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SYMMETRIES, TOPOLOGICAL DEGREE AND A THEOREM OF Z.Q. WANG

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ABSTRACT. The primary purpose of this paper is to give a technically simpler proof of a formula for the Leray-Schauder degree of compact perturbations of the identity covariant under the smooth action of a compact Lie group G, recently proposed by Z.Q. Wang in the Hilbert space setting. Doing so, we eliminate questions raised by a few gaps of variable importance in Wang's proof, we extend the validity of his result to arbitrary Banach space and arbitrary continuous linear actions, and we show that in all the cases when the formula is not both trivial and useless, it depends only upon the action of a finite group (the factor group $N(T)/T$ where T is any maximal torus of the identity component of G and $N(T)$ its normalizer in G) in some appropriate subspace (the fixed point space of T). This is important regarding the practical value of the formula.

1. Introduction. In the recent paper [**20**] devoted to the calculation of the Leray-Schauder degree in presence of symmetries, Wang gives the following result:

Theorem 1.1. *Let X be a real Hilbert space, and let G be a compact Lie group acting in X through a smooth orthogonal representation in $GL(X)$. On the other hand, let $\Omega \subset X$ be a bounded G-invariant subset and $f \in C^0(\overline{\Omega}; X)$ a G-covariant compact perturbation of the identity such that $0 \notin f(\partial\Omega)$. Then*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \mathcal{I}^G, \tag{1.1}$$

where $\Omega^G = \Omega \cap X^G$, X^G is the fixed point space of G, $f^G = f_{|\overline{\Omega} \cap X^G}$, and $\mathcal{I}^G$ denotes the ideal of $\mathbf{Z}$ generated by the Euler-Poincaré characteristics $\chi(G/G_x)$, $x \in X \backslash X^G$, G_x being the isotropy subgroup of x.

Formula (1.1) generalizes all previously known results, mostly devoted to finite $\wp$-groups or tori, regarding the calculation of $d(f, \Omega, 0)$ when

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covariance involves a single (or two equivalent) linear action(s). We refer to [**20**] for a brief comparison with other works in the literature. In particular, Borsuk's theorem corresponds to the case $G = \mathbf{Z}_2$ acting through $\{I, -I\}$ and $\mathcal{I}^G = 2\mathbf{Z}$. More generally, $\mathcal{I}^G \subset \wp\mathbf{Z}$ when G is a finite $\wp$-group, and (1.1) yields $d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \wp$. When G is a torus, then $\mathcal{I}^G = \{0\}$, so that $d(f, \Omega, 0) = d(f^G, \Omega^G, 0)$ in this case. Finally, if $X^G = \{0\}$, then (1.1) reads $d(f, \Omega, 0) = 1$ (respectively, 0) $\bmod \mathcal{I}^G$ if $0 \in \Omega$ (respectively, $0 \notin \Omega$).

The validity of (1.1) may be hinted from the work of Rubinzstein [**17**] who, modulo some modifications later made by Dancer [**4**], gave a proof of it in a particular case and for Brouwer's degree. One of the difficulties with mappings covariant under the action of a compact Lie group is that points in their zero sets are usually not isolated (since complete orbits lie in the zero sets) and hence the problem cannot be reduced to the case when 0 is a regular value. To overcome this difficulty, Wang introduces the concept of a regular zero-orbit and first proves Theorem 1.1 when all the zero-orbits are regular. He next extends the result to the general case through a denseness argument. However, at this stage, his proof contains a few gaps, especially when $\dim X = \infty$.

In particular, Wang's proof makes use of some unspecified topology for $C^r(X)$ ($\equiv C^r(X; X)$), $0 \le r \le 2$, but it is clear from the context and the notation used that the author has in mind a topology induced by C^r-type norms on closed bounded subsets. The problem, of course, is that no such norm exists if $\dim X = \infty$, and it exists only if $r = 0$ when attention is confined to compact perturbations of the identity (for example, if $k \in C^1(X)$ is compact, there is no guarantee that $Dk(B)$ is bounded on bounded subsets $B \subset X$). Thus, it is difficult to make sense of Lemma 2.3 in [**20**, p. 530] stating (without a proof) that "C^2 covariant compact perturbations of the identity form a closed subspace of $C^2(X)$, hence a second category complete metric space," a property naturally crucial to Wang's denseness argument based upon the Baire property. Other gaps in the proof may perhaps be viewed as technicalities which have been skipped to shorten the exposition, but this is not always clear either.

In this paper we give a completely rigorous and in any case technically simpler proof of (1.1), which is also valid when X is an arbitrary Banach space and for an arbitrary (not necessarily smooth or isometric) continuous linear action. This answers a question explicitly left open in [**20**]. When $X = \mathbf{R}^n$, smoothness of the action is not a restriction (see [**13**]), and neither is isometry when X is a Hilbert space, but it goes differently in general.

At a first sight, our main result is only vaguely reminiscent of (1.1): we prove that if T is any maximal torus of the identity component G^0 of G and $N(T)$ its normalizer in G, and if $X^{N(T)} = X^G$ (a condition independent of T), then

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \widetilde{\mathcal{I}}^G, \tag{1.2}$$

where $\widetilde{I}^G$ is the ideal of $\mathbf{Z}$ independent of T generated by the integers $[N(T)/T : \Gamma_x]$, $x \in X^T \backslash X^G$, and Γ_x is the isotropy subgroup of x relative to the (always well-defined) action of $N(T)/T$ in X^T. It is of practical importance to notice that (1.2) involves only the action of the *finite* group $N(T)/T$. When G is finite, (1.1) and (1.2) coincide; more generally, we show that (1.2) is always the "useful" part of (1.1), for it turns out that $\mathcal{I}^G = \mathbf{Z}$ (hence (1.1) is useless) if $X^{N(T)} \neq X^G$ and that $\mathcal{I}^G = \widetilde{\mathcal{I}}^G$ (hence (1.1) and (1.2) are the same) if $X^{N(T)} = X^G$.

Our approach relates to Wang's only when G is a finite $\wp$-group and $X = \mathbf{R}^n$. In particular, we need not make use of his concept of regular zero-orbit. Indeed, instead of trying to obtain (1.2) in one stroke, we start with $X = \mathbf{R}^n$ and G a finite $\wp$-group. Next we prove the validity of (1.2) for general finite groups, the problem being reduced to the case of $\wp$-groups via Sylow subgroups. From the case $G = \mathbf{Z}_\wp$, we easily derive (1.2) for tori by a limiting process. Together with the result previously established when G is finite, this yields the validity of (1.2) in general (when $X = \mathbf{R}^n$).

To handle the case when $\dim X = \infty$, we proceed by finite dimensional approximation, following closely the classical method in the "noncovariant" theory. Naturally, to account for covariance, a few extra considerations are needed. Finally, (1.1) is derived from (1.2) through old and new formulas for the Euler-Poincaré characteristics of homogeneous spaces.

Most of the tedious technicalities are confined to Section 2. Limitation to both the finite dimensional setting and the case when G is finite contributes to making their amount quite reasonable. Our proof of the denseness result there follows Wang's idea. Although this result is used later with $\wp$-groups only, it does not seem that this assumption allows for any simplification at that stage.

In Section 3 we introduce the new concept of (semi-)loose representation, intimately related to the notion of intrinsic isotropy subgroup recently developed by the author in [**14**] and [**16**]. Our motivation there is that properties of semi-loose representations of $\wp$-groups permit to avoid having to split the contributions of $f^{-1}(0) \cap X^G$ and $f^{-1}(0) \backslash X^G$ to calculate the degree (as Wang does). This eliminates unpleasant technicalities.

Sections 4 and 5 are devoted to the proof of (1.2) when $X = \mathbf{R}^n$, and G is finite or an arbitrary compact Lie group, respectively.

Preliminaries for the treatment of Leray-Schauder's degree are given in Section 6, and the extension of (1.2) to Banach spaces is presented in Section 7. Wang's theorem, properly generalized and complemented, is derived in Section 8.

The notation used throughout is fairly standard. Multiplication is chosen as the group operation, and hence the identity element of G is denoted by 1. We write $H \leq G$ to indicate that H is a closed subgroup of G (hence a Lie group), and $H < G$ if necessarily H is a proper subgroup. When G is finite, we use $|G|$ for the order of G and $[G : H]$ for the index of $H \leq G$, i.e., $[G : H] = |G|/|H|$.

As is customary, "G-invariant" and "G-covariant" mean "relative to the given action of G," which is accounted for by a representation of G denoted by R but often not explicitly mentioned. When $X = \mathbf{R}^n$ (or a Hilbert space, but we do not consider this case separately), it is not restrictive to assume that the representation R is orthogonal since this is true for some appropriate inner product. We thus make the

Blanket hypothesis. *When $X = \mathbf{R}^n$, it is always understood that the representation R is orthogonal for the usual inner product of $\mathbf{R}^n$.*

The isotropy subgroup of $x \in X$ is denoted by G_x, and X^G refers to the fixed point space of the action of G, even when $X = \mathbf{R}^n$. If $D \subset X$ is a G-invariant subset and $f : D \to X$ a G-covariant mapping, we abbreviate $D \cap X^G = D^G$ and $f_{|_{D^G}} = f^G$. This notation is consistent with the trivial and well-known fact that G-covariance implies $f^G(D^G) = f(D^G) \subset X^G$. If $\Omega \subset X$ is an open subset, we let $C^r(\overline{\Omega})$, $r = 0, 1$, denote the space $C^r(\overline{\Omega}; X)$. When Ω is G-invariant, and hence $\overline{\Omega}$ is G-invariant too, the subspace of $C^r(\overline{\Omega})$ of those G-

covariant mappings is denoted by $C^r_G(\overline{\Omega})$. We make use of $C^1_G(\overline{\Omega})$ only when $X = \mathbf{R}^n$ and Ω is bounded. If so, $C^1_G(\overline{\Omega})$ is a Banach space for the norm of $C^1(\overline{\Omega})$:

$$||f||_{1,\infty,\overline{\Omega}} = \max_{x\in\overline{\Omega}} |f(x)| + \max_{x\in\overline{\Omega}} |Df(x)|,$$

where $|\cdot|$ is the Euclidean norm. When X is an arbitrary Banach space with norm $||\cdot||$, $\Omega \subset X$ a bounded G-invariant subset and $k \in C^0_G(\overline{\Omega})$ is a compact (nonlinear) operator,

$$||k||_{0,\infty,\overline{\Omega}} = \max_{x\in\overline{\Omega}} ||k(x)||,$$

is well defined and a norm for k.

In this work, topological degree technicalities have been kept to a minimum by involving arguments from finite and compact Lie group theories as often as possible. All the group-theoretic results used without proper reference or proof are classical and "elementary" ones (e.g., properties of Sylow subgroups or of maximal tori) that can be found in most introductory textbooks (such as Scott [**18**] for finite groups or Bröcker and tom Dieck [**2**] for compact Lie groups, among many others).

2. A denseness result. This section is entirely devoted to the proof of the following technical result.

Theorem 2.1. *Let G be a finite group, and let $\Omega \subset \mathbf{R}^n$ be a G-invariant bounded open subset. Then, for every G-invariant open subset ω of Ω with $\bar{\omega} \subset \Omega$, the set*

$$Z_G(\bar{\omega}) = \{g \in C^1_G(\overline{\Omega}) : 0 \text{ is a regular value of } g_{|\bar{\omega}}\},$$

is dense in $C^1_G(\overline{\Omega})$ for the Banach space topology of $C^1(\overline{\Omega})$.

Proof. We argue by induction on the order of G. When $G = \{1\}$ and hence every mapping is G-covariant, the result follows from Sard's theorem (given $f \in C^1(\overline{\Omega})$, replace f by $f - y$ where y is a regular value of $f_{|\Omega}$ with $|y|$ arbitrarily small).

Suppose now that $|G| > 1$ and that Theorem 2.1 is true with all groups H with $|H| < |G|$. If so, we have

Lemma 2.1. *The set* $\{g \in C^1_G(\overline{\Omega}) : 0$ *is a regular value of* $g_{|_{\Omega\backslash\Omega^G}}\}$ *is dense in* $C^1_G(\overline{\Omega})$.

Proof. Let $x \in \Omega\backslash\Omega^G$. We claim that there is a G-invariant open neighborhood U_x of x in $\Omega\backslash\Omega^G$ such that $\overline{U}_x \subset \Omega\backslash\Omega^G$ and

$$Z_G(\overline{U}_x) = \{g \in C^1_G(\overline{\Omega}) : 0 \text{ is a regular value of } g_{|_{\overline{U}_x}}\} \tag{2.1}$$

is open and dense in $C^1_G(\overline{\Omega})$. Openness is clear irrespective of U_x by compactness of $\overline{U}_x \subset \Omega$ (argue by contradiction). We now find U_x such that $Z_G(\overline{U}_x)$ is dense in $C^1_G(\overline{\Omega})$.

Denote by $H = G_x$ the isotropy subgroup of x. Since $x \in \Omega\backslash\Omega^G$, we have $|H| < |G|$. The orbit of x consists of finitely many points $x_1 = x, x_2, \dots , x_m$ ($m = [G : H]$) with $x_i = R_{\gamma_i}x$, $\gamma_i \in G$, $1 \le i \le m$, and $\gamma_1 \in H$. As $\Omega\backslash\Omega^G$ is open in Ω, we may choose $\rho > 0$ such that the balls $\overline{B}(x_i, \rho)$ are disjoint and contained in $\Omega\backslash\Omega^G$. Note that $B(x_1, \rho)$ is H-invariant since for $y \in B(x_1, \rho)$ and $\gamma \in H$ we have $|R_\gamma y - x_1| = |R_\gamma(y - x_1)| = |y - x_1| < \rho$. (Recall the convention made in Section 1 that R is orthogonal when $X = \mathbf{R}^n$). Also, $B(x_i, \rho) = R_{\gamma_i}B(x_1, \rho)$, $1 \le i \le m$, and the union $V_x = \cup_{i=1}^m B(x_i, \rho) = \cup_{\gamma\in G} R_\gamma B(x_1, \rho)$ is G-invariant.

From H-invariance of $B(x_1, \rho)$, and since Theorem 2.1 is valid with H replacing G by hypothesis (because $|H| < |G|$), it follows that given $f \in C^1_G(\overline{\Omega}) \subset C^1_H(\overline{\Omega})$ and given $\varepsilon > 0$, there is a $g \in C^1_H(\overline{\Omega})$ such that $||f - g||_{1,\infty,\overline{\Omega}} \le \varepsilon$ and 0 is a regular value of $g_{|_{\overline{B}(x_1,\rho)}}$.

Define $h : \overline{V}_x = \cup_{i=1}^m \overline{B}(x_i, \rho) \to \mathbf{R}^n$ by

$$h(y) = \begin{cases} g(y) & \text{if } y \in \overline{B}(x_1, \rho), \\ 0 & \text{if } y \in \overline{B}(x_i, \rho),\ 2 \le i \le m. \end{cases}$$

This definition makes sense because the balls $\overline{B}(x_i, \rho)$ are disjoint, and $h \in C^1(\overline{V}_x)$ is obvious. Next, set

$$\tilde{h}(y) = \frac{1}{|H|} \sum_{\gamma\in G} R_\gamma^{-1} h(R_\gamma y), \qquad \forall\, y \in \overline{V}_x,$$

so that $\tilde{h} \in C^1_G(\overline{V}_x)$. Observe that $\tilde{h} = g$ in $\overline{B}(x_1, \rho)$. Indeed, if $\gamma \in G$, $\gamma \notin H$, then $R_\gamma x_1 \neq x_1$ (recall that H is the isotropy subgroup of $x = x_1$) whence $R_\gamma x_1 = x_i$ for some $2 \leq i \leq m$. It follows that $R_\gamma y \in \overline{B}(x_i, \rho)$ for every $y \in \overline{B}(x_1, \rho)$, and hence $h(R_\gamma y) = 0$. This shows that for $y \in \overline{B}(x_1, \rho)$ we have $\tilde{h}(y) = (1/|H|) \sum_{\gamma \in H} R_\gamma^{-1} h(R_\gamma y)$. As $\overline{B}(x_1, \rho)$ is H-invariant, we have $h(R_\gamma y) = g(R_\gamma y) = R_\gamma g(y)$ for $y \in \overline{B}(x_1, \rho)$ and $\gamma \in H$, so that $\tilde{h}(y) = g(y)$, as claimed. Since 0 is a regular value of $g_{|_{\overline{B}(x_1,\rho)}}$, we see that 0 is a regular value of $\tilde{h}$ in $\cup_{\gamma \in G} R_\gamma \overline{B}(x_1, \rho) = \overline{V}_x$.

As both f and $\tilde{h}$ are G-covariant, and since $\tilde{h} = g$ in $\overline{B}(x_1, \rho)$, we find

$$(2.2) \qquad \begin{aligned} ||f - \tilde{h}||_{1,\infty,\overline{V}_x} &= ||f - \tilde{h}||_{1,\infty,\overline{B}(x_1,\rho)} \\ &= ||f - g||_{1,\infty,\overline{B}(x_1,\rho)} \leq ||f - g||_{1,\infty,\overline{\Omega}} \leq \varepsilon. \end{aligned}$$

Let $\varphi : [0, \infty) \to \mathbf{R}$ be a smooth function with $\operatorname{supp} \varphi \subset [0, \rho^2)$ and $\varphi \equiv 1$ in $[0, \rho^2/4)$. Set

$$\psi(y) = \sum_{i=1}^{m} \varphi(|y - x_i|^2), \qquad \forall\, y \in \mathbf{R}^n.$$

As $\varphi(|y - x_i|^2) = 0$ for $y \notin B(x_i, \rho)$, we have $\operatorname{supp} \psi \subset \overline{V}_x$, and ψ is G-invariant. For the latter point, just note that every $\gamma \in G$ induces a permutation τ_γ of $\{1, \dots, m\}$ through $R_\gamma^{-1} x_i = x_{\tau_\gamma(i)}$, $1 \leq i \leq m$, and hence

$$\psi(R_\gamma y) = \sum_{i=1}^{m} \varphi(|y - x_{\tau_\gamma(i)}|^2) = \sum_{i=1}^{m} \varphi(|y - x_i|^2) = \psi(y).$$

For $y \in \overline{\Omega}$, set

$$\tilde{f}(y) = f(y) + \psi(y)(\tilde{h}(y) - f(y)),$$

where $\psi \tilde{h}$ is extended by 0 outside $\overline{V}_x$. Since $\operatorname{supp} \psi \subset \overline{V}_x$, this extension is C^1 and hence $\tilde{f} \in C^1_G(\overline{\Omega})$ from the properties of $\tilde{h}$ and ψ mentioned before.

Let $U_x = \cup_{i=1}^m B(x_i, \rho/2)$, a G-invariant open subset containing $x = x_1$. If $y \in \overline{U}_x$, then $\psi(y) = 1$ and hence $\tilde{f}(y) = \tilde{h}(y)$. As 0 is a regular value of $\tilde{h}$ in $\overline{V}_x \supset \overline{U}_x$, it follows that 0 is a regular value of $\tilde{f}_{|_{\overline{U}_x}}$. Finally, since $\tilde{f} - f = \psi(\tilde{h} - f)$ and $\psi \equiv 0$ outside $\overline{V}_x$, we have

$$\begin{aligned} ||\tilde{f} - f||_{1,\infty,\overline{\Omega}} &= ||\psi(\tilde{h} - f)||_{1,\infty,\overline{V}_x} \\ &\leq ||\psi||_{1,\infty,\overline{\Omega}} ||\tilde{h} - f||_{1,\infty,\overline{V}_x} \leq \varepsilon ||\psi||_{1,\infty,\overline{\Omega}} \end{aligned} \tag{2.3}$$

where the last inequality follows from (2.2). Since ε in (2.3) can be chosen arbitrarily small and ψ is independent of f, our proof of existence of U_x is complete.

To prove the lemma, cover $\Omega \backslash \Omega^G$ with all the neighborhoods U_x, $x \in \Omega \backslash \Omega^G$, found above and extract a countable covering $(U_{x^l})_{l \in \mathbf{N}}$. Since $C^1_G(\overline{\Omega})$ is a Banach space, hence a Baire space, the intersection $\cap_{l \in \mathbf{N}} Z_G(\overline{U}_{x^l})$ (see (2.1)) is dense in $C^1_G(\overline{\Omega})$. Thus, given $f \in C^1_G(\overline{\Omega})$ and given $\varepsilon > 0$, there is a $g \in \cap_{l \in \mathbf{N}} Z_G(\overline{U}_{x^l})$ such that $||f - g||_{1,\infty,\overline{\Omega}} \leq \varepsilon$. If $x \in \Omega \backslash \Omega^G$ and $g(x) = 0$, then $x \in \overline{U}_{x^l}$ for some index l. Since $g \in Z_G(\overline{U}_{x^l})$, 0 is a regular value of $g_{|_{\overline{U}_{x^l}}}$, whence $Dg(x) \in GL(\mathbf{R}^n)$. This shows that 0 is a regular value of $g_{|_{\Omega \backslash \Omega^G}}$. $\square$

We now complete the proof of Theorem 2.1. Given $f \in C^1_G(\overline{\Omega})$ and given $\varepsilon > 0$, we must find $g \in Z_G(\bar{\omega})$ such that $||f - g||_{1,\infty,\overline{\Omega}} \leq \varepsilon$ and 0 is a regular value of $g_{|_{\bar{\omega}}}$. Suppose that we can find $h \in C^1_G(\overline{\Omega})$ such that $||f - h||_{1,\infty,\overline{\Omega}} \leq \varepsilon/2$ and $Dh(x) \in GL(\mathbf{R}^n)$ if $x \in \bar{\omega} \cap X^G$ and $h(x) = 0$. By Lemma 2.1, for every $0 < \varepsilon' \leq \varepsilon$, there is a $g \in C^1_G(\overline{\Omega})$ such that 0 is a regular value of $g_{|_{\Omega \backslash \Omega^G}}$ and $||h - g||_{1,\infty,\overline{\Omega}} \leq \varepsilon'/2$. As $\bar{\omega} \backslash X^G \subset \Omega \backslash \Omega^G$, it is plain that $Dg(x) \in GL(\mathbf{R}^n)$ if $x \in \bar{\omega} \backslash X^G$ and $g(x) = 0$. But if $\varepsilon' > 0$ is small enough, we must also have $Dg(x) \in GL(\mathbf{R}^n)$ if $x \in \bar{\omega} \cap X^G$ and $g(x) = 0$; otherwise, there is a sequence g_i tending to h in $C^1_G(\overline{\Omega})$ and for each index i a point $x_i \in \bar{\omega} \cap X^G$ such that $g_i(x_i) = 0$ and $Dg_i(x_i) \notin GL(\mathbf{R}^n)$. By compactness of $\bar{\omega} \cap X^G$, this produces $x \in \bar{\omega} \cap X^G$ such that $h(x) = 0$ and $Dh(x) \notin GL(\mathbf{R}^n)$, a contradiction. Thus, g above satisfies $||f - g||_{1,\infty,\overline{\Omega}} \leq \varepsilon$ and 0 is a regular value of $g_{|_{\bar{\omega}}}$, as desired.

As the last step in the proof of Theorem 2.1, it remains to show how h above can be found. By Sard's theorem, there is a regular value $y \in X^G$ of $f^G = f_{|_{\Omega^G}}$ (recall $f(\Omega^G) \subset X^G$ by G-covariance) such that $|y| \leq \varepsilon/4$. Compactness of $\bar{\omega} \cap X^G$ implies that there are only finitely many points $x_1, \dots, x_p \in \bar{\omega} \cap X^G$ such that $f^G(x_i)$ $(= f(x_i)) = y$, $1 \le i \le p$. If $p = 0$, $h = f - y$ works, and we henceforth assume $p \ge 1$. Let $\eta \in C^1(\mathbf{R}^n)$ be any mapping satisfying the conditions $\eta(x_i) = 0$, $D\eta(x_i) = I$, $1 \le i \le p$ (existence of η is trivial). Replacing $\eta(x)$ by $(1/|G|) \sum_{\gamma \in G} R_\gamma^{-1} \eta(R_\gamma x)$, we may assume that $\eta \in C_G^1(\mathbf{R}^n)$: the conditions $\eta(x_i) = 0$ and $D\eta(x_i) = I$ are unaffected by this change because $x_i \in X^G$, $1 \le i \le p$.

For $(\lambda, x) \in \mathbf{R} \times \overline{\Omega}$, set

$$F(\lambda, x) = f(x) - y + \lambda \eta(x).$$

Clearly, $F(\lambda, \cdot) \in C_G^1(\overline{\Omega})$ and $F(\lambda, x_i) = 0$, $D_x F(\lambda, x_i) = Df(x_i) + \lambda I$, $1 \le i \le p$. Since y is a regular value of f^G, the implicit function theorem shows that there is a $\delta > 0$ such that $F(\lambda, x) = 0$ has no solution other than $x_1, \dots, x_p$ in $\cup_{i=1}^p B(x_i, \delta) \cap X^G$ if $|\lambda|$ is small enough. On the other hand, for $|\lambda|$ small, all the solutions $x \in \bar{\omega} \cap X^G$ of $F(\lambda, x) = 0$ must lie in $\cup_{i=1}^p B(x_i, \delta) \cap X^G$ (argue by contradiction). Thus, for $|\lambda|$ small, the solutions of $F(\lambda, x) = 0$ in $\bar{\omega} \cap X^G$ are exactly $x_1, \dots, x_p$. Furthermore, as $D_x F(\lambda, x_i) = Df(x_i) + \lambda I$, we have $D_x F(\lambda, x_i) \in GL(\mathbf{R}^n)$ for $1 \le i \le p$ as soon as $0 < |\lambda|$ (and $|\lambda|$ is small enough). In particular, λ may also be chosen such that $0 < |\lambda| \le \varepsilon/(4||\eta||_{1,\infty,\overline{\Omega}})$, and if so $h = F(\lambda, \cdot)$ satisfies $||f - h||_{1,\infty,\overline{\Omega}} \le \varepsilon/2$ and $Dh(x) \in GL(\mathbf{R}^n)$ when $x \in \bar{\omega} \cap X^G$ and $h(x) = 0$ (i.e., $x = x_i$, $1 \le i \le p$). This completes the proof of Theorem 2.1. $\square$

Remark 2.1. Evidently, denseness of $Z_G(\bar{\omega})$ in $C_G^1(\overline{\Omega})$ for the C^1 norm implies its denseness for the C^0 norm, which is the only property we shall need later. However, because the proximity of the derivatives is involved at various stages of the proof of Theorem 2.1, the C^0 norm cannot be substituted for the C^1 norm in that proof.

3. Loose and semi-loose representations. The concepts to be discussed here are closely related to the notion of intrinsic isotropy subgroup introduced in [**14**] and further studied in [**16**] in the case of finite groups. Below, we briefly recall their definition and the properties we shall use later.

Definition 3.1. Let G be a compact Lie group. A subgroup $H \le G$ is said to be an *intrinsic isotropy subgroup* of G (i.i.s. for short) if for every finite-dimensional real vector space X, every representation R of G in $GL(X)$ and every G-covariant linear isomorphism $A \in GL(X)$, we have

$$\operatorname{sgn}\det A = \operatorname{sgn}\det A_{|_{X^H}},$$

where $X^H \subset \mathbf{R}^n$ denotes the fixed point space of H relative to the representation R. An i.i.s. of G is said to be maximal (m.i.i.s. for short) if it is contained in no larger i.i.s. of G. This definition is unaffected by making the specific choices $X = \mathbf{R}^n$, $n \in \mathbf{N}$.

At first sight, the conditions required for a subgroup to be an i.i.s. appear extremely restrictive. Nevertheless, it is shown in [**14**] that every compact Lie group distinct from $\mathbf{Z}_2^k$, $k \geq 0$ an integer, possesses a nontrivial m.i.i.s. Here, "nontrivial" means "distinct from $\{1\}$" since $\{1\}$ is obviously an i.i.s. of G. In contrast, $\mathbf{Z}_2^k$ has no nontrivial i.i.s. It is also proved in [**14**] that finite groups of odd order and tori coincide with their unique m.i.i.s. The first result is complemented in [**16**], where it is shown that if G is a finite 2-nilpotent group (i.e., the elements of odd order form a subgroup of G) then G^2, the group generated by the elements γ^2, $\gamma \in G$, is the unique m.i.i.s. of G.

Definition 3.2. Let G be a compact Lie group and X a finite-dimensional real vector space. The representation R of G in $GL(X)$ is said to be loose if there is an m.i.i.s. H of G which is contained in no isotropy subgroup G_x of $x \in X\backslash\{0\}$. It is said to be *semi-loose* if there is an m.i.i.s. H of G which is contained in no isotropy subgroup G_x of $x \in X\backslash X^G$.

When $X = \mathbf{R}^n$ it is obvious from the above definition that R is semi-loose if and only if the subrepresentation of R in $GL((X^G)^\perp)$ is loose (recall that "representation" is understood as "orthogonal representation" when the space of the representation is $\mathbf{R}^n$). Clearly, a loose representation is fixed point free (i.e., $X^G = \{0\}$), but the converse is true only if G coincides with its unique m.i.i.s., e.g., when $|G|$ is finite and odd (*never* when $|G|$ is even; see [**16**], or when G is a torus. If $G \neq \mathbf{Z}_2^k$, then every free representation of G (i.e., $G_x = \{1\}$ for all $x \in \mathbf{R}^n\backslash\{0\}$) is loose. In fact, if $G = \mathbf{Z}_2^k$ acts freely in $\mathbf{R}^n$, $n \geq 1$, then $k = 1$ (see [**1**]) and $G = \mathbf{Z}_2$ acts through $\{I, -I\}$, which obviously is not a loose representation. This is the *only* free representation of a compact Lie group which is not loose. These remarks make it clear that loose representations are "somewhere" between free and fixed point free ones. For future use, note also that from previous remarks *every* representation of G is semi-loose when G is finite and $|G|$ is *odd.*

In this paper, semi-loose representations will be used through the following two lemmas.

Lemma 3.1. *Let $A \in GL(\mathbf{R}^n)$ be G-covariant relative to a semi-loose representation R of the compact Lie group G. Then*

$$\operatorname{sgn}\det A = \operatorname{sgn}\det A_{|_{X^G}}.$$

Proof. Both X^G and $(X^G)^\perp$ are G-invariant, and hence A has the decomposition

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

relative to the splitting $\mathbf{R}^n = X^G \oplus (X^G)^\perp$, where $A_1 = A_{|_{X^G}}$ and $A_2 = A_{|_{(X^G)^\perp}}$. Hence, $\det A = \det A_1 \det A_2$. As R is semi-loose, the subrepresentation of R in $(X^G)^\perp$ is loose, and hence there is an m.i.i.s. H of G such that $X^H \cap (X^G)^\perp = \{0\}$. By definition of an i.i.s., we have $\operatorname{sgn}\det A_2 = \operatorname{sgn}\det A_{2|_Y}$ where Y is the fixed point space of H in $(X^G)^\perp$, i.e., $Y = X^H \cap (X^G)^\perp = \{0\}$. Thus, $\det A_2 > 0$ and hence $\operatorname{sgn}\det A = \operatorname{sgn}\det A_1$. $\square$

Lemma 3.2. *Let G be a finite 2-group, i.e., $|G| = 2^k$ for some integer $k \geq 0$, and let R be a representation of G in $GL(\mathbf{R}^n)$. Suppose that R is not semi-loose. Then there is an $x \in \mathbf{R}^n$ such that $[G : G_x] = 2$.*

Proof. Obviously, a 2-group is 2-nilpotent (the only element of odd order is 1) and hence G^2, the group generated by the elements γ^2, $\gamma \in G$, is the unique m.i.i.s. of G. Thus, if R is not semi-loose, we have $X^G \subsetneq X^{G^2}$. Let $H < G$ be maximal with the property that $G^2 \leq H$ and $X^G \subsetneq X^H$. As is well known (and easily checked) G^2 is a normal subgroup of G and $G/G^2 \simeq \mathbf{Z}_2^k$, $k \geq 0$. Since $\mathbf{Z}_2^k$ is abelian, H/G^2 is normal in G/G^2 and hence H is normal in G. It follows that X^H is invariant under G and that the action of G in X^H factors through an action of G/H. Now this action is semi-free by maximality of H. As $(X^G)^\perp \cap X^H$ is a G/H-invariant complement of $(X^H)^{G/H} = X^G$ in X^H, the action of G/H in $(X^G)^\perp \cap X^H$ is free. But since $G/H \simeq (G/G^2)/(H/G^2)$, we have $G/H \simeq \mathbf{Z}_2^l$, $l \geq 0$. Necessarily, $l \geq 1$ since $H < G$, whereas $l \leq 1$ because G/H acts freely in $(X^G)^\perp \cap X^H \neq \{0\}$ (as recalled earlier, $\mathbf{Z}_2^l$, $l \geq 2$, has no free linear action in $\mathbf{R}^m$, $m \geq 1$). Thus, $l = 1$, i.e., $|G/H| = 2$. Since $X^G \subsetneq X^H$ by definition of H, there is an $x \in X^H \backslash X^G$, whence $H \leq G_x < G$. Thus, $G_x = H$ since every subgroup of G of index 2 is a maximal proper subgroup. $\square$

4. Brouwer's degree of covariant mappings: finite groups. We begin with the case when G is a finite $\wp$-group, $\wp \geq 2$ a prime number.

Lemma 4.1. *Let G be a finite $\wp$-group, and let $\Omega \subset \mathbf{R}^n$ be a G-invariant bounded open subset. Let $f \in C_G^0(\overline{\Omega})$ be such that $0 \notin f(\partial\Omega)$. Then,*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \mathcal{I}^G, \tag{4.1}$$

where d denotes Brouwer's degree and $\mathcal{I}^G$ is the ideal of $\mathbf{Z}$ generated by the integers $[G : G_x]$, $x \in X \backslash X^G$ (so that $\mathcal{I}^G = \wp^\alpha \mathbf{Z}$ for some $\alpha \geq 1$).

Proof. It is not restrictive to assume that $f \in C_G^1(\overline{\Omega})$. Indeed, extending f continuously outside $\overline{\Omega}$ and using a mollifier, we obtain $f_1 \in C^1(\mathbf{R}^n)$ such that $||f - f_1||_{0,\infty,\overline{\Omega}}$ is arbitrarily small, and replacing f_1 by $(1/|G|)\sum_{\gamma \in G} R_\gamma^{-1} f_1 \circ R_\gamma$, we may also assume that $f_1 \in C_G^1(\overline{\Omega})$. Since $d(f, \Omega, 0) = d(f_1, \Omega, 0)$ and $d(f^G, \Omega^G, 0) = d(f_1^G, \Omega^G, 0)$ is obvious provided that f_1 is close enough to f in $C_G^0(\overline{\Omega})$, we may then replace f by f_1 to prove (4.1).

Suppose then that $f \in C_G^1(\overline{\Omega})$. Since $0 \notin f(\partial\Omega)$, $f^{-1}(0)$ is a compact subset of Ω, and hence there is an open neighborhood ω of $f^{-1}(0)$ in Ω such that $\bar{\omega} \subset \Omega$. Replacing ω by $\cap_{\gamma \in G} R_\gamma \omega$ (which is possible since $f^{-1}(0)$ is G-invariant) we may assume that ω is G-invariant.

From Theorem 2.1, given $\varepsilon > 0$, there is a $g \in C_G^1(\overline{\Omega})$ such that $||f - g||_{1,\infty,\overline{\Omega}} \leq \varepsilon$ (hence $||f - g||_{0,\infty,\overline{\Omega}} \leq \varepsilon$) and 0 is a regular value of $g_{|\bar{\omega}}$. On the other hand, if $\varepsilon > 0$ is small enough, then $g^{-1}(0) \subset \bar{\omega}$: otherwise, there is a sequence g_i tending to f in $C^1_G(\overline{\Omega})$ (hence $C^0_G(\overline{\Omega})$) and for each index i a point $x_i \in \Omega \backslash \bar{\omega}$ such that $g_i(x_i) = 0$. By compactness of $\overline{\Omega}$, we find $x \in \overline{\Omega} \backslash \omega$ such that $f(x) = 0$, contradicting $f^{-1}(0) \subset \omega$. Thus, if $\varepsilon > 0$ is small enough, 0 is a regular value of $g_{|\Omega}$. Since, by further shrinking of ε if necessary, we have $d(f, \Omega, 0) = d(g, \Omega, 0)$ and $d(f^G, \Omega^G, 0) = d(g^G, \Omega^G, 0)$, we may replace f by g in the proof of (4.1), i.e., assume that 0 is a regular value of $f_{|\Omega}$.

If 0 is a regular value of $f_{|\Omega}$, then $f^{-1}(0)$ is finite and $d(f, \Omega, 0) = \sum_{x \in f^{-1}(0)} \operatorname{sgn} \det Df(x)$. For $x \in f^{-1}(0)$ and $\gamma \in G$, we have $R_\gamma x \in f^{-1}(0)$, and from $Df(R_\gamma x) = R_\gamma Df(x) R_\gamma^{-1}$ it follows that $\operatorname{sgn} \det Df(x) = \operatorname{sgn} \det Df(R_\gamma x)$. Thus, the sum

$$\sum_{\substack{x \in f^{-1}(0) \\ x \notin X^G}} \operatorname{sgn} \det Df(x)$$

can be rewritten as a sum of integer multiples of the indices $[G : G_x]$ of the isotropy subgroups G_x, $x \notin X^G$ (since $[G : G_x]$ equals the number of points in the orbit of x). This shows that

$$\sum_{\substack{x \in f^{-1}(0) \\ x \notin X^G}} \operatorname{sgn} \det Df(x) \in \mathcal{I}^G.$$

To prove (4.1), it then suffices to show that

$$d(f^G, \Omega^G, 0) = \sum_{x \in f^{-1}(0) \cap X^G} \operatorname{sgn} \det Df(x) \bmod \mathcal{I}^G. \tag{4.2}$$

If the representation R of G is semi-loose, in particular, if $\wp$ is odd (recall that every representation of a group of odd order is semi-loose, see Section 3), the conclusion follows at once from Lemma 3.1. Indeed, for $x \in X^G$, $Df(x)$ is G-covariant and $Df^G(x) = Df(x)_{|X^G}$, whence $\operatorname{sgn} \det Df^G(x) = \operatorname{sgn} \det Df(x)$ by Lemma 3.1. Thus,

$$d(f^G, \Omega^G, 0) = \sum_{x \in f^{-1}(0) \cap X^G} \operatorname{sgn} \det Df(x),$$

which, of course, implies (4.2).

If now R is not semi-loose, then $\wp = 2$, and $\mathcal{I}^G = 2\mathbf{Z}$ by Lemma 3.2. In this case, (4.2) holds because both $d(f^G, \Omega^G, 0)$ $(= \sum_{x\in f^{-1}(0)\cap X^G} \operatorname{sgn}\det Df^G(x))$ and $\sum_{x\in f^{-1}(0)\cap X^G} \operatorname{sgn}\det Df(x)$ represent a mod 2 count of the number of points in $f^{-1}(0)\cap X^G$ (since $1 = -1 \bmod 2$). This completes the proof. $\square$

Theorem 4.1. *Let G be a finite group, and let $\Omega \subset \mathbf{R}^n$ be a G-invariant bounded open subset. Let $f \in C^0_G(\overline{\Omega})$ be such that $0 \notin f(\partial\Omega)$. Then,*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \mathcal{I}^G, \tag{4.3}$$

where d denotes Brouwer's degree and $\mathcal{I}^G$ is the ideal of $\mathbf{Z}$ generated by the integers $[G : G_x]$, $x \in X\backslash X^G$.

Proof. If $\mathcal{I}^G = \mathbf{Z}$ (4.3) is trivial, so suppose $\mathcal{I}^G \neq \mathbf{Z}$, i.e., the g.c.d. Δ of the integers $[G : G_x]$, $x \in X\backslash X^G$ is greater than 1 and $\mathcal{I}^G = \Delta\mathbf{Z}$. Let $\Delta = \wp_1^{\alpha_1}\cdots\wp_k^{\alpha_k}$ be the decomposition of Δ into a product of distinct primes p_i (so that $\alpha_i \geq 1$), $1 \leq i \leq k$. For simplicity, call $\wp^\alpha$ $(\alpha \geq 1)$ any of the factors $\wp_i^{\alpha_i}$. Obviously, $\wp^\alpha$ divides the order $|G|$ of G, whence $|G| = \wp^\beta q$ with $\beta \geq \alpha$, $\wp \nmid q$.

Let S be a Sylow $\wp$-subgroup of G (i.e., a $\wp$-subgroup of G with order $|S| = \wp^\beta$). We claim that for every subgroup $\Sigma \leq S$ with order $|\Sigma| = \wp^\gamma$, $\gamma > \beta - \alpha$, we have $X^\Sigma = X^G$. Otherwise, $X^G \subsetneq X^\Sigma$ and hence there is an $x \in X\backslash X^G$ such that $\Sigma \leq G_x$. If so, $\wp^\gamma$ divides $|G_x|$. As $|G| = |G_x|[G : G_x]$ and $\wp^\gamma$ divides $|G_x|$, $\wp^\alpha$ divides $[G : G_x]$, we find that $\wp^{\alpha+\gamma}$ divides $|G|$. But $\alpha + \gamma > \beta$ and no power of $\wp$ larger than β divides $|G|$ (by definition of β), a contradiction. In particular, $X^S = X^G$ (since $|S| = \wp^\beta$ and $\beta > \beta - \alpha$ because $\alpha \geq 1$), and the relation $X^\Sigma = X^G$ for $|\Sigma| = \wp^\gamma$, $\gamma > \beta - \alpha$ also reads $X^\Sigma = X^S$ for $|\Sigma| = p^\gamma$, $\gamma > \beta - \alpha$. But this means that no isotropy subgroup S_x of $x \in \mathbf{R}^n\backslash X^S$ relative to the action of S has more than $\wp^{\beta-\alpha}$ elements. Equivalently, $\wp^\alpha$ divides $[S : S_x]$ for $x \in \mathbf{R}^n\backslash X^S = \mathbf{R}^n\backslash X^G$, i.e., $\mathcal{I}^S \subset \wp^\alpha\mathbf{Z}$ (actually, $\mathcal{I}^S = \wp^\alpha\mathbf{Z}$; see Remark 4.1). From Lemma 4.1 with S replacing G and using $X^S = X^G$, $\mathcal{I}^S \subset \wp^\alpha\mathbf{Z}$, we get

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \wp^\alpha\mathbf{Z},$$

for $\wp^\alpha = \wp_i^{\alpha_i}$, $1 \le i \le k$. Since the $\wp_i$'s are distinct, this yields

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \wp_1^{\alpha_1} \cdots \wp_k^{\alpha_k} \mathbf{Z},$$

and $\wp_1^{\alpha_1} \cdots \wp_k^{\alpha_k} = \Delta$, $\Delta\mathbf{Z} = \mathcal{I}^G$. This proves (4.3). $\square$

Remark 4.1. The relation $\mathcal{I}^S \subset \wp^\alpha \mathbf{Z}$ in the above proof suggests that some improvement of (4.3) could be available, i.e., that $\mathcal{I}^G$ could perhaps be replaced by a smaller ideal. But it is not so, because in fact $\mathcal{I}^S = \wp^\alpha \mathbf{Z}$. To see this, note first that by definition of $\wp^\alpha$, there is an $x \in X \backslash X^G$ such that $|G_x|$ is divisible by $\wp^{\beta-\alpha}$ (and no higher power of $\wp$). Next, every Sylow $\wp$-subgroup Σ of G_x is contained in some Sylow subgroup S' of G. By conjugacy of the Sylow $\wp$-subgroups of G, we may assume that $S' = S$ after changing x into $R_\gamma x$ for some $\gamma \in G$. But then $\Sigma \le S$ and $\Sigma \le G_x$ implies $\Sigma \le S_x$, whence $|\Sigma| = \wp^{\beta-\alpha} \le |S_x|$, i.e., $\wp^\alpha \mathbf{Z} \subset \mathcal{I}^S$. $\square$

5. Brouwer's degree of covariant mappings: compact Lie groups. For compact Lie groups with positive dimension, the simplest case is that of tori, considered in Lemma 5.1 below. The proof is similar to that of a special case in [**15**] (see also [**12**]).

Lemma 5.1. *Let G be a torus, and let $\Omega \subset \mathbf{R}^n$ be a G-invariant bounded open subset. Let $f \in C_G^0(\overline{\Omega})$ be such that $0 \notin f(\partial\Omega)$. Then*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0),$$

where d denotes Brouwer's degree.

Proof. Let γ_* be a generator of G, and let $\gamma_l \in G$ be a sequence such that $\lim_{l\to\infty} \gamma_l = \gamma_*$ and $G_l = (\gamma_l)$ (the subgroup of G generated by γ_l) has prime order $|G_l| = \wp_l$ with $\lim_{l\to\infty} \wp_l = \infty$. Existence of such a sequence γ_l is easily seen and may be called a standard result (for more details, see [**15**] where a similar procedure is used).

We claim that for l large enough, we have $X^{G_l} = X^G$. Otherwise, there is a sequence $x_l \in (X^G)^\perp \cap X^{G_l}$ such that $|x_l| = 1$. Extracting a subsequence, we may assume that $\lim_{l\to\infty} x_l = x$. Obviously, $x \in (X^G)^\perp$ and $|x| = 1$. Also, taking the limit in $R_{\gamma l} x_l = x_l$, we find $R_{\gamma *}x = x$. Thus, $R_\gamma x = x$ for γ in the dense subset of G of the powers of $\gamma *$, and hence for every $\gamma \in G$ by continuity. This shows that $x \in X^G$, a contradiction.

From the above and Theorem 4.1, we find that for l large enough,

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \mathcal{I}^{G_l}. \tag{5.1}$$

Now if $H \simeq \mathbf{Z}_\wp$, $\wp$ a prime number, it is obvious that $\mathcal{I}^H = \wp\mathbf{Z}$. Thus, (5.1) reads

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \wp_l \mathbf{Z}, \tag{5.2}$$

for l large enough. As $\lim_{l\to\infty} \wp_l = \infty$, we may choose l such that $\wp_l > |d(f, \Omega, 0) - d(f^G, \Omega^G, 0)|$. For this choice, (5.2) holds only if $d(f, \Omega, 0) = d(f^G, \Omega^G, 0)$ and we are done. □

Suppose now that G is an arbitrary compact Lie group with identity component G^0. Let T denote a maximal torus of G^0 and $N(T)$ its normalizer in G. Obviously, the fixed point space X^T is invariant under the action of $N(T)$, and since T acts trivially in X^T, this action factors through an action of $N(T)/T$ (when G is connected, i.e., $G = G^0$, $N(T)/T$ is called the Weyl group of G; some authors, e.g., Bredon [**1**], use this terminology even when G is not connected, but this does not seem to be the rule).

Theorem 5.1. *Let G be a compact Lie group, and let $\Omega \subset \mathbf{R}^n$ be a G-invariant bounded open subset. Let $T \leq G^0$ be a maximal torus of the identity component G^0 of G and $N(T)$ its normalizer in G. Suppose that $X^{N(T)} = X^G$, and let $f \in C^0_G(\overline{\Omega})$ be such that $0 \notin f(\partial\Omega)$. Then*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \mathcal{I}^{N(T)/T}, \tag{5.3}$$

where $\mathcal{I}^{N(T)/T}$ is the ideal of $\mathbf{Z}$ generated by the integers $[N(T)/T : \Gamma_x]$, $x \in X^T \backslash X^G$, and Γ_x is the isotropy subgroup of x relative to the action of $N(T)/T$ in X^T.

Note. The condition $X^{N(T)} = X^G$ and the ideal $\mathcal{I}^{N(T)/T}$ are independent of the maximal torus T. This simple result will be proved in Theorem 7.1, where $\mathbf{R}^n$ is replaced by any real Banach space X.

Proof. By Lemma 5.1 with T replacing G, and since $C^0_G(\overline{\Omega}) \subset C^0_T(\overline{\Omega})$, we get $d(f,\Omega,0) = d(f^T,\Omega^T,0)$. Next, $f^T \in C^0_{N(T)/T}(\overline{\Omega}^T)$ is obvious, and $(X^T)^{N(T)/T} = X^{N(T)}$ (hence $(\Omega^T)^{N(T)/T} = \Omega^{N(T)}$ and $(f^T)^{N(T)/T} = f^{N(T)}$). Because the group $N(T)/T$ is finite ([**1**, p. 26]; a more precise result is proved in Lemma 8.1 later) we infer from Theorem 4.1 that $d(f^T,\Omega^T,0) = d(f^{N(T)},\Omega^{(N(T)},0) \bmod \mathcal{I}^{N(T)/T}$, and (5.3) follows from the hypothesis $X^{N(T)} = X^G$. $\square$

6. Preliminary results for the infinite-dimensional case. In this section G is a compact Lie group and X an arbitrary real Banach space. We assume throughout that a representation R of G in $GL(X)$ is given. Recall that this means that R is a group homomorphism: $R_{\gamma\gamma'} = R_\gamma R_{\gamma'}$, for all $\gamma,\gamma' \in G$, and that for each $x \in X$, the mapping $\gamma \in G \mapsto R_\gamma x \in X$ is continuous.

If $\varphi : G \to X$ is a continuous mapping, the integral $\int_G \varphi(\gamma)\,d\gamma$ (invariant integral) is well-defined, and invariant integration possesses the same properties as in the better known case $\dim X < \infty$. We shall not list these properties here since a full account of invariant integration in Banach spaces can be found in Lang [**11**]. Although the following lemma is certainly not new, we have found no convenient reference for it. We give its proof for completeness.

Lemma 6.1. (i) *There is a constant $M > 0$ such that*

$$||R_\gamma|| \leq M, \qquad \forall\, \gamma \in G. \tag{6.1}$$

(ii) *Let $\Omega \subset X$ be a G-invariant open subset, and let $f \in C^0(\overline{\Omega})$. For $x \in \overline{\Omega}$, set $\tilde{f}(x) = \int_G R_\gamma^{-1} f(R_\gamma x)\,d\gamma$. Then we have $\tilde{f} \in C^0_G(\overline{\Omega})$. Furthermore, if $g \in C^0(\overline{\Omega})$ and $\tilde{g}(x) = \int_G R_\gamma^{-1} g(R_\gamma x)\,d\gamma$, then*

$$\sup_{x\in\overline{\Omega}} ||\tilde{f}(x) - \tilde{g}(x)|| \leq M \sup_{x\in\overline{\Omega}} ||f(x) - g(x)||, \tag{6.2}$$

with M as in (6.1).

(iii) *Let $\Omega \subset X$ be a G-invariant open subset, and let $k \in C^0(\overline{\Omega})$ be compact. Then $\tilde{k} \in C^0_G(\overline{\Omega})$ defined by $\tilde{k}(x) = \int_G R_\gamma^{-1} k(R_\gamma x)\,d\gamma$ is compact too.*

Proof. (i) is a trivial application of the uniform boundedness principle. For (ii), note first that $\tilde{f}(x)$ is well-defined since $\gamma \in G \mapsto R_\gamma^{-1} f(R_\gamma x)$ is continuous for each $x \in \overline{\Omega}$ (use (i), continuity of $\gamma \in G \mapsto \gamma^{-1} \in G$ and $R_\gamma^{-1} = R_{\gamma-1}$). To prove continuity of $\tilde{f}$, let $x_* \in \overline{\Omega}$ be fixed and let $x_m \in \overline{\Omega}$ be a sequence such that $\lim_{m\to\infty} x_m = x_*$. Using (i), it is straightforward to check that the set C consisting of all the orbits $\{R_\gamma x_m : \gamma \in G\}$, $m \geq 0$, and the orbit $\{R_\gamma x_* : \gamma \in G\}$ is a compact subset of $\overline{\Omega}$. By uniform continuity of f on C and by (i) it follows that for every $\varepsilon > 0$ we have $||f(R_\gamma x_m) - f(R_\gamma x_*)|| \leq \varepsilon$ for m large enough and every $\gamma \in G$. Hence, since $||R_\gamma^{-1}|| = ||R_{\gamma-1}|| \leq M$ for all $\gamma \in G$, we have

$$||\tilde{f}(x_m) - \tilde{f}(x_*)|| \leq M \sup_{\gamma\in G} ||f(R_\gamma x_m) - f(R_\gamma x_*)|| \leq M\varepsilon,$$

for m large enough. This shows that $\lim_{m\to\infty} \tilde{f}(x_m) = \tilde{f}(x_*)$ and we are done. That (6.2) holds is trivial.

Finally, if $k \in C^0(\overline{\Omega})$ is compact, then $\tilde{k} \in C^0_G(\overline{\Omega})$ from (ii). To prove that $\tilde{k}$ is compact, it suffices to show that $\tilde{k}(B)$ is relatively compact in X whenever $B \subset \overline{\Omega}$ is bounded. From (i), $D = \cup_{\gamma\in G} R_\gamma B$ is bounded and hence $k(R_\gamma B) \subset k(D) \subset K$, where K is some compact subset of X. Using (i), we find at once that $C = \cup_{\gamma\in G} R_\gamma(K)$ is compact (and not merely relatively compact, although this is a minor point). Now for $\gamma \in G$ and $x \in B$ we have $R_\gamma^{-1} k(R_\gamma x) = R_{\gamma-1} k(R_\gamma x) \in C$, whence $\int_G R_\gamma^{-1} k(R_\gamma x)\, d\gamma \in \tilde{C}$, the closed convex hull of C. As C is compact, $\tilde{C}$ is compact, and since $\tilde{k}(B) \subset \tilde{C}$ from the above, it follows that $\tilde{k}(B)$ is relatively compact. □

When X is a Hilbert space, R is orthogonal and Y is a closed G-invariant subspace, it is well-known that $Y^\perp$ is a closed G-invariant complement of Y in X. Of course, it is hopeless to expect a full generalization of this result to the case when X is an arbitrary Banach space since there is no guarantee that a given G-invariant subspace Y admits any closed complement, let alone a G-invariant one. However, it is true that if Y is split (i.e., possesses some closed complement in X), then it admits a closed G-invariant complement (see e.g. [**19**]). We record this property for future reference:

Lemma 6.2. (Maschke's theorem): *Let $Y \subset X$ be a split G-invariant subspace. Then there is a closed G-invariant subspace Z of X such that $X = Y \oplus Z$.*

Remark 6.1. From G-invariance of Y and Z in Lemma 6.2, it follows at once that the (continuous) projection operators onto Y and Z relative to $X = Y \oplus Z$ are G-covariant. □

7. Leray-Schauder's degree of covariant compact perturbations of the identity. In this section X is a real Banach space equipped with a continuous linear G-action R. We extend Theorem 5.1 to Leray-Schauder's degree.

Theorem 7.1. *Let G be a compact Lie group, and let $\Omega \subset X$ be a G-invariant bounded open subset. Let $T \leq G^0$ be a maximal torus of the identity component G^0 of G and $N(T)$ its normalizer in G. We have*:

(i) *The condition $X^{N(T)} = X^G$ is independent of T.*

(ii) *If $X^{N(T)} = X^G$, the ideal of $\mathbf{Z}$ generated by the integers $[N(T)/T : \Gamma_x]$, $x \in X^T \backslash X^G$, where Γ_x is the isotropy subgroup of x relative to the action of $N(T)/T$ in X^T is independent of T and denoted by $\widetilde{\mathcal{I}}^G$.*

(iii) *If $X^{N(T)} = X^G$ and $f \in C^0_G(\overline{\Omega})$ is a compact perturbation of the identity such that $0 \notin f(\partial\Omega)$, then*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \widetilde{\mathcal{I}}^G, \tag{7.1}$$

where $\widetilde{\mathcal{I}}^G$ is defined in (ii) *above and d denotes Leray-Schauder's degree.*

Note. We emphasize that the validity of (7.1) is subjected to the condition $X^{N(T)} = X^G$. For an equivalent formulation of this condition, see Theorem 8.1.

Proof. (i) Suppose $X^{N(T)} = X^G$, and let $T' \leq G^0$ be another maximal torus, so that $T' = \gamma T \gamma^{-1}$ for some $\gamma \in G^0$ (conjugacy of maximal tori) and hence $N(T') = \gamma N(T) \gamma^{-1}$. This implies $X^{N(T')} = R_\gamma X^{N(T)} = R_\gamma X^G = X^G$.

(ii) This is trivial by (i) and conjugacy of maximal tori.

(iii) Set $f(x) = x - k(x)$ with $k \in C^0_G(\overline{\Omega})$ compact. Given $\varepsilon > 0$, we show that there are a G-invariant finite-dimensional subspace $X_\varepsilon \subset X$ and $\tilde{k}_\varepsilon \in C^0_G(\overline{\Omega})$ compact such that $\tilde{k}_\varepsilon(\overline{\Omega}) \subset X_\varepsilon$ and $||k - \tilde{k}_\varepsilon||_{0,\infty,\overline{\Omega}} \leq M\varepsilon$ where $M > 0$ is a constant depending only upon the representation R. Once this is done, $\varepsilon > 0$ can be chosen so that $d(f, \Omega, 0) = d(f_\varepsilon, \Omega, 0)$ where $f_\varepsilon = I - \tilde{k}_\varepsilon$, and also such that $d(f^G, \Omega^G, 0) = d(f^G_\varepsilon, \Omega^G, 0)$ (since obviously $||k^G - \tilde{k}^G_\varepsilon||_{0,\infty,\overline{\Omega}} \leq ||k - \tilde{k}_\varepsilon||_{0,\infty,\overline{\Omega}} \leq M\varepsilon$). Thus, replacing k by $\tilde{k}_\varepsilon$, it suffices to prove (7.1) when $f = I - k$ and $k \in C^0_G(\overline{\Omega})$ is compact and maps into a finite-dimensional G-invariant subspace.

We now show how X_ε and $\tilde{k}_\varepsilon$ above can be obtained. Our procedure follows closely the standard method when no covariance is involved, e.g., Proposition 8.1 in Deimling [**5**, p. 55], with a few extra steps. First, we choose $z_1, \ldots, z_m$ such that $\overline{k(\overline{\Omega})} \subset \cup_{i=1}^m B(z_i, \varepsilon/2)$. Next, call "irreducible G-module" any closed G-invariant subspace Y of X having no nontrivial proper G-invariant subspace. It is known (see, e.g., [**2**, pp. 141 and 143]) that for a compact Lie group G every irreducible G-module is finite-dimensional and that the algebraic sum of the irreducible G-modules is dense in X. Thus, for $1 \leq i \leq m$, there is a finite sum Y_i of irreducible G-modules and $y_i \in Y_i$ such that $||y_i - z_i|| < \varepsilon/2$. Evidently, $\dim Y_i < \infty$ and Y_i is G-invariant, and $\overline{k(\overline{\Omega})} \subset \cup_{i=1}^m B(y_i, \varepsilon)$. As in [**5**], set $\varphi_i(y) = \max(0, \varepsilon - ||y - y_i||)$ and $\psi_i(y) = \varphi_i(y)/\sum_{j=1}^m \varphi_j(y)$, for all $y \in \overline{k(\overline{\Omega})}$, and define

$$k_\varepsilon(x) = \sum_{i=1}^m \psi_i(k(x))y_i, \qquad \forall\, x \in \overline{\Omega}.$$

Then $k_\varepsilon \in C^0(\overline{\Omega})$, $k_\varepsilon(\overline{\Omega}) \subset X_\varepsilon = Y_1 + \cdots + Y_m$. Note that $\dim X_\varepsilon < \infty$ and X_ε is G-invariant. In particular, since $k_\varepsilon(\overline{\Omega})$ is obviously bounded, it is relatively compact in X_ε and therefore in X. Also, $||k - k_\varepsilon||_{0,\infty,\overline{\Omega}} \leq \varepsilon$ is trivial. Now k_ε need not be G-covariant, but from Lemma 6.1 (iii), $\tilde{k}_\varepsilon(x) = \int_G R_\gamma^{-1} k_\varepsilon(R_\gamma x)\, d\gamma$, $x \in \overline{\Omega}$, is compact and G-covariant. Furthermore, $\tilde{k}_\varepsilon(\overline{\Omega}) \subset X_\varepsilon$ by G-invariance of X_ε, and $||k - \tilde{k}_\varepsilon||_{\infty,\overline{\Omega}} \leq M\varepsilon$ by Lemma 6.1 (ii) and G-covariance of k, where $M > 0$ is a constant depending only upon the representation R of G in $GL(X)$.

From the above, we may then limit ourselves to proving (7.1) when $f = I - k$, $k \in C^0_G(\overline{\Omega})$ is compact and maps into some finite dimensional G-invariant subspace Y of X. With no loss of generality, we may assume that k is defined over the entire space X. Indeed, since $k(\overline{\Omega}) \subset Y$ is relatively compact, its closed convex hull C is a compact subset of Y, and it follows from Dugundji's theorem that k can be extended as an element of $C^0(X)$ with values in C, and hence compact. Again, this procedure may destroy G-covariance, which however may be reinstated by replacing k by $\tilde{k}(x) = \int_G R_\gamma^{-1} k(R_\gamma, x)\, d\gamma$, $x \in X$ (Lemma 6.1 (iii) with $\Omega = X$).

Since $\dim Y < \infty$, the space Y is split. As Y is also G-invariant, Lemma 6.2 and Remark 6.1 ensure the existence of a continuous G-covariant projection $P : X \to Y$. Set

$$h(t, x) = x - tk(Px) - (1 - t)k(x), \qquad x \in \overline{\Omega},$$

a G-covariant compact perturbation of the identity such that $h(0, \cdot) = f$ and $h(1, \cdot) = I - k \circ P \equiv g$. Let $(t, x) \in [0, 1] \times \overline{\Omega}$ be such that $h(t, x) = 0$. Because $Pk = k$, we have $x = Px \in Y$ and hence $0 = h(t, x) = x - k(x) = f(x)$. Thus, $x \in \Omega$ since $0 \notin f(\partial\Omega)$. This shows that $0 \notin h([0, 1] \times \partial\Omega)$, and hence by homotopy invariance of the degree,

$$d(f, \Omega, 0) = d(g, \Omega, 0).$$

Similarly, because G-covariance of P implies $P(X^G) \subset Y^G = Y \cap X^G$, we find

$$d(f^G, \Omega^G, 0) = d(g^G, \Omega^G, 0).$$

Thus, replacing f by g, we may as well prove (7.1) when $x = (y, z)$ and f has the form $f(y, z) = (y - k(y), z)$ relative to a (G-invariant) splitting $X = Y \oplus Z$, and $k \in C^0_G(Y)$. If so, it is a standard property of Leray-Schauder's degree that

$$d(f, \Omega, 0) = d(I_Y - k, \Omega \cap Y, 0), \tag{7.2}$$

where the degree is Brouwer's in the right-hand side. By similar arguments,

$$d(f^G, \Omega^G, 0) = d(I_{Y^G} - k^G, \Omega^G \cap Y^G, 0), \tag{7.3}$$

where the right-hand side is Brouwer's degree. Clearly, $\Omega^G \cap Y^G = \omega \cap Y^G \equiv \omega^G$, where $\omega \equiv \Omega \cap Y$, and $I_{Y^G} - k^G = (I_Y - k)^G$. Thus, by (7.2) and (7.3), proving (7.1) amounts to showing that

$$(7.4) \qquad d(I_Y - k, \omega, 0) = d((I_Y - k)^G, \omega^G, 0) \bmod \widetilde{\mathcal{I}}^G$$

the degree being Brouwer's.

Since $Y^{N(T)} = Y \cap X^{N(T)}$ and $Y^G = Y \cap X^G$, the hypothesis $X^{N(T)} = X^G$ implies $Y^{N(T)} = Y^G$. Thus, by Theorem 5.1, we know that

$$(7.5) \qquad d(I_Y - k, \omega, 0) = d((I_Y - k)^G, \omega^G, 0) \bmod i^{N(T)/T},$$

where $i^{N(T)/T}$ is the ideal of $\mathbf{Z}$ generated by the integers $[N(T)/T : \Gamma_x]$, $x \in Y^T \backslash Y^G$, and Γ_x is the isotropy subgroup of x relative to the action of $N(T)/T$ in Y^T. But Γ_x is unchanged if x is viewed in $X^T \supset Y^T$ instead of Y^T, i.e., each generator of $i^{N(T)/T}$ is in $\widetilde{\mathcal{I}}^G$. Thus, $i^{N(T)/T} \subset \widetilde{\mathcal{I}}^G$, and (7.4) thus follows from (7.5). □

When G is finite, we have $T = \{1\}$ and $N(T) = G$, so that $X^{N(T)} = X^G$ always holds. In this case, $\widetilde{\mathcal{I}}^G$ is simply the ideal $\mathcal{I}^G$ of $\mathbf{Z}$ generated by the integers $[G : G_x]$, $x \in X \backslash X^G$, and Theorem 7.1 generalizes Theorem 4.1 to the Banach space setting.

When $X^{N(T)} = \{0\}$ in Theorem 7.1, then obviously $X^G = \{0\} = X^{N(T)}$ and we obtain

$$(7.6) \qquad d(f, \Omega, 0) = 1 \text{ (respectively, 0)} \bmod \widetilde{\mathcal{I}}^G$$

if $0 \in \Omega$ (respectively, $0 \notin \Omega$). The condition $X^{N(T)} = \{0\}$ is satisfied, e.g., when $G = \mathbf{Z}_2$ is represented by $\{I, -I\}$, and (7.6) is just Borsuk's theorem since $\widetilde{\mathcal{I}}^G = 2\mathbf{Z}$ in this case.

If $\dim G \geq 1$ and $\operatorname{rank} G_x < \operatorname{rank} G$ for $x \in X \backslash X^G$, i.e., no isotropy subgroup of $x \in X \backslash X^G$ contains a maximal torus, then $X^T = X^G$ for every maximal torus, whence $X^{N(T)} = X^G$. Furthermore, as $X^T \backslash X^G = \varnothing$, we have $\widetilde{\mathcal{I}}^G = \{0\}$ (the ideal generated by the empty set of generators) and (7.1) reads

$$(7.7) \qquad d(f, \Omega, 0) = d(f^G, \Omega^G, 0),$$

a formula generalizing Lemma 5.1 as well as [**20**, Corollary 2.2] where (7.7) is obtained under the assumption that G and $\{0\}$ are the only isotropy subgroups and $\dim G \geq 1$ (whence $X^T = X^G$). The validity of (7.7) under the condition $X^T = \{0\}$ $(= X^G)$ was proved in [**15**] for Brouwer's degree and reads $d(f, \Omega, 0) = 1$ (respectively, 0) if $0 \in \Omega$ (respectively, $0 \notin \Omega$).

8. Relationship with Wang's theorem. To show that Theorem 7.1 implies (a generalization of) Wang's Theorem 1.1, we need two lemmas. The first one is a simple exercise in elementary Lie group theory but because normalizers of maximal tori are rarely used when the group is not connected, we have found no reference for it in the literature.

Lemma 8.1. *Let G be a compact Lie group with identity component G°, and let $T \leq G^\circ$ be a maximal torus. Denote by $N(T)$ and $N_{G^\circ}(T)$ the normalizers of T in G and G°, respectively. Then*:

(i) *$N(T)$ intersects every component of G.*

(ii) *$N(T)/T$ and $N_{G^\circ}(T)/T$ are finite and*

$$|N(T)/T| = |G/G^\circ|\,|N_{G^\circ}(T)/T|.$$

Proof. It was mentioned before that finiteness of $N(T)/T$ and $N_{G^\circ}(T)/T = W(G^\circ)$ (Weyl group) is well known [**1**]. Let $C \subset G$ be a connected component of G. We claim that there is a $\sigma \in C$ such that $\sigma T \sigma^{-1} = T$. Indeed, irrespective of $\eta \in C$, $\eta T \eta^{-1} \leq G^\circ$ is a maximal torus of G° ($\eta T \eta^{-1} \leq G^\circ$ by connectedness of $\eta T \eta^{-1}$ and $1 \in \eta T \eta^{-1}$). Maximal tori being conjugate in G°, there is a $\gamma \in G^\circ$ Such that $\eta T \eta^{-1} = \gamma^{-1} T \gamma$. This yields $\gamma\eta T (\gamma\eta)^{-1} = T$, and $\sigma = \gamma\eta \in C$ since multiplication by G° leaves C invariant.

We now show that $N(T) \cap C = \sigma N_{G^\circ}(T) \subset C$ (which incidentally proves (i)). First, if $\gamma \in N_{G^\circ}(T)$ and hence $\gamma T \gamma^{-1} = T$, then $(\sigma\gamma)T(\sigma\gamma)^{-1} = \sigma(\gamma T \gamma^{-1})\sigma^{-1} = \sigma T \sigma^{-1} = T$. Thus, $\sigma N_{G^\circ}(T) \subset N(T)$, and $\sigma N_{G^\circ}(T) \subset \sigma G^\circ = C$, so that $\sigma N_{G^\circ}(T) \subset N(T) \cap C$. Conversely, let $\gamma \in N(T) \cap C$, i.e., $\gamma \in C$ and $\gamma T \gamma^{-1} = T$. Since $\sigma T \sigma^{-1} = T = \sigma^{-1} T \sigma$, we find $(\sigma^{-1}\gamma)T(\sigma^{-1}\gamma)^{-1} = T$, i.e., $\sigma^{-1}\gamma \in$

$N(T)$. But $\sigma^{-1}\gamma \in G^\circ$ because $\gamma \in C = \sigma G^\circ$. Hence, $\sigma^{-1}\gamma \in N(T) \cap G^\circ = N_{G^\circ}(T)$, and $\gamma = \sigma(\sigma^{-1}\gamma) \in \sigma N_{G^\circ}(T)$.

The torus T is the identity component of $N_{G^\circ}(T) = N(T) \cap G^\circ$, thus also that of $N(T)$. It follows that $|N(T)/T|$ (respectively, $|W(G^\circ)|$) is the number of connected components of $N(T)$ (respectively, $N_{G^\circ}(T)/T$). Since $N(T)$ is the disjoint union $\cup N(T) \cap C$ over all the connected components C of G, and $N(T) \cap C = (\sigma N_{G^\circ}(T)$ for some $\sigma \in C$ from the above) has the same number of connected components as $N_{G^\circ}(T)$, we find that $|N(T)/T| = |G/G^\circ|\,|W(G^\circ)|$. This proves (ii). $\square$

Lemma 8.2. *Let G be a compact Lie group with identity component G°, and let X be a real Banach space equipped with a continuous linear action of G.*

(i) *Let $T \le G^\circ$ be a maximal torus. For $x \in X^T$, denote by G_x and Γ_x the isotropy subgroups of x relative to the actions of G in X and $N(T)/T$ in X^T, respectively, where $N(T)$ is the normalizer of T in G. Then, the Euler-Poincaré characteristic of G/G_x is given by:*

$$\chi(G/G_x) = [N(T)/T : \Gamma_x].$$

(ii) *Let $x \in X$ be such that* $\operatorname{rank} G_x < \operatorname{rank} G$ *(i.e., G_x contains no maximal torus of G°). Then,*

$$\chi(G/G_x) = 0.$$

Proof. (i) According to Dancer [**3**], it follows from Greub et al., [**9**, p.182], that if $K \le G$ is a closed subgroup with $\operatorname{rank} K = \operatorname{rank} G$, then

$$\chi(G/K) = \frac{[G : G^\circ]|W(G^\circ)|}{[K : K^\circ]|W(K^\circ)|}, \tag{8.1}$$

where K° is the identity component of K and $W(G^\circ)$, $W(K^\circ)$ are the Weyl groups of G° and K°, respectively. However, only the case when $G = G^\circ$ and $K = K^\circ$ are connected is considered in [**9**], and Dancer gives no hint as to what is involved to derive (8.1) in general. The argument is as follows: if $M, \tilde{M}$ are compact manifolds and $\tilde{M}$

is a p-fold covering of M, then $\chi(\tilde{M}) = p\chi(M)$ (see Hu [**10**, p. 277]). Here, this can be used with the three coverings by canonical maps: $G^\circ/K^\circ \to G^\circ/(K \cap G^\circ)$, $G/(K \cap G^\circ) \to G^\circ/(K \cap G^\circ)$ and $G/(K \cap G^\circ) \to G/K$, which are p-fold coverings with $p = [K \cap G^\circ : K^\circ]$, $[G : G^\circ]$ and $[K : K \cap G^\circ]$, respectively. This yields $\chi(G/K) = ([G : G^\circ]/[K : K^\circ])\chi(G^\circ/K^\circ)$, and $\chi(G^\circ/K^\circ) = |W(G^\circ)|/|W(K^\circ)|$ by the result in [**9**], which proves (8.1).

With K as above, let $T \leq K^\circ$ be a maximal torus. Since $\operatorname{rank} K = \operatorname{rank} G$, T is also a maximal torus of G° and it follows from Lemma 8.1 that $|N(T)/T| = [G : G^\circ]|W(G^\circ)|$, $|N_K(T)/T| = [K : K^\circ]|W(K^\circ)|$ where of course $N_K(T)$ is the normalizer of T in K. With this, (8.1) may be rewritten as

$$\chi(G/K) = \frac{|N(T)/T|}{|N_K(T)/T|}. \tag{8.2}$$

Clearly, (8.2) can be used with $K = G_x$ and x, T as in part (i) of the lemma, so that

$$\chi(G/G_x) = \frac{|N(T)/T|}{|N_{G_x}(T)/T|}. \tag{8.3}$$

We claim that the groups G_x and Γ_x are related through

$$\Gamma_x = N_{G_x}(T)/T. \tag{8.4}$$

If so, it follows from (8.3) that $\chi(G/G_x) = |N(T)/T|/|\Gamma_x| = [N(T)/T : \Gamma_x]$, as desired.

To prove (8.4), note first that by definition of Γ_x we have $\Gamma_x = H/T$ where $T \leq H \leq N(T)$. Since Γ_x is the isotropy subgroup of x for the action of $N(T)/T$ in X^T, we must have $R_\gamma x = x$, for all $\gamma \in H$, i.e., $H \leq G_x$. Also, T is normal in H since $H \leq N(T)$, whence $H \leq N_{G_x}(T)$. This yields $\Gamma_x \leq N_{G_x}(T)/T$. Conversely, it follows from $N_{G_x}(T) \leq G_x$ that the action of $N_{G_x}(T)$ on x is trivial, so that $N_{G_x}(T)/T \leq \Gamma_x$.

(ii) When G and $K \leq G$ are connected and $\operatorname{rank} K < \operatorname{rank} G$, it is shown in [**9**, p. 182] that $\chi(G/K) = 0$. In general, assuming only $K \leq G$ and $\operatorname{rank} K < \operatorname{rank} G$, the arguments of the proof of

(8.2) show that $\chi(G/K) = ([G : G^\circ]/[K : K^\circ])\chi(G^\circ/K^\circ) = 0$ since $\chi(G^\circ/K^\circ) = 0$, and (ii) follows by taking $K = G_x$. $\square$

Theorem 8.1. *Let G be a compact Lie group and X a real Banach space equipped with a continuous linear G-action. Let $\Omega \subset X$ be a G-invariant bounded open subset with $f \in C_G^\circ(\overline{\Omega})$ a compact perturbation of the identity such that $0 \notin f(\partial\Omega)$. Then,*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \mathcal{I}^G, \tag{8.5}$$

where $\mathcal{I}^G$ denotes the ideal of $\mathbf{Z}$ *generated by the integers $\chi(G/G_x)$, $x \in X\backslash X^G$, rank G_x = rank G. Moreover, if T is any maximal torus of the identity component G° of G, and $N(T)$ denotes the normalizer T in G, we have*

(i) *$X^{N(T)} \neq X^G$ if and only if there is $x \in X\backslash X^G$ such that $\chi(G/G_x) = 1$.*

(ii) *If $X^{N(T)} \neq X^G$, then $\mathcal{I}^G = \mathbf{Z}$ and* (8.5) *is vacuous.*

(iii) *If $X^{N(T)} = X^G$, then $\mathcal{I}^G = \widetilde{\mathcal{I}}^G$ (see Theorem* 7.1) *and* (8.5) *and* (7.1) *coincide.*

Proof. We begin with the proof of (i). If $X^{N(T)} \neq X^G$, there is an $x \in X$ such that $N(T) \leq G_x < G$. Obviously, $x \in X^T\backslash X^G$ and the isotropy subgroup Γ_x of x relative to the action of $N(T)/T$ in X^T is the whole group $N(T)/T$. Thus $\operatorname{rank} G_x = \operatorname{rank} G$ and $\chi(G/G_x) = 1$ by Lemma 8.2 (i). Conversely, if $x \in X\backslash X^G$ is such that $\chi(G/G_x) = 1$, then $\operatorname{rank} G_x = \operatorname{rank} G$ by Lemma 8.2 (ii), i.e., there is a maximal torus T' of G° such that $T' \leq G_x$. From Lemma 8.2 (i), the isotropy subgroup Γ_x relative to the action of $N(T')/T'$ in $X^{T'}$ satisfies $[N(T')/T' : \Gamma_x] = \chi(G/G_x) = 1$, whence $\Gamma_x = N(T')/T'$. This implies $N(T') \leq G_x$, i.e., $x \in X^{N(T')}$. Thus, $X^{N(T')} \neq X^G$ since $x \notin X^G$, and $X^{N(T)} \neq X^G$ by Theorem 7.1 (i).

To prove (8.5) it suffices of course to prove (ii) and (iii), and (ii) follows at once from (i). Suppose then that $X^{N(T)} = X^G$, so that $\chi(G/G_x) \neq 1$ for every $x \in X\backslash X^G$ from (i). Let then $x \in X\backslash X^G$ with $\operatorname{rank} G_x = \operatorname{rank} G$, and let T' be a maximal torus of G° such that $T' \leq G_x$. It follows from Lemma 8.2 (i) that the isotropy subgroup Γ_x of x relative to the action of $N(T')/T'$ in $X^{T'}$ is a proper subgroup of $N(T')/T'$, whence $x \notin X^{N(T')} = X^G$ (Theorem 7.1 (i)) and $\chi(G/G_x) = [N(T')/T' : \Gamma_x] \in \widetilde{\mathcal{I}}^G$. This shows that $\mathcal{I}^G \subset \widetilde{\mathcal{I}}^G$. Conversely, let $x \in X^T \backslash X^G$, so that $[N(T)/T : \Gamma_x] \in \widetilde{\mathcal{I}}^G$. By Lemma 8.2 (i), $\chi(G/G_x) = [N(T)/T : \Gamma_x]$, and $\chi(G/G_x) \in \mathcal{I}^G$ since $\operatorname{rank} G_x = \operatorname{rank} G$ and $x \notin X^G$. Thus, $[N(T)/T : \Gamma_x] \in \mathcal{I}^G$, i.e., $\widetilde{\mathcal{I}}^G \subset \mathcal{I}^G$. $\square$

Remark 8.1. By Lemma 8.2 (ii), $\mathcal{I}^G$ is also the ideal of $\mathbf{Z}$ generated by *all* the integers $\chi(G/G_x)$, $x \in X \backslash X^G$, the definition used by Wang.

Corollary 8.1. *Suppose that in Theorem* 8.1 *both the component group G/G° and the Weyl group $W(G^\circ)$ ($\simeq N_{G^\circ}(T)/T$ for any maximal torus T) are $\wp$-groups, $\wp$ a prime number. Suppose also that $\chi(G/G_x) \neq 1$ for every $x \in X \backslash X^G$. Then*

$$d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod \wp\mathbf{Z}.$$

Proof. From Lemma 8.1 (ii), $N(T)/T$ is a $\wp$-group whereas $\mathcal{I}^G = \widetilde{\mathcal{I}}^G$ from Theorem 8.1 (i) and (iii). Now $\widetilde{\mathcal{I}}^G \subset \wp\mathbf{Z}$ if $N(T)/T$ is a $\wp$-group, and the conclusion follows. $\square$

Corollary 8.1 generalizes a result of Wang [**20**, p. 527] stating that if $\dim G \leq 3$, G has 2^m components, $m \geq 0$, and the orbits of the action of G are orientable, then $d(f, \Omega, 0) = d(f^G, \Omega^G, 0) \bmod 2\mathbf{Z}$. This follows at once from Corollary 8.1 since $\dim G \leq 3$ implies that G° is a torus or $G^\circ \simeq SU(2)$ or $SO(3)$ (hence $W(G^\circ) = \{1\}$ or $\mathbf{Z}_2$ is a 2-group), and $\chi(G/G_x) = 1$ is impossible for $x \in X \backslash X^G$ since $\chi(G/G_x) = 0$ if $\operatorname{rank} G_x < \operatorname{rank} G$ (Lemma 8.2 (ii)) whereas G/G_x is either finite and different from $\{1\}$ (when G° is a torus) or an orientable compact surface (when $G^\circ \simeq SU(2)$ or $SO(3)$ and hence has rank 1) if $\operatorname{rank} G_x = \operatorname{rank} G$.

As shown in Theorem 8.1, the condition $X^{N(T)} = X^G$ must hold for formula (8.5) to be of any use at all. We now investigate this condition more closely.

Theorem 8.2. *The notation being as in Theorem* 8.1, *we have*

$X^{N(T)} \neq X^G$ if and only if there is an $x \in X \backslash X^G$ such that $G/G_x = G^\circ/(G^\circ)_x$ and $\chi(G^\circ/(G^\circ)_x) = 1$. In particular, $X^{N(T)} = X^G$ if

(i) *G is not connected and G/G_x is not connected for every $x \in X\backslash X^G$ with* $\operatorname{rank} \ G_x = \operatorname{rank} G$, *or if*

(ii) $X^{N_{G^\circ}(T)} = X^{G^\circ}$.

Proof. By Theorem 8.1 (i), $X^{N(T)} \neq X^G$ if and only if there is an $x \in X\backslash X^G$ such that $\chi(G/G_x) = 1$. By [**2**, p. 138] G_x can be viewed as the isotropy subgroup G_ξ of a *finite dimensional* representation of G. If so, $(G^\circ)_x = G_x \cap G^\circ = G_\xi \cap G^\circ = (G^\circ)_\xi$. Furthermore, G/G_ξ $(= G/G_x)$ is diffeomorphic to $G \cdot \xi$ ([**2**, p. 35]) which in turn is the disjoint union of a finite number $q \geq 1$ of copies of $G^\circ \cdot \xi$. Thus, $\chi(G/G_x) = \chi(G/G_\xi) = \chi(G \cdot \xi) = q\chi(G^\circ \cdot \xi) = q\chi(G^\circ/(G^\circ)_\xi) = q\chi(G^\circ/(G^\circ)_x)$, and $\chi(G/G_x) = 1$ if and only if $q = 1$ (i.e., $G/G_x = G^\circ/(G^\circ)_x$) and $\chi(G^\circ/(G^\circ)_x) = 1$.

That $X^{N(T)} = X^G$ if condition (i) of Theorem 8.2 holds now follows at once from Lemma 8.2 (ii) and connectedness of (G°/G_x°) for every $x \in X$. That condition (ii) of Theorem 8.2 implies $X^{N(T)} = X^G$ is trivial from the above and Theorem 8.1 (i). $\square$

If G is connected, Theorem 8.2 does not help checking the condition $X^{N(T)} = X^G$. However, if G is a "classical" group, the Weyl group $N(T)/T$ is known explicitly (see [**2**] or [**6**]), and this can be used to calculate $X^{N(T)} = (X^T)^{N(T)/T}$.

Remark 8.2. With $\mathcal{I}^G$ replaced by $\sqrt{\mathcal{I}^G}$, Theorem 8.1 can be extended to covariant proper C^2 Fredholm mappings with index 0 and the degree introduced in [**7**] (see [**8**]). The proof is only vaguely reminiscent of the one given here because the degree of [**7**] is not defined by finite-dimensional approximation and because other complications arise from "G-regularity" conditions which are vacuous in the case of compact perturbations of the identity.

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