNO}\,(\mathbf{n}) = \begin{cases} \mathbf{P(n)} & \text{for } n \text{ even} \\ \mathbf{Q(n)} & \text{for } n \text{ odd.} \end{cases}$$

Proof. (i) follows from (2.7) and Figure 1 while (ii) follows from $\det(T) = (-1)^n$ and Figure 1. □

Further geometric significance of the groups $\mathbf{P(n)}$ and $\mathbf{Q(n)}$ derives from the notion of **G**-orientation (cf. O'Neill [**8**, pp. 240–241]). Let $B(n)$ be the collection of all pseudo-orthonormal bases $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ of $\mathbf{R}^{n,n}$ such that the v_i's are all time-like and the w_i's are all space-like or the v_i's are all space-like and the w_i's are all time-like. Such bases will be called *configured* and, in the former case, the basis is said to have *standard configuration* while in the latter case it is said to have "nonstandard configuration." $\mathbf{NO}\,(\mathbf{n})$ acts transitively and freely on $B(n)$, and this action establishes an obvious noncanonical bijection between $\mathbf{NO}\,(\mathbf{n})$ and $B(n)$. Any subgroup **G** of $\mathbf{NO}\,(\mathbf{n})$ of index two defines an equivalence relation on $B(n)$ with two equivalence classes. One says two elements of $B(n)$ have the same G-orientation if one is the image of the other by an element of G, i.e., if they belong to the same equivalence class.

There are four obvious choices for $\mathbf{G}$: $\mathbf{SNO}\,(\mathbf{n})$, $\mathbf{O}\,(\mathbf{n},\mathbf{n})$, $\mathbf{P}(\mathbf{n})$, and $\mathbf{Q}(\mathbf{n})$. The first gives the usual notion of orientation. $\mathbf{O}\,(\mathbf{n},\mathbf{n})$-orientation defines the notion of configuration class for $B(n)$ already introduced above. Note that $\mathbf{P}(\mathbf{n}) \cap \mathbf{Q}(\mathbf{n}) = \mathbf{SO}\,(\mathbf{n},\mathbf{n})$. Identifying $\mathbf{R}^{n,n}$ with $\mathbf{R}^n \times \mathbf{R}^n$ via the preferred basis, the orientations induced on the factors by the preferred orientation of $\mathbf{R}^{n,n}$ are called the preferred semi-orientations of $\mathbf{R}^{n,n}$. The action of $\mathbf{SO}\,(\mathbf{n},\mathbf{n})$ either preserves or reverses both semi-orientations while the action of R reverses the first semi-orientation and preserves the second. It follows that the $\mathbf{P}(\mathbf{n})$-orientation class of the preferred basis consists of those elements of $B(n)$ for which the semi-orientations are either both preferred or both nonpreferred, while the other $\mathbf{P}(\mathbf{n})$-orientation class consists of those bases for which one semi-orientation is preferred and the other nonpreferred. The $\mathbf{Q}(\mathbf{n})$-orientation class of the preferred basis consists of those elements of $B(n)$ which, if of standard configuration, have both semi-orientations either preferred or nonpreferred but, if of nonstandard configuration, have one semi-orientation preferred and the other nonpreferred. The other $\mathbf{Q}(\mathbf{n})$-orientation class is the opposite of this. Because of (2.10)(ii), either $\mathbf{P}(\mathbf{n})$- or $\mathbf{Q}(\mathbf{n})$-orientation is always just ordinary orientation.

Finally, note that both $\mathbf{P}(\mathbf{n})$ and $\mathbf{Q}(\mathbf{n})$ act transitively on $\mathbf{S}^{n,n}$ with isotropy group $\mathbf{SO}\,(\mathbf{n}-\mathbf{1},\mathbf{n})$.

3. Angle in $\mathbf{R}^{1,1}$. When $n = 1$, certain simplifications occur. For example, I is the negative of the identity transformation while $B(1)$ consists simply of all pseudo-orthonormal bases. Any normal subgroup $\mathbf{G}$ of $\mathbf{NO}\,(\mathbf{1})$ defines a partition of $\mathbf{NO}\,(\mathbf{1})$ by its cosets and a corresponding partition of $B(1)$. With $\mathbf{G} = \mathbf{SO}\,(\mathbf{1},\mathbf{1})$, one gets $B(1) = B^+_+ \amalg B^+_- \amalg B^-_+ \amalg B^-_-$ where the superscript indicates preferred or nonpreferred orientation and the subscript indicates standard or nonstandard configuration. $\mathbf{P}\,(\mathbf{1})$-orientation combines B^+_+ and B^-_- into one class, and B^+_- and B^-_+ into one class to give the partition $B(1) = (B^+_+ \amalg B^-_-) \amalg (B^+_- \amalg B^-_+)$. $\mathbf{Q}\,(\mathbf{1})$-orientation is ordinary orientation by (2.10)(ii). The following facts are readily confirmed.

Lemma 3.1. (i) *T commutes with each element of $\mathbf{SO}\,(\mathbf{1},\mathbf{1})$ and itself but anticommutes with R,*

FIGURE 2.

(ii) $\mathbf{P}\,(\mathbf{1})$ *is Abelian,*

(iii) T *has invariant matrix form on each* $\mathbf{P}\,(\mathbf{1})$*-orientation class*:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

on $B^+_+ \amalg B^-_-$ *and* $B^+_- \amalg B^-_+$*, respectively.*

(iv) $\mathbf{Q}\,(\mathbf{1})$ *is not Abelian, e.g., for any* L *in* $\mathbf{SO}\,(\mathbf{1},\mathbf{1})$*,* $RLR = L^{-1}$ *so* $(TR)L = L^{-1}(TR)$*.*

Let $\{\mathbf{t}, \mathbf{x}\}$ be any element of B^+_+. The two null directions in $\mathbf{R}^{1,1}$ divide $\mathbf{R}^{1,1}$ into quadrants, each containing one component of $\mathbf{S}^{1,1}$. The action of $\mathbf{SO}^+\,(\mathbf{1},\mathbf{1})$ on $\pm\mathbf{t}$ and $\pm\mathbf{x}$ provides parametrizations and orientations for the components of $\mathbf{S}^{1,1}$ which are natural in the context of Lorentzian geometry. With respect to $\{\mathbf{t}, \mathbf{x}\}$, the components are labelled as in Figure 2.

Definition 3.2. Let u and v be unit vectors. If they both belong to the same component of $\mathbf{S}^{1,1}$, then there exists a unique L in $\mathbf{SO}^+\,(\mathbf{1},\mathbf{1})$ such that $u = L(v)$, say. With respect to $\{\mathbf{t}, \mathbf{x}\}$, L takes the form $L(\beta)$

(recall (1.2)), with β independent of the choice of $\{\mathbf{t}, \mathbf{x}\}$ in B^+_+, and $|g(u,v)| = \cosh|\beta|$. Thus, $|\beta| := \cosh^{-1}(|g(u,v)|)$ is unambiguously associated with the pair u and v and is called the unoriented angle between u and v. If u and v are of the same character, but lie in distinct components of $\mathbf{S}^{1,1}$, then $u = IL(\beta)(v)$ for unique β and $|g(u,v)| = \cosh|\beta|$. Define the unoriented angle between u and v to be $|\beta| + i\pi$. Finally, if u and v are of opposite character, $u = TL(\beta)(v)$ or $TIL(\beta)(v)$, for unique β, and $|g(u,v)| = \sinh|\beta|$. Define the unoriented angle between u and v to be $|\beta| + i\pi/2$.

Remark 3.3. If u and v are of opposite character and orthogonal, the unoriented angle between them is defined above as $i\pi/2$. This value is not determined by $\mathbf{NO}\,(\mathbf{1})$ (since T is disconnected from the identity). Choosing this angle to be imaginary is important because the transformations which generate it are independent of the boost transformations of $\mathbf{SO}^+\,(\mathbf{1},\mathbf{1})$. Definition (3.2) is an extension of that in Birman and Nomizu [**2**] rather than Dzan's [**4**] notion of unoriented angle but the choice of $i\pi/2$ as the angle between orthogonal unit vectors of opposite character leads to a consistency with Dzan's notion of oriented angle and an appealing statement of the Gauss-Bonnet formula.

In order to introduce the notion of oriented angle from v to u, one should consider $\mathbf{SNO}\,(\mathbf{1}) = \mathbf{Q}\,(\mathbf{1})$ which acts transitively on $\mathbf{S}^{1,1}$ with trivial isotropy subgroup. Write $\{E_1, E_2\}$ for the preferred basis of $\mathbf{R}^{1,1}$, recalling that the preferred orientation of $\mathbf{R}^{1,1}$ is opposite to the standard orientation of $\mathbf{R}^2$. Notice that $TRI(E_1) = E_2$, $TRI(E_2) = -E_1$, $TRI(-E_1) = -E_2$, and $TRI(-E_2) = E_1$, i.e., successive applications of TRI to E_1 produce a discrete motion through the four quadrants and back to itself in the clockwise direction, which is the positive sense according to the prescribed orientation. Similarly, TR has the same effect on E_1, but in the counterclockwise direction. Note that this phenomenon is a consequence of the fact that $\mathbf{Q}\,(\mathbf{1})$ lacks analogues of the subgroups of $\mathbf{P}\,(\mathbf{1})$ evident in Figure 1. More generally, for any basis $\{\mathbf{t}', \mathbf{x}'\}$ which is the image of the preferred basis by an element L of $\mathbf{SO}^+\,(\mathbf{1},\mathbf{1})$, define an anti-rotation R_L by $R_L(\mathbf{t}') := -\mathbf{t}'$ and $R_L(\mathbf{x}') := \mathbf{x}'$ (so, for L the identity transformation, R_L is just R). Obviously, successive applications of TR_LI and TR_L on $\mathbf{t}'$ produce a similar phenomenon. Notice that the basis $\{-\mathbf{t}', -\mathbf{x}'\}$ gives rise to the same mapping R_L. For any anti-rotation S, TS and TSI both belong to $\mathbf{Q}\,(\mathbf{1})$.

Lemma 3.4. *Let $\{\mathbf{t}, \mathbf{x}\}$ be an element of B^+_+. Then there exists a unique $L(\beta)$ in $\mathbf{SO}^+\,(\mathbf{1},\mathbf{1})$ such that either $\{\mathbf{t}', \mathbf{x}'\}$ or $\{-\mathbf{t}', -\mathbf{x}'\}$ is the image of $\{\mathbf{t}, \mathbf{x}\}$ under $L(\beta)$. Suppose that S is the element of $\mathbf{SO}^+\,(\mathbf{1},\mathbf{1})$ such that S applied to the preferred basis yields either $\{\mathbf{t}, \mathbf{x}\}$ or $\{-\mathbf{t}, -\mathbf{x}\}$. Let R_L be the anti-rotation defined as above by $\{\mathbf{t}', \mathbf{x}'\}$. Then, $C_+(\beta) := TR_LI = L(\beta)TR_SIL(-\beta)$ and $C_-(\beta) := TR_L = L(\beta)TR_SL(-\beta)$. Moreover,*

(i) $C_+(\beta)^{-1} = C_-(\beta)$

(ii) $C_+(\beta)^2 = C_-(\beta)^2 = I$

(iii) $C_+(\beta)^3 = C_-(\beta)$ *and* $C_-(\beta)^3 = C_+(\beta)$

(iv) $C_+(\beta)^4 = C_-(\beta)^4 = 1$

(v) $L(\alpha)C_+(\beta)^n = C_+(\alpha+\beta)^n L(\alpha)$ *and* $L(\alpha)C_-(\beta)^n = C_-(\alpha+\beta)^n L(\alpha)$

(vi) $C_+(\alpha)C_+(\beta) = C_-(\alpha)C_-(\beta) = IL(2(\alpha-\beta))$

(vii) $C_+(\alpha)C_-(\beta) = C_-(\alpha)C_+(\beta) = L(2(\alpha-\beta))$

Proof. With respect to $\{\mathbf{t}, \mathbf{x}\}$, T, I, and Rs have matrix representations identical, respectively, with those of T, I, and R with respect to the preferred basis and so (2.7) with R replaced by Rs is valid, as are the assertions involving R in (3.1)(i) and (iv). With these facts, checking the various assertions consists of straightforward calculations. The formulae for TR_LI and TR_L need only be checked on the elements $\mathbf{t}'$ and $\mathbf{x}'$. For (iv), the case $n = 1$ is an easy calculation and the result follows by induction on n. Of course, the seven assertions (i)–(vii) are not all independent of each other. $\square$

Remark 3.5. Note that the symbols $C_+(\beta)$ and $C_-(\beta)$ are unambiguous only with reference to a specified element $\{\mathbf{t}, \mathbf{x}\}$ of B^+_+.

Lemma 3.6. *Let u and v be two unit vectors. Using any $\{\mathbf{t}, \mathbf{x}\}$ in B^+_+ to define the symbols $C_+(\beta)$ and $C_-(\beta)$, there is a unique $L(\alpha)$ in*

$\mathbf{SO}^+(\mathbf{1},\mathbf{1})$, *a unique* n *in* $\mathbf{Z}_4$, *and a unique real number* β *such that*

$$u = L(\alpha)C_+(\beta)^n(v).$$

The quantities α *and* n *are independent of* $\{\mathbf{t},\mathbf{x}\}$ *but* β *is not.* $L(\alpha+\beta)$ *is the unique element of* $\mathbf{SO}^+(\mathbf{1},\mathbf{1})$ *mapping that one of* $\pm\mathbf{t}$, $\pm\mathbf{x}$ *in the same component of* $\mathbf{S}^{1,1}$ *as* u *to* u. *Furthermore,*

(i) $u = L(\alpha)C_-(\beta)^{4-n}(v)$; *indeed,* $L(\alpha)C_+(\beta)^n = L(\alpha)C_-(\beta)^{4-n}$,

(ii) $v = L(-\alpha)C_-(\alpha+\beta)^n(u) = L(-\alpha)C_+(\alpha+\beta)^{4-n}(u)$

(iii) *If* w *is a unit vector such that* $w = L(\theta)C_+(\alpha+\beta)^m(u)$, *then* $w = L(\theta+\alpha)C_+(\beta)^{n+m}(v)$ (*there is an analogous result for* C_- *which may be derived by an application of* (i),

(iv) $\{v, C_+(\beta)(v)\}$ *is a pseudo-orthonormal basis with preferred orientation while* $\{v, C_-(\beta)(v)\}$ *is a pseudo-orthonormal basis with nonpreferred orientation.*

Proof. One of $\pm\mathbf{t}$, $\pm\mathbf{x}$ belongs to the same component of $\mathbf{S}^{1,1}$ as v and there is a unique $L(\beta)$ in $\mathbf{SO}^+(\mathbf{1},\mathbf{1})$ mapping it to v, thus determining β. The quantity n is the number of successive applications of $C_+(\beta)$ that takes v to the component of $\mathbf{S}^{1,1}$ in which u lies, and $L(\alpha)$ is the unique element in $\mathbf{SO}^+(\mathbf{1},\mathbf{1})$ mapping the image of v to u. (i) is obtained similarly, or from (3.4)(i)–(iv). For (ii), solve for v in terms of u and use (3.4)(v). (iii) and (iv) are straightforward. □

If two unit vectors u and v lie in the same component of $\mathbf{S}^{1,1}$, then $u = L(\alpha)(v)$ for some unique $L(\alpha)$ in $\mathbf{SO}^+(\mathbf{1},\mathbf{1})$ and the oriented angle from v to u is defined to be α. Such an angle may be called a "boost angle." As already noted in Section 1, this agrees with Birman and Nomizu [**2**]. If u and v are orthogonal, unit vectors of opposite character, choose any element $\{\mathbf{t},\mathbf{x}\}$ of B^+_+ and find β as in the proof of (3.6). Then, either $u = C_+(\beta)(v)$ or $u = C_-(\beta)(v)$. Define the oriented angle from v to u to be $i\pi/2$ or $-i\pi/2$, respectively. Following Dzan [**4**], any of the above oriented angles will be called a fundamental (oriented) angle. In general, for two arbitrary unit vectors, the oriented angle from v to u can be thought of as a *sum* of fundamental angles. Thus, the representation of the element of $\mathbf{Q}(\mathbf{1})$ which maps v to u given in (3.6) permits a suitable definition of oriented angle.

Definition 3.7. Let u and v be unit vectors. Choose any element $\{\mathbf{t}, \mathbf{x}\}$ of B^+_+ and, as in (3.6), write $u = L(\alpha)C_+(\beta)^n(v) = L(\alpha)C_-(\beta)^{4-n}(v)$. The oriented angle from v to u in the positive/negative sense is defined by

$$(v, u)_\pm := \begin{cases} \alpha + n(i\pi/2) \\ \alpha + (4-n)(-i\pi/2). \end{cases}$$

It is clear from (3.6) that this definition is independent of $\{\mathbf{t}, \mathbf{x}\}$.

Proposition 3.8. (i) $(u, v)_\pm = -(v, u)_\mp$,

(ii) $(u, v)_\pm = -(v, u)_\pm \pm 2\pi i$,

(iii) *Oriented angles are additive*; *for example, if* v, u *and* w *are three unit vectors such that "*u *lies between* v *and* w *in the clockwise direction from* v *to* w*," then*

$$\begin{aligned} (v, u)_+ + (u, w)_+ &= (v, w)_+ \\ (v, w)_- + (w, u)_- &= (v, u)_- \\ (v, w)_+ + (w, u)_- &= (v, u)_+ \\ (v, u)_- + (u, w)_+ &= (v, w)_-, \end{aligned}$$

(iv) $(v, u)_\pm = (-v, -u)_\pm$,

(v) $(v, -u)_\pm = (-v, u)_\pm = [(v, u)_\pm + i\pi]_{\mathrm{mod}4}$, *where* $[\quad]_{\mathrm{mod}4}$ *means the number of multiples of* $\pm i\pi/2$ *in the imaginary part of the angle must be read modulo four.*

Proof. (i) follows from (3.6)(ii). For (ii), suppose u and v are related as in (3.6). By (3.6)(ii), $v = L(-\alpha)C_+(\alpha + \beta)^{4-n}(u)$ so $(u, v)_+ = -\alpha + (4-n)(i\pi/2) = -\alpha - n(i\pi/2) + 2\pi i = -(v, u)_+ + 2\pi i$. Similarly, and again by (3.6)(ii), $(u, v)_- = -\alpha + n(-i\pi/2)$, while $(v, u)_- = \alpha + (4-n)(-i\pi/2)$ by (3.6)(i). Thus, $(u, v)_- = -(v, u)_- - 2\pi i$. For (iii), the first equation follows from (3.6)(iii) while the second is analogous. The remaining two follow from the first two and (3.8)(i). It is important to remember that exponents of C_+ and C_- are read modulo four in (3.7). Finally, (iv) and (v) are straightforward, with (3.4)(ii) relevant to (v). $\square$

Remark 3.9. (i) Although the action of $\mathbf{Q}\,(\mathbf{1})$ has been used to define oriented angle in $\mathbf{R}^{1,1}$, the action of $\mathbf{P}\,(\mathbf{1})$ suffices for computations.

Given two unit vectors, there is a unique A in $\mathbf{P}(\mathbf{1})$ such that $u = A(v)$. A can be written as $(T)(I)L(\alpha)(v)$, where the brackets around T and I indicate that they may or may not occur in this decomposition of A. The real number α is the real part of $(v, u)_{\pm}$, while the imaginary part is determined by the presence of T (contributes $i\pi/2$) and I (contributes $i\pi$) with the sign determined by the orientation of the oriented angle.

(ii) If the conventions of Dzan [**4**] are modified so as to agree with those employed in the current formalism, then Dzan's directed sectorial measure Φ of a given oriented angle is related to the angle Ω obtained from (3.7) by $\Phi = i\overline{\Omega}$.

4. Some trigonometry.

Proposition 4.1 (Dzan's [**4**] Umlaufsatz). *Let Γ be a simple closed nonnull piecewise-smooth curve in $\mathbf{R}^{1,1}$. Then, including the jumps at the vertices, the tangent turns through $\pm 2\pi i$.*

Proof. At a point on Γ, the Euclidean unit tangent T_E defines a point on the Euclidean unit circle. By the Euclidean Umlaufsatz, the point on the circle travels around exactly once as Γ is traversed once. The Lorentzian unit tangent T_L of a given point on Γ is the point on $\mathbf{S}^{1,1}$ obtained by extending the ray from the origin through the point on the unit circle corresponding to T_E. It follows that T_L turns through $\pm 2\pi i$, the real part of the oriented angle through which T_L turns being zero. $\square$

Proposition 4.2. *Let Γ be a simple closed nonnull polygon. A subscript $\pm$ indicates that the relevant oriented angle is measured by $(\ ,\ )_{\pm}$. At each vertex, one can measure oriented interior and exterior angles as indicated in Figure 3.*

(i) $\text{interior}_{-} = \text{exterior}_{+} - \pi i$ *and* $\text{interior}_{+} = \text{exterior}_{-} + \pi i$,

(ii) *The sum of the oriented exterior angles of Γ is $\pm 2\pi i$ according to whether one traverses Γ clockwise/counterclockwise (since in this direction the exterior angle is naturally measured by $(\ ,\ )_{\pm}$).*

(iii) *The sum of the oriented interior angles of Γ is $\pm(n-2)\pi i$*

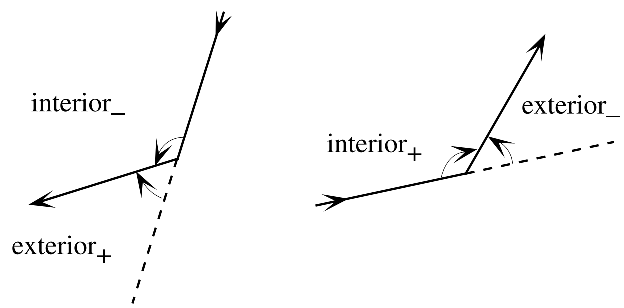

FIGURE 3.

according to whether Γ *is traversed counterclockwise/clockwise* (*since in this direction the interior angle is naturally measured by* $(\quad , \quad)_{\pm}$), *where* n *is the number of vertices.*

Proof. (i) is tedious to describe because not every oriented angle can be interpreted as an oriented interior angle of a polygon. Nevertheless, it is easy to check, case by case. (ii) follows from (4.1)(i) since for a polygon the only contributions to the turning of the tangent are the jumps at the vertices, i.e., the exterior angles. (iii) follows from (i) and (ii). □

Corollary 4.3. *Let* Γ *be a nonnull triangle with two sides orthogonal to each other. Then the angle between these two sides is* $\pm i\pi/2$ *and the other two sides add up to* $\pm i\pi/2$. *If these latter two oriented angles are called* θ *and* ϕ, *then "complementarity" is expressed analytically by* $\sinh(\theta) = -i\cosh(\phi)$.

Proof. Follows from (4.2)(iii) and elementary calculations. □

Remark 4.4. The results given above are close analogues of standard results in plane Euclidean geometry. In essence, these analogues occur because in (4.1) there is zero net real angle while an imaginary angle of $\pm 2\pi i$ arises from finite subgroups of $\mathbf{Q}\,(\mathbf{1})$ which act discretely on $\mathbf{S}^{1,1}$. Each point of any given component of $\mathbf{S}^{1,1}$ gives rise to a finite subgroup $\{1, TR_L I, I, TR_L\}$. In the Euclidean case, each point of the unit circle also defines an analogous finite subgroup, but each element of it is, of course, connected with the identity. Notice that when discussing accumulation of the angle in the Euclidean case, as in the Umlaufsatz, one essentially utilizes the covering of $\mathbf{S}^1$ by $\mathbf{R}$ to get away from angles modulo 2π. Loosely speaking, at least, the analogue for $\mathbf{R}^{1,1}$ is the covering of $\mathbf{Z}_4$ by $\mathbf{Z}$.

5. The Gauss-Bonnet theorem.

Theorem 5.1. *Let D be a domain in a two-dimensional Lorentzian manifold M with metric g. Assume that D has piecewise smooth boundary Γ consisting of a finite number of nonnull smooth curves. If K is the Gaussian curvature and k_g geodesic curvature, both defined in the usual way with respect to g, then*

$$\int_D K\,dV + \int_\Gamma k_g\,ds + \sum \text{exterior}_\pm = \pm 2\pi i$$

where $\text{exterior}_\pm$ *denotes the oriented exterior angle at the nonsmooth points of the boundary and the choice of sign corresponds to whether the boundary is traversed clockwise or counterclockwise.*

Proof. As already noted in Section 1, the essentially new ingredient as compared to the Riemannian version of this theorem is the theory of angle in $\mathbf{R}^{1,1}$. With it in hand, the differential-geometric aspects of the proof of the Riemannian version may be carried over. The essence of that part of the argument is contained in Birman and Nomizu, for example, though one can also carry over the detailed arguments in Singer and Thorpe [**10**, pp. 157–177]. I shall only note that the crucial difference between the above formula and that in Birman and Nomizu is that the exterior angles have imaginary parts and these balance the nonzero term on the right hand side of the equation. □

Corollary 5.2. *If M is a compact, oriented, two-dimensional Lorentzian manifold with Gaussian curvature K and Euler characteristic χ, then*

$$\int_M K\,dV = \pm 2\pi i \chi$$

whence χ must be zero and $\int_M K\,dV = 0$ since the left-hand side is real and the right-hand side imaginary. These deductions are well known.

Proof. There is a standard argument using a triangulation of M and adding up the formula in (5.1) over the two-simplices of the triangulation. Here, one needs to pick a triangulation in which the edges are all nonnull curves. Briefly, suppose there are n_o vertices, n_1 1-simplices, and n_2 2-simplices in the triangulation. Each 2-simplex has 3 edges, but each edge is shared by 2 2-simplices so $n_1 = 3n_2/2$. Thus, $\chi = n_0 - n_2/2$. Using (4.2)(i), rewrite the formula in (5.1) as

$$\int_S K\,dV = -\int_{\partial S} k_g\,ds - \sum \text{interior}_{\mp} - (\pm 3\pi i) \pm 2\pi i$$

where S is a 2-simplex. At each vertex, $\sum \text{interior}_{\mp} = \mp 2\pi i$. Adding over the 2-simplices, the line integrals cancel since each edge is traversed twice, once with each orientation. Thus, one finds

$$\int_M K\,dV = -n_0(\mp 2\pi i) - n_2(\pm \pi i) = \pm 2\pi i \chi. \qquad \square$$

Further applications of the neutral orthogonal group will be presented elsewhere.

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