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ON A CLASSICAL BOUNDARY VALUE PROBLEM INVOLVING A SMALL PARAMETER

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ABSTRACT. Differential inequality techniques are used to provide accurate information *throughout the interval* $[a, b]$ on boundary layer solutions of the problem

$$\varepsilon y'' = f(t,y)y' + g(t,y) \quad \text{for} \quad a \le t \le b$$
$$y(a) = A \quad \text{and} \quad y(b) = B,$$

subject to weak regularity requirements on the data ($\varepsilon > 0$ is a small positive parameter). Such accurate information has been previously obtained by asymptotic expansion techniques coupled with the contraction mapping method, but only subject to more severe regularity requirements on the data, whereas the differential inequality technique has previously given such accurate information subject to weak regularity requirements, but only outside the boundary layer, with a loss of accuracy occurring inside the boundary layer. The detailed approximations of solutions obtained here may be very useful in studying solutions with other types of singularity perturbed behavior, such as shock or interior layer behavior. Problems of this type arise in fluid dynamics.

1. Introduction. In this paper is studied the singularly perturbed scalar boundary value problem

$$\begin{aligned} \varepsilon y'' = f(t,y)y' + g(t,y), \qquad a < t < b, \\ y(a) = A \quad \text{and} \quad y(b) = B, \end{aligned} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and where y, f, g, A, and B are real valued quantities. This problem has been much studied in the literature, mainly by the method of differential inequalities and the method of contraction mappings. The former method has provided the existence of solutions and detailed information on solutions away from the boundary layer, subject to relatively weak smoothness requirements

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on the data, but detailed information has not been obtained inside the boundary layer. The latter method has provided existence and detailed information on solutions throughout the interval $[a, b]$, but only subject to relatively stronger smoothness requirements. Differential inequalities are used here to obtain detailed information on solutions throughout the interval including precise and detailed information inside the boundary layer, subject to relatively weak smoothness requirements on the data. This result is obtained by defining lower and upper solutions (α and β) in a fruitful way, which may provide useful insight in attacking other classes of singularly perturbed differential equations.

Solutions possessing a single boundary layer are considered for (1.1), and in particular interior layers and (interior) turning points are excluded. Such solutions of (1.1) possessing a single boundary layer have been studied by many authors including Coddington and Levinson [**5**], Brish [**2**], Wasow [**17**], Erdelyi [**7, 8**], O'Malley [**14, 15**], Chang [**3, 4**], Howes [**10**], van Harten [**9**], and Smith [**16**]. Most of these works, however, have made rather strong assumptions on the "boundary layer jump," the sign of the function $f(t, y)$, or on the regularity of the function f, or have not provided the fine details, quantitatively speaking, for the solutions $y = y(t, \varepsilon)$ inside the boundary layer.

A very general sufficient condition for boundary layer behavior was given many years ago by Coddington and Levinson [**5**]. We use the same assumptions as Coddington and Levinson [**5**], except that we define a certain domain $\mathcal{D}$ in a slightly different manner which seems more natural when using differential inequalities. Since we show the existence of a solution $y = y(t, \varepsilon)$ of (1.1) that is bounded by functions $\alpha(t, \varepsilon)$ and $\beta(t, \varepsilon)$, namely $\alpha(t, \varepsilon) \le y(t, \varepsilon) \le \beta(t, \varepsilon)$ for t in $[a, b]$, we define the domain $\mathcal{D}$ as follows:

$$\mathcal{D} = \{(t, y) : a \le t \le b, \alpha(t, \varepsilon) \le y \le \beta(t, \varepsilon)\},$$

where $\alpha(t, \varepsilon)$ and $\beta(t, \varepsilon)$ are lower and upper solutions of problem (1.1) given explicitly by equations (1.4) and (1.5), respectively. We will show that $\beta(t, \varepsilon) - \alpha(t, \varepsilon) = O(\varepsilon)$ for all t in $[a, b]$ and hence obtain very detailed approximations for solutions throughout the interval $[a, b]$, including the boundary layer. See Figure 1.1.

The following theorem is a classic result given by Coddington and Levinson [**5**]. It gives sufficient conditions for the existence of a solution with boundary layer behavior at the left endpoint ($t = a$).

FIGURE 1.1(a). $[A < u(a)]$.

Theorem 1.1 (Coddington and Levinson). *Assume that*

(1) *the reduced problem* $0 = f(t,u)u' + g(t,u)$, $u(b) = B$, *has a solution* $u = u(t)$ *of class* $C^{(2)}[a,b]$;

(2) f *and* g *are of class* $C^{(1)}$ *with respect to* t *and* y *for* (t,y) *in* $\mathcal{D}$;

(3) *the reduced solution* $u = u(t)$ *is globally stable, that is, there exists a positive constant* k *such that* $f(t,u) \le -k$ *for* t *in* $[a,b]$;

(4) *the inequality*

$$(u(a) - A) \cdot \int_{u(a)}^{n} f(a,s)\,ds > 0$$

holds for $A \le n < u(a)$ *if* $A < u(a)$, *or for* $u(a) < n \le A$ *if* $u(a) < A$.

Then, for each sufficiently small $\varepsilon > 0$, *(1.1) has a solution* $y = y(t,\varepsilon)$

FIGURE 1.1(b). $[A > u(a)]$.

in $\mathcal{D}$ such that, for each fixed t in $(a, b]$,

$$\lim_{\varepsilon \to 0} y(t, \varepsilon) = u(t)$$

and

$$\lim_{\varepsilon \to 0} y'(t, \varepsilon) = u'(t).$$

Under these assumptions Coddington and Levinson showed that this solution is unique in the sense that there is no other solution of (1.1) which satisfies the stated limiting relations.

Using the method of differential inequalities, Howes [**10**] proved the following theorem which provides only limited information inside the boundary layer.

Theorem 1.2 (Howes). *Under the assumptions of Theorem* 1.1 *we can further conclude that for t in $[a,b]$,*

$$y(t,\varepsilon) = u(t) + w_L(t,\varepsilon) + O(\varepsilon)$$

and

$$y'(t,\varepsilon) = u'(t) + w_L'(t,\varepsilon) + O(\varepsilon),$$

where $w_L(t,\varepsilon)$ is an $O(\varepsilon)$-boundary layer function at $t = a$. (That is, $w_L(t,\varepsilon) = O(|A - u(a)|)$ for $a \le t \le a + c_1\varepsilon$ and $\lim_{\varepsilon\to 0} w_L(t,\varepsilon) = 0$ for each fixed t in $(a,b]$.)

In proving Theorem 1.2, Howes defined lower and upper solutions α and β, respectively, as

$$\alpha(t,\varepsilon) = u(t) + w_L(t,\varepsilon) + O(\varepsilon) \tag{1.2}$$

and

$$\beta(t,\varepsilon) = u(t) + O(\varepsilon), \tag{1.3}$$

where w_L is a boundary layer function. He then showed problem (1.1) has a solution $y = y(t,\varepsilon)$ satisfying $\alpha(t,\varepsilon) \le y(t,\varepsilon) \le \beta(t,\varepsilon)$ for t in $[a,b]$. Thus, detailed information inside the boundary layer is lacking.

We approach the problem by defining bounding pairs such as

$$\alpha(t,\varepsilon) = u(t) + w_1(t,\varepsilon) + O(\varepsilon) \tag{1.4}$$

and

$$\beta(t,\varepsilon) = u(t) + w_2(t,\varepsilon) + O(\varepsilon), \tag{1.5}$$

where w_1 and w_2 are unique solutions to the equations

$$\varepsilon w_1''(t) = f(a, u(a) + w_1(t))w_1'(t) + ce^{-h(t-a)/2\varepsilon} \tag{1.6}$$

and

$$\varepsilon w_2''(t) = f(a, u(a) + w_2(t))w_2'(t) - ce^{-h(t-a)/2\varepsilon} \tag{1.7}$$

satisfying $w_j(a,\varepsilon) = A - u(a)$ and $\lim_{\varepsilon\to 0} w_j(t,\varepsilon) = 0$ for each fixed t in $(a,b]$ for $j = 1,2$. c and h are positive constants to be determined below. We show that

$$\max_{t\in[a,b]} |w(t,\varepsilon) - w_j(t,\varepsilon)| = O(\varepsilon) \quad \text{for } j = 1,2, \tag{1.8}$$

where w is the unique solution of the equation

$$\varepsilon w''(t,\varepsilon) = f(a, u(a) + w(t))w'(t) \tag{1.9}$$

satisfying $w(a) = A - u(a)$ and $\lim_{\varepsilon\to 0} w(t,\varepsilon) = 0$ for t in $(a,b]$. Then, upon showing $\alpha(t,\varepsilon) \le y(t,\varepsilon) \le \beta(t,\varepsilon)$, we are able to conclude that

$$y(t,\varepsilon) = u(t) + w(t,\varepsilon) + O(\varepsilon)$$

and

$$y'(t,\varepsilon) = u'(t) + w'(t,\varepsilon) + O(e^{-k(t-a)/\varepsilon}) + O(\varepsilon),$$

which provides detailed information throughout the interval $[a,b]$.

In Section 2 we define the lower and upper solutions (bounding functions) and state a basic existence result from the theory of differential inequalities. Lower and upper solutions are constructed in Section 3 and used to prove the main result of this paper (Theorem 3.1). Example 3.1 is a model problem which has been studied by many authors in singular perturbation theory. The analysis proving that equations (1.6), (1.7), and (1.9) have solutions satisfying condition (1.8) is contained in the Appendix.

2. A preliminary result. In this section we state a standard result from the theory of differential inequalities which is used in the proof of our theorems.

Definition 2.1. Twice continuously differentiable functions $\alpha(t)$ and $\beta(t) \in C^{(2)}[a,b]$ are said to be a bounding pair of problem (1.1) if the following properties hold for t in $[a,b]$:

$$\begin{gathered} \alpha(t,\varepsilon) \le \beta(t,\varepsilon), \\ \alpha(a) \le A \le \beta(a), \qquad \alpha(b) \le B \le \beta(b), \end{gathered} \tag{2.1}$$

and

$$\varepsilon\alpha'' \geq f(t,\alpha)\alpha' + g(t,\alpha), \qquad \varepsilon\beta'' \leq f(t,\beta)\beta' + g(t,\beta).$$

α and β are also known as lower and upper solutions, respectively.

Theorem 2.1. *Assume that there exist bounding functions $\alpha(t)$ and $\beta(t)$ with the properties* (2.1)*, and assume that the functions $f(t,y)$ and $g(t,y)$ are continuous in the region $\mathcal{D} = [a,b] \times [\alpha,\beta]$. Then the Dirichlet problem* (1.1) *has a solution $y = y(t,\varepsilon)$ of class $C^{(2)}[a,b]$ satisfying $\alpha(t) \leq y(t) \leq \beta(t)$ for t in $[a,b]$.*

Proof. This result can be found in many references. See Nagumo [**12**], Bernfeld and Lakshmikantham [**1**], Jackson [**11**], or O'Donnell [**13**]. □

3. Existence and approximations. The following theorem gives sufficient conditions for the existence of a solution $y = y(t,\varepsilon)$ exhibiting boundary layer behavior at the left endpoint ($t = a$); detailed approximations are also obtained.

Theorem 3.1. *Suppose the assumptions of Theorem* 1.1 *hold. Then there exists a solution $y = y(t,\varepsilon)$ of problem* (1.1) *for each sufficiently small $\varepsilon > 0$ such that for t in $[a,b]$*

$$y(t,\varepsilon) = u(t) + w(t,\varepsilon) + O(\varepsilon) \tag{3.1}$$

and

$$y'(t,\varepsilon) = u'(t) + w'(t,\varepsilon) + O(e^{-q(t-a)/\varepsilon}) + O(\varepsilon), \tag{3.2}$$

where $w(t,\varepsilon)$ is the unique solution of the equation $\varepsilon w'' = f(a,u(a)+w)w'$ satisfying $w(a,\varepsilon) = A - u(a)$ and $\lim_{\varepsilon\to 0} w(t,\varepsilon) = 0$ for each fixed $t > a$. q is a positive constant.

Proof. The proof is similar to that of Howes [**10**]. For definiteness, we construct a bounding pair (α,β) under the assumption that $A < u(a)$.

If $A > u(a)$, the bounding pair would be defined analogously. Define for t in $[a,b]$ and $\varepsilon > 0$ the functions

$$\alpha(t,\varepsilon) = u(t) + w_1(t,\varepsilon) - \varepsilon\gamma l^{-1}[e^{\lambda(t-2|b|)} - 1]$$

and

$$\beta(t,\varepsilon) = u(t) + w_2(t,\varepsilon) + \varepsilon\gamma l^{-1}[e^{\lambda(t-2|b|)} - 1].$$

Here $w_1(t,\varepsilon)$ and $w_2(t,\varepsilon)$ are exponentially decaying solutions of problems (1.6) and (1.7) satisfying $w_1(t,\varepsilon) \le w_2(t,\varepsilon)$ for t in $[a,b]$ (see the Appendix). The positive constant l is an upper bound on

$$|f_y(t,y)u' + g_y(t,y)| \quad \text{for } (t,y) \text{ in } \mathcal{D}$$

and

$$\lambda = -lk^{-1} + O(\varepsilon) < 0$$

is a root of $\varepsilon\lambda^2 + k\lambda + l = 0$. Finally, M is an upper bound on $|u''(t)|$ for t in $[a,b]$, and γ is a positive constant to be chosen below ($\gamma > M$).

It is easy to see that $\alpha \le \beta$, $\alpha(a,\varepsilon) \le A \le \beta(a,\varepsilon)$, and $\alpha(b,\varepsilon) \le B \le \beta(b,\varepsilon)$. The conditions of Theorem 2.1 will be satisfied if, for t in $[a,b]$,

$$\varepsilon\alpha'' \ge f(t,\alpha)\alpha' + g(t,\alpha) \quad \text{and} \quad \varepsilon\beta'' \le f(t,\beta)\beta' + g(t,\beta).$$

We only consider the inequality for α since the proof for β is analogous. For simplicity, we define $\varepsilon_1 = \varepsilon\gamma l^{-1}\exp(\lambda(t-2|b|))$, so that $\alpha = u + w_1 + \varepsilon\gamma l^{-1} - \varepsilon_1$.

$$\begin{aligned}\varepsilon\alpha'' - f(t,\alpha)\alpha' - g(t,\alpha) = {} & \varepsilon u'' + \varepsilon w_1'' - \varepsilon\lambda^2\varepsilon_1 \\ & - [f(t,u+w_1) + f_y(\varepsilon\gamma l^{-1} - \varepsilon_1)] \\ & \cdot (u' + w_1' - \lambda\varepsilon_1) \\ & - [g(t,u+w_1) + g_y(\varepsilon\gamma l^{-1} - \varepsilon_1)],\end{aligned}$$

where f_y and g_y are evaluated at appropriate intermediate points. From assumption (3), we can write

$$f(t,u+w_1)\lambda\varepsilon_1 \ge -k\lambda\varepsilon_1 + f_y\lambda\varepsilon_1 w_1, \tag{3.3}$$

where f_y is evaluated at an appropriate intermediate point. Using (3.3), $|u''| \leq M$, $|f_y u' + g_y| \leq l$, and $\varepsilon\lambda^2 + k\lambda + l = 0$, we can write

$$\begin{aligned}\varepsilon\alpha'' - f(t,\alpha)\alpha' - g(t,\alpha) = {} & \varepsilon w_1'' - f(t, u + w_1)w_1' \\ & - [f(t, u + w_1)u' + g(t, u + w_1)] \\ & + f_y(\varepsilon_1 - \varepsilon\gamma l^{-1})w_1' + f_y\lambda\varepsilon_1 w_1 \\ & + (\gamma - M)\varepsilon + O(\varepsilon^2).\end{aligned}$$

At this point, it is convenient to note that

$$\varepsilon w_1'' - f(t, u + w_1)w_1' = \varepsilon w_1'' - f(a, u(a) + w_1)w_1' - [f_t + f_y u'](t - a)w_1',$$

where the terms f_t, f_y, and u' are evaluated at intermediate points. Using assumption (1), we also note that

$$f(t, u + w_1)u' + g(t, u + w_1) = [f_y u' + g_y]w_1,$$

where f_y and g_y are evaluated at intermediate points. The inequality can now be written

(3.4)

$$\begin{aligned}\varepsilon\alpha'' - f(t,\alpha)\alpha' - g(t,\alpha) \geq {} & \varepsilon w_1'' - f(a, u(a) + w_1)w_1' \\ & - [f_t + f_y u'](t - a)w_1' - [f_y u' + g_y]w_1 \\ & + f_y(\varepsilon_1 - \varepsilon\gamma l^{-1})w_1' + f_y\lambda\varepsilon_1 w_1 \\ & + (\gamma - M)\varepsilon + O(\varepsilon^2).\end{aligned}$$

Assumptions (3) and (4) imply problem (1.6) has a solution $w_1 = w_1(t, \varepsilon)$, as shown in the Appendix. Substituting

$$\varepsilon w_1'' - f(a, u(a) + w_1)w_1' = ce^{-h(t-a)/2\varepsilon},$$

along with estimates for w_1 and w_1' which can be found from equations (A.1) and (A.2), into inequality (3.4), the result

$$\alpha'' - f(t,\alpha)\alpha' - g(t,\alpha) \geq 0$$

can be easily obtained. Similarly, we find

$$\varepsilon\beta'' \leq f(t,\beta)\beta + g(t,\beta),$$

and by Theorem 2.1, there exists a solution $y = y(t,\varepsilon)$ of problem (1.1) satisfying

$$\alpha(t,\varepsilon) \le y(t,\varepsilon) \le \beta(t,\varepsilon) \quad \text{for } t \text{ in } [a,b].$$

The estimates (1.8), proven in the Appendix, then imply the desired result, namely,

$$y(t,\varepsilon) = u(t) + w(t,\varepsilon) + O(\varepsilon) \quad \text{for } t \text{ in } [a,b].$$

The estimate (3.2) on y' can be obtained by setting $z = y - u - w = O(\varepsilon)$ and substituting into problem (1.1). Using $\varepsilon w'' = f(a, u(a)+w)w'$ and $f(t,u)u' + g(t,u) = 0$, we get the transformed problem

$$\begin{aligned} \varepsilon z'' - f(t,y)z' = {} & [f_t + f_y u'(\cdot)](t-a)w' + f_y w' z \\ & + [f_y u' + g_y]w + [f_y u' + g_y]z - \varepsilon u'' \end{aligned} \tag{3.5}$$

$$z(a) = 0, \qquad z(b) = -w(b),$$

where f_t, f_y, g_y, and $u'(\cdot)$ are evaluated at appropriate intermediate points. The equations (A.7) and (A.6) for w and w' can then be used in (3.5), and the result follows by arguing in a similar manner to Coddington and Levinson [**5**], or Howes [**10**]. □

Example 3.1. We consider the following model problem which has interested many writers on singular perturbations; see, for example, Cole [**6**], and Howes [**10**].

$$\begin{aligned} \varepsilon y'' &= -yy' + y, \qquad 0 < t < 1 \\ y(0,\varepsilon) &= A, \qquad y(1,\varepsilon) = B. \end{aligned} \tag{3.6}$$

We find conditions on A and B which yield boundary layer behavior at $t = 0$. The reduced problem has the solution $u(t) = t + B - 1$, which is globally stable for $B > 1$, namely $f(t,u) = -(t+B-1) < 0$ for t in $[0,1]$. If $A > u(0) = B - 1$, then one sees directly that assumption (4) is satisfied, namely,

$$\int_{(B-1)}^{n} (-s)\, ds < 0 \qquad \text{for} \quad B - 1 < n \le A.$$

Since all four assumptions of Theorem 3.1 are satisfied, there exists a solution $y = y(t,\varepsilon)$ of problem (3.6). The boundary layer function

$w(t,\varepsilon)$ satisfies the problem $\varepsilon w'' = -[u(0)+w]w'$, $w(0) = A - u(0)$, which can be solved by quadratures. From (3.1) and (3.2), we then have the estimates

$$y(t,\varepsilon) = t + (B-1)\coth[(B-1)(t+C_1)(2\varepsilon)^{-1}] + O(\varepsilon)$$

and

$$y'(t,\varepsilon) = 1 - ((B-1)^2/2\varepsilon)\operatorname{csch}^2[(B-1)(t+C_1)(2\varepsilon)^{-1}] \\ + O(e^{-qt/\varepsilon}) + O(\varepsilon),$$

where $C_1 = 2\varepsilon(B-1)^{-1}\coth^{-1}[A(B-1)^{-1}]$. Similarly, if $A < u(0) = B-1$, one directly sees that assumption (4) is satisfied provided $|A| < B-1$. Here one obtains the estimates

$$y(t,\varepsilon) = t + (B-1)\tanh[(B-1)(t+C_2)(2\varepsilon)^{-1}] + O(\varepsilon)$$

and

$$y'(t,\varepsilon) = 1 + ((B-1)^2/2\varepsilon)\operatorname{sech}^2[(B-1)(t+C_2)(2\varepsilon)^{-1}] \\ + O(e^{-qt/\varepsilon}) + O(\varepsilon),$$

where $C_2 = 2\varepsilon(B-1)^{-1}\tanh^{-1}[A(B-1)^{-1}]$. Other choices of A and B yield solutions with boundary layers at $t = 1$. Interior (shock) layers are also possible. See Figure 3.1.

If we set $t' = a + b - t$, Theorem 3.1 can be transformed into a theorem which allows solutions to have boundary layers at the right endpoint. This result can be obtained without difficulty, and its statement is therefore left to the reader. Also, the results follow *mutatis mutandis* for problems with an ε-dependent right-hand side and ε-dependent boundary conditions, namely, $f = f(t,y,\varepsilon)$, $g = g(t,y,\varepsilon)$, $A = A(\varepsilon)$ and $B = B(\varepsilon)$. Only slight modifications are needed to state the more generalized theorems. For example, we require $f(t,u,0)u' + g(t,u,0) = 0$, and f, g, A and B to be class $C^{(1)}$ with respect to ε.

Remark. The assumption that g is of class $C^{(1)}$ with respect to t can be relaxed. From the proof of Theorem 3.1, one sees that g needs to be continuous only with respect to t.

FIGURE 3.1.

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Appendix

First we show that the boundary layer equations (1.6), (1.7), and (1.9) each have unique solutions $w_1 = w_1(t, \varepsilon)$, $w_2 = w_2(t, \varepsilon)$ and $w = w(t, \varepsilon)$, respectively, satisfying the conditions $w(a, \varepsilon) = A - u(a)$ and $\lim_{\varepsilon \to 0} w(t, \varepsilon) = 0$ for each fixed $t > a$. Integrating equation (1.6) from positive infinity to t, we see that a first integral is given by

$$\varepsilon w_1'(t, \varepsilon) = G(w_1) w_1(t, \varepsilon) - 2c\varepsilon h^{-1} e^{-h(t-a)/(2\varepsilon)}, \tag{A.1}$$

where

$$G(w) = \int_0^1 f(a, u(a) + sw(t))\, ds.$$

Equation (A.1) along with the initial condition $w_1(a, \varepsilon) = A - u(a)$ can be integrated to give

$$\begin{aligned} w_1(t,\varepsilon) &= w_1(a) e^{(1/\varepsilon)\int_a^t G(w_1)\,ds} \\ &\quad - \frac{2c}{h}\int_a^t e^{-h(s-a)/(2\varepsilon)} \cdot e^{(1/\varepsilon)\int_s^t G(w_1)\,dp}\, ds. \end{aligned} \tag{A.2}$$

Conditions (3) and (4) of Theorem 1.1 imply that there exists a positive constant h such that $G(w_1) \le -h$, which shows that $w_1(t, \varepsilon)$ decays exponentially. Also, using the identity

$$f(a, u(a) + sw(t)) = f(a, u(a) + sw_1(t) + s(w(t) - w_1(t))), \tag{A.3}$$

one can easily show

$$G(w) = G(w_1) + Q(t) \cdot [w(t) - w_1(t)], \tag{A.4}$$

where $Q(t) = \int_0^1 f_y(\cdot) s\, ds$, and $(\cdot) = u(a) + sw_1(t) + \theta s(w - w_1)$ for $0 < \theta < 1$; it follows that the function $G(w(t))w(t)$ satisfies a Lipschitz condition in w, namely,

$$|G(w_1)w_1 - G(w)w| \le H \cdot |w_1(t) - w(t)|, \tag{A.5}$$

where H is a bound on $|G(w_1) - Qw|$. The representation (A.1) and a standard continuation result then prove that $w_1 = w_1(t, \varepsilon)$ exists for all $t > a$ and is unique. Similarly, for (1.9) the unique solution vanishing at infinity satisfies

$$\varepsilon w'(t, \varepsilon) = G(w(t)) w(t, \varepsilon) \tag{A.6}$$

and

$$w(t, \varepsilon) = w(a) e^{(1/\varepsilon)\int_a^t G(w(s))\,ds}. \tag{A.7}$$

It is easy to see that $w_1(t, \varepsilon) \le w_2(t, \varepsilon)$ for t in $[a, b]$. Equations (A.1) and (A.6) imply that $w_1'(a) < w'(a)$, and if $w_1(t_1) = w(t_1)$, for some

$t_1 > a$, (A.1) and (A.6) would also imply that $w_1'(t_1) < w'(t_1)$. Hence, $w_1(t,\varepsilon) \le w(t,\varepsilon)$. A similar argument shows that $w_1(t,\varepsilon) \le w_2(t,\varepsilon)$ for t in $[a,b]$.

Next we show that

$$\max_{t\in[a,b]} |w_j(t,\varepsilon) - w(t,\varepsilon)| = O(\varepsilon) \quad \text{for } j = 1, 2,$$

where w_1, w_2 and w are described above. Note that the function $|w_1(t) - w(t)|$ will have a maximum value when $w_1'(t) - w'(t) = 0$. From equations (A.1) and (A.6), we have

$$\text{(A.8)} \qquad \varepsilon w_1' - \varepsilon w' = G(w_1)w_1 - G(w)w - 2c\varepsilon h^{-1} e^{-h(t-a)/(2\varepsilon)},$$

and a maximum occurs at some $t = T > a$, where

$$G(w_1(T))w_1(T) - G(w(T))w(T) = 2c\varepsilon h^{-1} e^{-h(T-a)/(2\varepsilon)}.$$

Using identity (A.4), one can easily obtain

$$|w_1(T) - w(T)| = O(\varepsilon)/[G(w_1(T)) + w(T)\cdot Q(T)].$$

The result will follow if we show the quantity $[G(w_1(T)) + w(T)Q(T)]$ is bounded away from 0. Since $G(w_1(t)) \le -h$, $|Q(t)|$ is bounded, and $w(T)$ decays exponentially, we see that the quantity $[G_1(t) + w(t)Q(t)]$ is bounded away from 0 for t in the interval $[a + p\varepsilon, \infty)$, where p is a suitably chosen positive constant. Hence, if T is in $[a + p\varepsilon, \infty)$, the result follows. Now we must show

$$\max_{t\in[a,a+p\varepsilon]} |w_1(t,\varepsilon) - w(t,\varepsilon)| = O(\varepsilon).$$

Integrating (A.8) from a to t, we see that

$$\text{(A.9)} \quad \varepsilon w_1(t,\varepsilon) - \varepsilon w(t,\varepsilon) - \int_a^t [G(w_1)w_1(s) - G(w)w(s)]\, ds = O(\varepsilon^2).$$

Using identity (A.4), one easily obtains from (A.9) the inequality

$$\text{(A.10)} \qquad \varepsilon|w_1(t,\varepsilon) - w(t,\varepsilon)| \le H \cdot \int_a^t |w_1(s) - w(s)|\, ds + H_1 \cdot \varepsilon^2,$$

where H is defined in (A.5) and H_1 is a suitable positive constant.

Let $r(t) = |w_1(t,\varepsilon) - w(t,\varepsilon)|$, so that (A.10) becomes

$$\varepsilon r(t) \le H \int_a^t r(s)\,ds + H_1\varepsilon^2. \tag{A.11}$$

Setting $R(t) = \int_a^t r(s)\,ds$, equation (A.11) becomes

$$\varepsilon R'(t) - HR(t) \le H_1 \cdot \varepsilon^2, \tag{A.12}$$

which, upon integrating from a to t, yields

$$R(t) \le H_1 \cdot H^{-1} \cdot \varepsilon^2 [e^{H(t-a)/\varepsilon} - 1]. \tag{A.13}$$

Substituting (A.13) into (A.11), we obtain

$$|w_1(t,\varepsilon) - w(t,\varepsilon)| \le H_1 \cdot \varepsilon \cdot e^{H(t-a)/\varepsilon}.$$

This shows that, for t in $[a, a+p\varepsilon]$,

$$|w_1(t,\varepsilon) - w(t,\varepsilon)| \le H_1 \cdot \varepsilon \cdot e^{H\cdot p} = O(\varepsilon),$$

as desired. A similar procedure shows

$$|w_2(t,\varepsilon) - w(t,\varepsilon)| = O(\varepsilon)$$

for t in $[a,\infty)$.

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