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RESTRICTIONS OF THE SPECIAL REPRESENTATION OF AUT(TREE$_3$) TO TWO COCOMPACT SUBGROUPS

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ABSTRACT. Let $\mathcal{T}$ be a homogeneous tree of degree 3, let G be the automorphism group of $\mathcal{T}$, and let π_+ and π_- be the special representations of G. We consider two discrete subgroups of G isomorphic to $\mathbf{Z}_3 * \mathbf{Z}_3$ and $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ and show how to decompose into irreducibles the restrictions of π_+ and π_- to these subgroups. We also present a general formula relating continuous dimension for representations of discrete groups and formal dimension for representations of continuous groups.

1. General introduction. Let $\mathcal{T}$ be the homogeneous tree of degree 3, that is, a connected combinatorial graph with no loops and with three edges leaving each vertex. The tree $\mathcal{T}$ is of course infinite. Let $G = \mathrm{Aut}\,(\mathcal{T})$ and topologize G by letting pointwise fixers of finite subtrees form a neighborhood base for the identity. [**5, 6**] classifies its irreducible unitary representations. In its representation theory, as in other ways, G is analogous to $SL(2, \mathbf{R})$, with $\mathcal{T}$ analogous to the hyperbolic disk.

Consider the special discrete series representations π_+ and π_- of G (to be defined later). When these representations are restricted to any discrete cocompact subgroup Γ of G they are continuously reducible. This follows from [**4**] since π_+ and π_- are square integrable on G and are easily seen to be square integrable on Γ as well. The purpose of this paper is to exhibit particular decompositions of $\pi_\pm|_\Gamma$ when Γ is either of two particular discrete subgroups.

The subgroups in question are $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ as constructed in [**3**] and $\mathbf{Z}_3 * \mathbf{Z}_3$ as considered in [**12, 14, 19**]. These two groups will be

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described later, but in short, the first acts simply transitively on the vertices of $\mathcal{T}$ and the second acts simply transitively on the edges.

Since these groups are *not* type I, the decompositions of $\pi_{\pm}|_{\Gamma}$ which we exhibit are *not* unique. Nonetheless, there are points of interest. The restricted representations turn out to be equivalent to very natural subrepresentations of $\ell^2(\Gamma)$. In fact, the orthogonal projections onto these subrepresentations of $\ell^2(\Gamma)$ are given by right convolution with finitely supported functions; in particular, these projections are in $C^*_{\text{reg}}(\Gamma)$. This raises the question of whether, in similar circumstances, restrictions of square integrable representations to cofinite subgroups Γ always give rise to representations which agree with the images of projections in $C^*_{\text{reg}}(\Gamma)$ (or $C^*_{\text{reg}}(\Gamma) \otimes \mathcal{K}$, $\mathcal{K}$ being the algebra of compact operators on some generic Hilbert space). To put this question in perspective, observe that if Γ is $\mathbf{Z}_3 * \mathbf{Z}_3$ or $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$, then Γ has no nontrivial finite conjugacy classes, and consequently, $VN_{\text{reg}}(\Gamma)$ is a factor. Since $VN_{\text{reg}}(\Gamma)$ also has a finite faithful trace, the results of [**5**, III.1.1 and III.1.2] imply that square integrable representations of Γ are completely characterized by their continuous dimensions.

A quick check (using Lemma 1) shows that for π one of the special representations of $SL(2, \mathbf{R})$ and for Γ a free group with two generators cofinitely embedded in $SL(2, \mathbf{R})$, the continuous dimension of $\pi|_{\Gamma}$ is $1/2$. On the other hand, [**17**] says that for the free group the only projections in $C^*_{\text{reg}}(\Gamma)$ are 0 and I, corresponding to continuous dimensions 0 and 1. This, then, is a case where the restriction of a square integrable representation to a cofinite subgroup does not agree with the image of any projection in $C^*_{\text{reg}}(\Gamma)$.

Another point of interest is the formula of Lemma 1, relating continuous dimension for representations of discrete groups to formal dimension for representations of continuous groups. This lemma is known; it appears in the manuscript [**9**] and a somewhat different form occurs in [**3**, Theorem 3.3.2].

Finally, observe that this paper complements the result (from [**18**]) that the restrictions to Γ of principal and complementary series representations of G are (with one simple exception) themselves irreducible representations.

The geometry of the tree $\mathcal{T}$ is very easy to grasp and the reader should draw diagrams as he/she reads, in order to appreciate its simplicity.

This work now proceeds with the definitions of the two representations and the two discrete subgroups, followed by the very general Lemma 1, and finally the decomposition. We work with only one of the two groups and one of the two representations, giving indications in the last section of how to proceed in the other three cases.

Be forewarned that our decomposition proceeds by showing that after some twists the spectral decompositions of, on the one hand, [**14** or **19**], and on the other hand [**3**] following [**7**] are applicable. Thus, the reader will have to consult at least one other paper if his/her goal is to see the most explicit form of the decomposition.

We thank Dan Voiculescu for helpful background, for information about K-theory and for the references on Lemma 1.

2. The representations $\pi_\pm$. The representations $\pi_\pm$ are called the *special representations* and are analogous to the special discrete series representations of $SL(2,\mathbf{R})$. (But note that π_- is analogous to the sum of the two special representations of $SL(2,\mathbf{R})$ while π_+ has no precise analogue.) To define these representations, first let $\mathcal{E}$ be the set of oriented edges of $\mathcal{T}$, that is, the set of pairs (v_1,v_2) where v_1 and v_2 are adjacent vertices of $\mathcal{T}$. Each oriented edge s has an opposite edge s' obtained by reversing the orientation. The representations $\pi_\pm$ can be realized as subrepresentations of the regular representation of G on $\ell^2(\mathcal{E})$. Define

$$\mathcal{H}_\pm = \left\{ f \in \ell^2(\mathcal{E});\ \begin{array}{l} f(s) = \pm f(s') \text{ whenever } s \text{ and } s' \text{ are opposite and} \\ f(s_1)+f(s_2)+f(s_3)=0 \text{ whenever } s_1,s_2,s_3 \\ \text{are the three edges leaving a vertex} \end{array} \right\}.$$

Define $\pi_\pm$ as the restriction of the regular representation to the invariant subspace $\mathcal{H}_\pm$.

In passing, note that $f \in \mathcal{H}_-$ can be written $f(v_1,v_2) = F(v_2) - F(v_1)$ where F is a harmonic function on the vertices of $\mathcal{T}$, that is, a function so that $F(v_1)+F(v_2)+F(v_3) = 3F(v_0)$ whenever v_1, v_2, v_3 are the three vertices neighboring v_0.

3. The groups $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ and $\mathbf{Z}_3 * \mathbf{Z}_3$. Define distances on the tree in the obvious way: the distance between two vertices is the number of edges in the shortest path which connects them; the distance between two edges (oriented or unoriented) is one less than the number of edges in the shortest path containing both. Two vertices or two edges are called *nearest neighbors* if they are at distance one. Note that a pair of nearest neighbor edges has a common vertex.

Fix an abstract group $\Gamma = \mathbf{Z}_3 * \mathbf{Z}_3$ with generators a and b for the two cyclic factors. Thus, $\Gamma = \langle a, b; a^3 = b^3 = e\rangle$. In order to realize Γ as a subgroup of G, start by fixing an oriented edge $s_0 = (v_0, v_1)$. Send a to any automorphism of order 3 in G which fixes v_0 and cyclically permutes the 3 edges around v_0; send b to a similar automorphism with respect to v_1. This defines a homomorphism of Γ into G which is in fact an injection.

Indeed, let u_0 be the unoriented edge corresponding to s_0, and observe first that the nearest neighbor edges of u_0 are $au_0, a^2u_0, bu_0, b^2u_0$. Since any edge of $\mathcal{T}$ is connected to u_0 by a sequence of edges, each the nearest neighbor of the next, induction easily shows that Γ acts transitively on the (unoriented) edges of $\mathcal{T}$.

Pick $\gamma \in \Gamma$ and let $c_1^{j_1} c_2^{j_2} \cdots c_K^{j_K}$ be the reduced word representing γ, so that $j_k \in \{1,2\}$, $c_k \in \{a,b\}$, and $c_k \neq c_{k+1}$ (making the c_k's a sequence of alternating a's and b's.) Call K the *length* or *block length* of γ and denote it by $|\gamma|$. We will show

$$d(\gamma u_0, u_0) = |\gamma|.$$

This implies that Γ acts simply transitively on the set of (unoriented) edges of $\mathcal{T}$, and in particular that the map from Γ to G is injective.

Consider the sequence $(u_0, u_1, \ldots, u_K)$ defined by $u_k = c_1^{j_1} \cdots c_k^{j_k} u_0$. The edge u_k is a nearest neighbor of the edges u_{k+1} and u_{k-1}, and the vertex which u_k and u_{k+1} have in common differs from the vertex which u_k and u_{k-1} have in common. This is because (u_{k-1}, u_k, u_{k+1}) is the translation by $c_1^{j_1} \cdots c_k^{j_k}$ of $(c_k^{-j_k} u_0, u_0, c_{k+1}^{j_{k+1}} u_0)$. The geometry of trees and simple induction show that $d(u_0, u_k) = k$, and in particular $d(u_0, \gamma u_0) = d(u_0, u_K) = K$.

Identify Γ as a subgroup of G. Call a vertex of $\mathcal{T}$ *even* or *odd* according to whether its distance from v_0 is even or odd. Any automorphism in G either preserves the set of even vertices or interchanges it with the set of odd vertices, and since both a and b preserve the set of even vertices, so does any $\gamma \in \Gamma$. Since Γ acts transitively on the edges of

$\mathcal{T}$, and since each edge of $\mathcal{T}$ has one even and one odd vertex, Γ acts transitively on the set of even (respectively odd) vertices.

The injection of Γ into G constructed here is canonical in the sense that any two injections constructed in the way we described are conjugate by an element of G fixing s_0. One can see this by constructing a canonical tree from Γ whose unoriented edges are elements of Γ, whose even vertices are triples $\{\gamma, \gamma a, \gamma a^2\}$ and whose odd vertices are triples $\{\gamma, \gamma b, \gamma b^2\}$. One can map this canonical tree to $\mathcal{T}$ in exactly one way which preservers the Γ-action and takes $e \in \Gamma$ to u_0. The fact that Γ's action on $\mathcal{T}$ is canonical, (given the choice of s_0) isn't necessary in the following. It is illuminating to draw a diagram labelling the edges of $\mathcal{T}$ with the words of Γ according to the correspondence $\gamma \mapsto \gamma u_0$.

Now we reverse field and let Γ be the abstract group $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ with generators a, b, c for the three copies of $\mathbf{Z}_2$. Thus, $\Gamma = \langle a, b, c; a^2 = b^2 = c^2 = e\rangle$. Let v_1, v_2, v_3 be the three vertices neighboring v_0 and choose three involutions of $\mathcal{T}$ which interchange v_0 with v_1, v_2, v_3, respectively. Map $\mathcal{T}$ to G by sending a, b, c to these three involutions. In analogy with the previous paragraphs, show that Γ acts transitively and simply transitively on the set of vertices of $\mathcal{T}$. The construction is again canonical and it is again helpful to draw a diagram, labelling the vertices of $\mathcal{T}$ with the elements of γ according to the correspondence $\gamma \mapsto \gamma v_0$.

4. Continuous dimension versus formal dimension. For the next lemma only, let G be any separable locally compact group, let Γ be a discrete subgroup of G, and suppose that F is a measurable (left) fundamental domain for Γ in G, i.e., $G = \Gamma \cdot F$ with no redundancy. Let π be an irreducible square integrable representation of G with representation space $\mathcal{H}_\pi$. Under these conditions

Lemma 1. $\mathrm{DIM}\,(\pi|_\Gamma) = \mathrm{vol}\,(G/\Gamma)\dim_F(\pi)$ *where* DIM *means continuous dimension and* $\dim_F$ *means formal dimension.*

In this circumstance, *continuous dimension* is easily defined. Let $\mathcal{H}$ be separable Hilbert space and suppose P is a bounded operator on $\ell^2(\Gamma) \otimes \mathcal{H}$ which commutes with the left action of Γ. Fix a basis $(e_j)_{j=1}^\infty$ for $\mathcal{H}$, and define $P_{jk} : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $\langle P_{jk} f_1, f_2\rangle =$

$\langle P(f_1 \otimes e_k), (f_2 \otimes e_j)\rangle$ for $f_1, f_2 \in \ell^2(\Gamma)$. Then $P_{jk} \in VN_{\text{reg}}(\Gamma)$, that is, P_{jk} is given by right convolution with some function $p_{jk} \in \ell^2(\Gamma)$. One may consider P to be a matrix with entries in $VN_{\text{reg}}(\Gamma)$. Now suppose that P is positive. Then each P_{jj} is positive and in particular $p_{jj}(e) = \langle P_{jj}\delta_e, \delta_e\rangle$ is positive, so we may define

$$\text{TR}\,(P) = \sum_{j=1}^{\infty} p_{jj}(e) = \sum_{j=1}^{\infty} \langle P(\delta_e \otimes e_j), (\delta_e \otimes e_j)\rangle \tag{1}$$

where $\delta_e \in \ell^2(\Gamma)$ is the Kronecker δ at the identity. This is a faithful trace (see [**5**, I.6.1]) on the commutant of the left Γ-action on $\ell^2(\Gamma)\otimes\mathcal{H}$. Now if $\mathcal{H}_1$ is a subrepresentation of $\ell^2(\Gamma)\otimes\mathcal{H}$, (that is, an invariant subspace of $\ell^2(\Gamma)\otimes\mathcal{H}$), then the orthogonal projection P_1 onto $\mathcal{H}_1$ commutes with the left G-action, so we may define

$$\text{DIM}\,(\mathcal{H}_1) = \text{TR}\,(P_1).$$

Moreover, if $\mathcal{H}_2$ is another subrepresentation of $\ell^2(\Gamma)\otimes\mathcal{H}$, equivalent as a representation of Γ to $\mathcal{H}_1$, then we may find a partial isometry E, commuting with the left Γ-action, so that $P_1 = EE^*$ and $P_2 = E^*E$, and the properties of the trace then show that $\text{TR}\,(P_1) = \text{TR}\,(P_2)$, i.e., the continuous dimension of a subrepresentation depends only on the abstract representation, not on the particular embedding in $\ell^2(\Gamma)\otimes\mathcal{H}$. Write

$$\text{DIM}\,(\pi) = \text{DIM}\,(\mathcal{H}_1)$$

if π is any abstract representation equivalent to the left action of Γ on $\mathcal{H}_1 \subseteq \ell^2(\Gamma)\otimes H$. Of course, the continuous dimension of an abstract representation is defined only if it can be embedded in the tensor product. Lemma 1 asserts in particular that $\pi|_\Gamma$ has a continuous dimension when π is a square integrable representation of G.

Recall the notation of Lemma 1: π is a representation of G on the Hilbert space $\mathcal{H}_\pi$ and F is a fundamental domain for a discrete subgroup Γ. Let w_0 be any element of norm 1 in $\mathcal{H}_\pi$. The basic proposition on irreducible square integrable representations (see **6**, Chapter 14]) asserts that there exists a positive number $\dim_F(\pi)$, the *formal dimension* of π, so that the map $\alpha : \mathcal{H}_\pi \to \ell^2(G)$ given by

$$\alpha(w) = \dim{}_F^{1/2}(\pi)\langle w, \pi(\cdot)w_0\rangle_{H_\pi}$$

is a unitary map of representations.

Let $(e_j)_j$ be a basis for $L^2(F)$ and identify $L^2(G)$ with $\ell^2(\Gamma) \otimes L^2(F)$ by sending $\pi_{\mathrm{reg}}(\gamma)e_j$ to $\delta_\gamma \otimes e_j$. The natural left actions of Γ on the two spaces coincide. Let P be the orthogonal projection from $L^2(G)$ to $\alpha(\mathcal{H}_\pi)$. We can calculate continuous dimension with (1) by using $L^2(F)$ for $\mathcal{H}$.

$$\begin{aligned}
\mathrm{DIM}\,(\pi) = \mathrm{TR}\,(P) &= \sum_j \langle P(\delta_e \otimes e_j), \delta_e \otimes e)_j \rangle_{\ell^2(\Gamma)\otimes L^2(F)} \\
&= \sum_j \langle Pe_j, e_j \rangle_{L^2(G)} = \sum_j \sum_k \langle \langle e_j, \alpha(f_k)\rangle_{L^2(G)} \alpha(f_k), e_j \rangle_{L^2(G)} \\
&= \sum_j \sum_k |\langle e_j, \alpha(f_k)\rangle_{L^2(G)}|^2 = \sum_k \sum_j |\langle \alpha(f_k), e_j \rangle_{L^2(G)}|^2 \\
&= \sum_k \int_F |\alpha(f_k)^2|\, dg \qquad (\text{since } (e_j)_j \text{ is a basis for } L^2(F)) \\
&= \int_F \sum_k \dim{}_F(\pi) |\langle f_k, \pi(g)w_0\rangle_{\mathcal{H}_\pi}|^2\, dg \\
&= \dim{}_F(\pi) \int_F 1\, dg \qquad (\text{since } (f_k)_k \text{ is a basis for } \mathcal{H}_\pi) \\
&= \dim{}_F(\pi) \cdot \mathrm{vol}\,(G/F).
\end{aligned}$$

This proves Lemma 1. □

Now return to the specific situation where $G = \mathrm{Aut}\,(\mathcal{T})$ and consider the special representations $\pi_\pm$.

Lemma 2.

(1) *If* $\Gamma \subseteq G$ *is* $\mathbf{Z}_3 * \mathbf{Z}_3$ *as above, then* $\mathrm{DIM}\,(\pi|_\Gamma) = 1/3$.

(2) *If* $\Gamma \subseteq G$ *is* $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ *as above, then* $\mathrm{DIM}\,(\pi|_\Gamma) = 1/2$.

Proof. Theorem 2 of [**16**] gives the formula

$$\dim{}_F(\pi_\pm) = \frac{1}{6\,\mathrm{vol}\,(U(s_0))}$$

where $U(s_0) \subseteq G$ is the stabilizer of the oriented edge s_0. Consider $\Gamma = \mathbf{Z}_3 * \mathbf{Z}_3$, which acts simply transitively on the (unoriented) edges and therefore has $U(u_0)$, the stabilizer of the unoriented version of s_0, as a fundamental domain. According to Lemma 1,

$$\mathrm{DIM}\,(\pi_\pm|_\Gamma) = \frac{1}{6}\frac{\mathrm{vol}\,(U(u_0))}{6\mathrm{vol}\ \ (U(s_0))} = \frac{1}{3}$$

since $U(s_0)$ is of index 2 in $U(u_0)$.

Now take $\Gamma = \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ and use $U(v_0)$, the stabilizer of v_0 as a fundamental domain. Because $U(v_0)$ acts transitively on the three edges leaving v_0, we have $[U(v_0) : U(s_0)] = 3$ and $\mathrm{DIM}\,(\pi_\pm|_\Gamma) = 1/2$. □

5. Embedding $\mathcal{H}_+$ in $\ell^2(\Gamma)$. For the next two sections we will work exclusively with π_+ and with $\Gamma = \mathbf{Z}_3 * \mathbf{Z}_3$. Since $\pi_+|_\Gamma$ has square integrable matrix coefficients, the basic ideas from [**8**] (also described in [**6**, Section 14.1]) tell us that $\mathcal{H}_+$ can be embedded in the direct sum of copies of $\ell^2(\Gamma)$ as a Γ-representation. The objective of this section is to construct a particularly useful embedding of $\mathcal{H}_+$ in a single copy of $\ell^2(\Gamma)$.

Let P be the orthogonal projection from $\ell^2(\mathcal{E})$ to $\mathcal{H}_+$ and let $\phi_0 = P(\delta_{s_0})$. Since $\mathcal{H}_+$ is G-invariant, P commutes with the G-action and we have, for $f \in \mathcal{H}_+$ and $\gamma \in \Gamma$,
(2)
$$\langle f, \pi_+(\gamma)\phi_0\rangle = \langle f, \pi_+(\gamma)P\delta_{s_0}\rangle = \langle f, P\pi_{\mathrm{reg}}(\gamma)\delta_{s_0}\rangle = \langle f, \delta_{\gamma_{s_0}}\rangle = f(\gamma s_0).$$

Since, up to orientation, Γ acts transitively on $\mathcal{E}$, this shows that ϕ_0 is cyclic for $\pi_+|_\Gamma$. Also, for $g \in U(s_0)$ (the stabilizer of s_0) $\pi_+(g)\phi_0 = P\pi_+(g)\delta_{s_0} = P\delta_{s_0} = \phi_0$. This shows that $\phi_0(s)$ depends only on $d(s, s_0)$ and (conceivably) on the side of s_0 to which s belongs. Knowing also that $\phi_0 \in \mathcal{H}_+$, one can easily compute that

$$\phi_0(s) = \left(\frac{-1}{2}\right)^{d(s,s_0)} \phi_0(s_0).$$

Consider next the function

$$\begin{aligned}\phi_\omega &= \phi_0 + \omega\pi_+(a)\phi_0 + \omega^2\pi_+(a^2)\phi_0 \\ &= P(\delta_{s_0} + \omega\delta_{as_0} + \omega^2\delta_{a^2s_0}),\end{aligned}$$

where ω is a cube root of 1.

Lemma 3. *ϕ_ω is cyclic for $\pi_+|_\Gamma$.*

The proof depends upon

Lemma 4. *If $f \in \ell^2(\mathcal{E})$ satisfies*

(1) $|f(s)| = |f(s')|$ *when s and s' are opposite,*

(2) $|f(s_1)| = |f(s_2)| = |f(s_3)|$ *when s_1, s_2, and s_3 are the three oriented edges leaving any* even *vertex, and*

(3) $f(s_1)+f(s_2)+f(s_3) = 0$ *when s_1, s_2, and s_3 are the three oriented edges leaving any* odd *vertex,*

then $f = 0$.

Proof of Lemma 4. Let $S_0 = \{s_0, s_0'\}$ and for $n \geq 1$, let S_n be the set of oriented edges at distance n to s_0 which are closer to v_0 than to v_1. The sets $(S_n)_{n=0}^{n=\infty}$ make up roughly *one half* of the tree. We shall prove

(3)
$$2 \sum_{s \in S_{2n}} |f(s)|^2 \geq \sum_{s \in S_{2n-1}} |f(s)|^2 \geq 2 \sum_{s \in S_{2n-2}} |f(s)|^2 \quad \text{for every } n \geq 1.$$

Since $f \in \ell^2(\mathcal{E})$, this implies that $f(s_0) = 0$. Now repeat the argument using in place of s_0 any oriented edge from an even to an odd vertex, and conclude that f is identically zero.

Let s be any element of S_{2n-1} pointing towards v_1, let v be the vertex which s leaves and let s_1 and s_2 be the two elements of S_{2n} which also leave v. Since $f(s) + f(s_1) + f(s_2) = 0$, Schwarz's inequality gives

$$|f(s)|^2 \leq 2(|f(s_1)|^2 + |f(s_2)|^2)$$

and therefore

$$|f(s')|^2 \leq 2(|f(s_1')|^2 + |f(s_2'|^2)$$

where $'$ denotes opposite. The first half of (3) follows by summing these two inequalities over $s, s' \in S_{2n-1}$.

Now suppose $s \in S_{2n-2}$, suppose s points towards v_1, let v be the vertex which s leaves, and let s_1 and s_2 be the two oriented edges in S_{2n-1} which also leave v. Since $d(v, v_0) = 2n-2$, v is an even vertex, so $|f(s_1)| = |f(s_2)| = |f(s)|$,

$$2|f(s)|^2 = |f(s_1)|^2 + |f(s_2)|^2 \quad \text{and} \quad 2|f(s')|^2 = |f(s_1')|^2 + |f(s_2')|^2.$$

Summing these two over $s, s' \in S_{2n-2}$ gives the second half of (3) and proves Lemma 4. □

Proof of Lemma 3. Suppose $f \in \mathcal{H}_+$ satisfies $\langle f, \pi_+(\gamma)\phi_\omega \rangle = 0$ for each $\gamma \in \Gamma$, or equivalently, using (2)

$$\begin{aligned} 0 &= \langle f, \pi_+(\gamma)(\phi_0 + \omega\pi_+(a)\phi_0 + \omega^2\pi_+(a^2)\phi_0)\rangle \\ &= f(\gamma s_0) + \omega f(\gamma a s_0) + \omega^2 f(\gamma a^2 s_0). \end{aligned}$$

Since $\gamma s_0, \gamma a s_0$, and $\gamma a^2 s_0$ are the three oriented edges leaving γv_0, the defining relations for $\mathcal{H}_+$ give

$$f(\gamma s_0) + f(\gamma a s_0) + f(\gamma a^2 s_0) = 0$$

so $f(\gamma s_0) = \omega^2 f(\gamma a s_0) = \omega f(\gamma a^2 s_0)$. Since Γ acts transitively on the set of even vertices, this gives condition (2) of Lemma 4, while conditions (1) and (3) follow from the definition of $\mathcal{H}_+$. Thus $f = 0$ and Lemma 3 is proved. □

In order to produce an inclusion i of $\pi_+|_\Gamma$ into the left regular representation of Γ consider the matrix coefficient $\tilde{\phi}_\omega(\gamma) = \langle \phi_\omega, \pi_+(\gamma)\phi_\omega \rangle$, and likewise the matrix coefficient $\tilde{\phi}_0(\gamma) = \langle \phi_0, \pi_+(\gamma)\phi_0 \rangle$.

Lemma 5. *$\tilde{\phi}_\omega$ and $\tilde{\phi}_0$ are bounded convolvers of $\ell^2(\Gamma)$.*

Proof. Since $\tilde{\phi}_\omega$ is the sum of nine translations on both the left and the right of constant multiples of $\tilde{\phi}_0$, we need only consider $\tilde{\phi}_0$. We apply Haagerup's inequality (from [**10**]) as generalized by [**11**] to the free product of cyclic groups

$$||f||_{C^*_{\mathrm{reg}}(\Gamma)} \leq C \sum_{n=0}^{\infty} (n+1) \left(\sum_{|\gamma|=n} |f(\gamma)|^2 \right)^{1/2} \tag{4}$$

where $||\cdot||_{C^*_{\rm reg}(\Gamma)}$ is convolver norm. ([**11**] gives value for the constant C.) To see that the right-hand side is finite when we take f to be $\tilde{\phi}_0$, observe that

$$\begin{aligned}\tilde{\phi}_0(\gamma) &= \langle \phi_0, \pi_+(\gamma)\phi_0\rangle = \phi_0(\gamma s_0)\\ &= \left(\frac{-1}{2}\right)^{d(\gamma s_0, s_0)} \phi(s_0) = \left(\frac{-1}{2}\right)^{|\gamma|} \phi(s_0),\end{aligned}$$

and that $\#\{\gamma; |\gamma| = n\} = O(2^n)$. □

Following [**8**], observe that $\tilde{\phi}_\omega$ is necessarily a positive operator, let ψ_ω be the positive square root of $\tilde{\phi}_\omega$ in $C^*_{\rm reg}(\Gamma)$, and set $i(\phi_\omega) = \psi_\omega$. Since

$$\langle \phi_\omega, \pi_+(\gamma)\phi_\omega\rangle = \tilde{\phi}_\omega(\gamma) = \langle \psi_\omega, \pi_{\rm reg}(\gamma)\psi_\omega\rangle,$$

i extends uniquely to a Γ-isomorphism of $\mathcal{H}_+$ with the subspace, $\ell^2_+(\Gamma)$, generated by ψ_ω. Next, we describe $\ell^2_+(\Gamma)$.

Let $\ell^2_\omega(\Gamma)$ be the subspace of $\ell^2(\Gamma)$ consisting of functions satisfying

$$f(\gamma a) = \omega f(\gamma) \quad \text{for } \gamma \in \Gamma. \tag{5}$$

The orthogonal projection $P_\omega : \ell^2(\Gamma) \to \ell^2_\omega(\Gamma)$ is given by right convolution with $p_\omega = (1/3)(\delta_e + \omega\delta_a + \omega^2\delta_{a^2})$, so that

$$\mathrm{DIM}\,(\ell^2_\omega(\Gamma)) = 1/3.$$

(Here applying the definition of DIM to the case of a Γ-invariant subspace of a single copy of $\ell^2(\Gamma)$.) On the other hand, it is easy to verify that $\tilde{\phi}_\omega(\gamma a) = \omega\tilde{\phi}_\omega(\gamma)$, i.e., that $\tilde{\phi}_\omega = \tilde{\phi}_\omega * p_\omega$, and since ψ_ω, the square root of $\tilde{\phi}_\omega$ is the limit of polynomials in $\tilde{\phi}_\omega$ without constant terms, it is also true that $\psi_\omega = \psi_\omega * p_\omega$. Thus, $\ell^2_+(\Gamma)$ is contained in $\ell^2_\omega(\Gamma)$, but Lemma 2 says that $\mathrm{DIM}\,(\ell^2_+(\Gamma)) = \mathrm{DIM}\,(\pi_+|_\Gamma) = 1/3 = \mathrm{DIM}\,(\ell^2_\omega(\Gamma))$, hence $\ell^2_+(\Gamma) = \ell^2_\omega(\Gamma)$.

Proposition 1. *The representation $\pi_+|_\Gamma$ on $\mathcal{H}_+$ is equivalent via i to the left regular representation of Γ on $\ell^2_\omega(\Gamma)$.*

Please observe that insofar as we are unable to calculate the convolution square root ψ_ω explicitly, we are unable in the end to give explicit formulas for our decomposition.

6. The decomposition of $\pi_+|_{\mathbf{Z}_3 * \mathbf{Z}_3}$. Taking advantage of the previous proposition, we will decompose $\ell^2_\omega(\Gamma)$ instead of $\mathcal{H}_+$. For any $\gamma \in \Gamma$ the number of times a occurs in any representation of γ as a product of a's and b's defines a residue class modulo 3, denoted $\eta(\gamma)$, which is independent of the representation. Let Γ_0 be the subgroup consisting of $\gamma \in \Gamma$ such that $\eta(\gamma) = 0$. (Clearly, η is the unique homomorphism from $\mathbf{Z}_3 * \mathbf{Z}_3$ to $\mathbf{Z}_3$ which takes a to 1 and b to 0.) One can see [**13**] that Γ_0 is isomorphic to the free product of three copies of $\mathbf{Z}_3$ with generators b, aba^2, and a^2ba. Let

$$\mu_1 = (1/6)(\delta_b + \delta_{b^2} + \delta_{aba^2} + \delta_{ab^2a^2} + \delta_{a^2ba} + \delta_{a^2b^2a}).$$

The function μ_1 is considered by [**12, 14** and **19**], who describe its spectrum as a (right) convolver of $\ell^2(\Gamma_0)$ and the associate spectral resolution. The first and last of these works show also that this spectral resolution induces a direct integral decomposition of $\ell^2(\Gamma_0)$ into irreducible components.

Next observe that ρ, the restriction map from $\ell^2_\omega(\Gamma)$ to $\ell^2(\Gamma_0)$, is unitary up to a factor of $\sqrt{3}$, since each triple $(\gamma, \gamma a, \gamma a^2)$ contains exactly one element of Γ_0. Let T be right convolution with μ_1 on $\ell^2(\Gamma)$ and let T_0 be right convolution with μ_1 on $\ell^2(\Gamma_0)$. T preserves $\ell^2_\omega(T)$ since $p_\omega * \mu_1 = \mu_1 * p_\omega$. It is clear that $\rho T = T_0 \rho$. Thus ρ^{-1} takes the spectral decomposition of $\ell^2(\Gamma_0)$ with respect to T_0, as described in the above works, to the spectral decomposition of $\ell^2_\omega(\Gamma)$ with respect to T. Let π denote the regular representation of Γ on $\ell^2_\omega(\Gamma)$ and π_0 the regular representation of Γ_0 on $\ell^2(\Gamma_0)$. Then $\rho\pi(\gamma) = \pi_0(\gamma)\rho$ for $\gamma \in \Gamma_0$, so the components of the spectral decomposition of $\ell^2_\omega(\Gamma)$ are irreducible as representations of Γ_0, a fortiori as representations of Γ. This proves

Proposition 2. *The spectral decomposition of $\ell^2_\omega(\Gamma)$ with respect to right convolution by μ_1 induces a direct integral decomposition into irreducibles.*

We mention another inclusion of $\pi_+|_\Gamma$ in $\ell^2(\Gamma)$, namely the inclusion mapping $\phi_0 \in \mathcal{H}_+$ to $\tilde{\phi}_0 \in \ell^2(\Gamma)$. This works because $\tilde{\phi}_0 * \tilde{\phi}_0 = \tilde{\phi}_0$, so that $\tilde{\phi}_0$ is its own square root in $C^*_{\text{reg}}(\Gamma)$. The image of this inclusion is the subspace of $\ell^2_0(\Gamma)$ generated by $\tilde{\phi}_0$. This subspace also comes up in the spectral resolution of the operator of right convolution by $\kappa = (1/4)(\delta_a + \delta_{a^2} + \delta_b + \delta_{b^2})$ on $\ell^2(\Gamma)$. [**14**] shows that the continuous spectrum of κ consists of a closed interval I while the point spectrum is $\{-1/2\}$. The piece of $\ell^2(\Gamma)$ corresponding to the continuous spectrum resolves as the direct integral of irreducible components [**12**] while the eigenspace of $-1/2$ is nothing but $\ell_0^2(\Gamma)$. This last assertion follows easily from the description of the orthogonal projection onto the eigenspace which one can reconstruct from [**13**].

Thus, the decomposition of $\mathcal{H}_+ \simeq \ell_0^2(\Gamma)$ into irreducibles completes the project of finding at least one particular decomposition of $\ell^2(\Gamma)$ into irreducibles.

7. The other three cases. First consider $\pi_-|_\Gamma$ when $\Gamma = \mathbf{Z}_3 * \mathbf{Z}_3$ (as in Section 2). [**16**] says that

$$\pi_- \cong \mathrm{sgn} \otimes \pi_+$$

where $\mathrm{sgn}\,(g)$ is $+1$ or -1 according to whether g preserves the set of even vertices or sends it to the set of odd vertices. As noted in Section 2, $\mathrm{sgn}\,(\gamma) = +1$ for all γ in Γ, so $\pi_-|_\Gamma \cong \pi_+|_\Gamma$ and this case reduces trivially to the previous case.

We now show how to work out a similar construction for $\Gamma = \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ (as in Section 2). Recall that $\Gamma = \langle a, b, c; a^2 = b^2 = c^2 = e \rangle$ and let $|\gamma|$ denote the length of the shortest product of generators giving γ.

Consider $\pi_+|_\Gamma$ first. One shows that ϕ_0 is cyclic for $\pi_+|_\gamma$ by showing that no function of $\mathcal{H}_+$ can be orthogonal to $\pi_+(\gamma)\phi_0$ for all γ in Γ. Let $\phi_0(\gamma) : \Gamma \to \mathbf{C}$ be the matrix coefficient

$$\tilde{\phi}_0(\gamma) = \langle \phi_0, \pi_+(\gamma)\phi_0 \rangle = \phi_0(\gamma s_0).$$

Use Haagerup's inequality (Section 5.(4)) to show that $\tilde{\phi}_0$ is a bounded convolver, observing that

$$|d(\gamma \cdot s_0, s_0) - |\gamma|| \leq 1$$

and so

$$|\tilde{\phi}_0(\gamma)| \leq 2\tilde{\phi}_0(e)(1/2)^{|\gamma|},$$

and again $\#\{\gamma; |\gamma| = n\} = O(2^n)$. A slight modification of the arguments in Section 5 shows that $\pi_+|_\Gamma$ is equivalent to the left regular representation of Γ restricted to the subspace

$$\ell^2_+(\Gamma) = \{f \in \ell^2(\Gamma) : f(\gamma a) = f(\gamma)\ \forall \gamma \in \Gamma\}.$$

(The generator a appears here because a is that generator which flips the oriented edge s_0, and ϕ_0 is the projection to $\mathcal{H}_+$ of the delta function at s_0.)

Now let $\eta(\gamma)$ be the residue class modulo 2 of the number of times the generator a occurs in any expression of γ as a product of generators, and let Γ_0 be the subgroup of all $\gamma \in \Gamma$ such that $\eta(\gamma) = 0$. Then Γ_0 is the free product of four copies of $\mathbf{Z}_2$ with generators $\{b, c, aba, aca\}$. As in Section 6, the spectral decomposition of $\ell^2_+(\Gamma)$ with respect to right convolution by

$$\mu = (1/4)(\delta_b + \delta_c + \delta_{aba} + \delta_{aca})$$

(given explicitly in [**3**]) is a decomposition into irreducibles. This completes the decomposition of $\ell^2_+(\Gamma)$ and hence of $\pi_+|_\Gamma$.

Lastly, consider the case of $\pi_-|_\Gamma$ with $\Gamma = \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$. One must redefine ϕ_0 as the projection of δ_{s_0} onto $\mathcal{H}_- \subseteq \ell^2(\mathcal{E})$. Then the preceding constructions show that $\pi_-|_\Gamma$ is equivalent to the regular representation of Γ restricted to

$$\ell^2_-(\Gamma) = \{f \in \ell^2(\Gamma) : f(\gamma a) = -f(\gamma)\ \forall \gamma \in \Gamma\}$$

and that $\ell^2(\Gamma)$ also decomposes into irreducibles according to the spectral decomposition for right convolution by

$$\mu = (1/4)(\delta_a + \delta_b + \delta_{aba} + \delta_{aca}).$$

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