ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 1, Winter 1995

IMPULSIVE STABILIZATION AND APPLICATIONS TO POPULATION GROWTH MODELS

XINZHI LIU

ABSTRACT. This paper establishes some stability criteria for impulsive differential systems. It is shown that impulses do contribute to yield stability properties even when the corresponding differential system without impulses does not enjoy any stability behavior. As an application, these results are applied to some population growth models.

1. Introduction. Many physical systems are characterized by the fact that at certain moments of time they experience a sudden change of their state. For example, when a mass on a spring is given a blow by a hammer, it experiences a sharp change of velocity; and a pendulum in a mechanical clock undergoes a drastic increase of momentum every time when it crosses its equilibrium position. These systems are subject to short-term perturbations which are often assumed to be in the form of impulses in the modeling process. Consequently, impulsive differential equations provide a natural description of such systems [**3**].

In this paper, we investigate the problem of stability for impulsive differential systems by Lyapunov's direct method. In Section 2 we describe impulsive differential systems and introduce some notations and definitions. We establish, in Section 3, some stability criteria which may be considered as impulsive stabilization of the underlying continuous physical system. It may provide a greater prospect to solving problems that are basically defined by continuous dynamical systems, but on which only discrete-time actions are exercised. As an application, we apply our results, in Section 4, to some population growth models.

2. Preliminaries. Let a physical system be described by the following system of differential equations

$$x' = f(t, x). \tag{2.1}$$

Received by the editors on August 11, 1992.
Research partially supported by NSERC Canada.

Let $x(t) = x(t, t_0, x_0)$ be any solution of the system (2.1) starting at (t_0, x_0). The point $P_t(t, x(t))$ begins its motion from the initial point $P_{t_0} = (t_0, x_0)$ and moves along the curve $[(t, x); t \geq t_0, x = x(t)]$ until the time $t_1 > t_0$ at which the point $P_{t_1}(t_1, x(t_1))$ is transferred immediately to $P_{t_1^+} = (t_1, x_1^+)$, where $x_1^+ = x(t_1) + I_1(x(t_1))$. Then the point P_t continues to move further along the curve with $x(t) = x(t, t_1, x_1^+)$ until it triggers a second transfer at $t_2 > t_1$. Once again, the point $P_{t_2} = (t_2, x(t_2))$ is mapped into the point $P_{t_2^+} = (t_2, x_2^+)$, where $x_2^+ = x(t_2) + I_2(x(t_2))$. As before, the point P_t continues to move forward with $x(t) = x(t, t_2, x_2^+)$ as the solution of (2.1) starting at (t_2, x_2^+). Clearly, this process continues as long as the solution of (2.1) exists and it results in a piecewise continuous trajectory $x(t)$ which satisfies the following relations

$$\begin{cases} x' = f(t, x), & t \neq t_k, \\ \Delta x = I_k(x), & t = t_k, \\ x(t_0) = x_0, & k = 1, 2, \dots, \end{cases} \tag{2.2}$$

where $\Delta x(t_k) = x_k^+ - x(t_k)$. We call (2.2) an impulsive differential system.

We denote by Γ the class of maps $h : R_+ \times R^n \to R_+$, which are continuous and $\inf h(t, x) = 0$. Let $h, h_0 \in \Gamma$. We shall discuss the qualitative behavior of the map h along solutions of (2.2) whose initial values are measured by the second map h_0. By doing this, we are able to deal with, in a unified way, several concepts and associate problems, which are usually considered separately. We shall assume, for simplicity, that the functions $f(t, x)$ and $I_k(x)$ satisfy all required conditions so that all solutions $x(t) = x(t, t_0, x_0)$ of (2.2) exist for all $t \geq t_0$. For a detailed discussion of this point, see [**5**].

Definition 2.1. Let $h, h_0 \in \Gamma$. Then the impulsive differential system (2.2) is called

(i) (h_0, h)-*stable* if for any $\varepsilon > 0$ and $t_0 \in R_+$ given, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$, $t \geq t_0$, where $x(t) = x(t, t_0, x_0)$ is any solution of (2.2);

(ii) (h_0, h)-*attractive* if for $t_0 \in R_+$ there exists a $\sigma = \sigma(t_0)$ such that $h(t_0, x_0) < \sigma$ implies $\lim_{t \to \infty} h(t, x(t)) = 0$;

(iii) (h_0, h)-*asymptotically stable* if (i) and (ii) hold together;

(iv) (h_0, h)-*unstable* if (i) fails to hold.

The concepts of (h_0, h)-stability enable us to unify a variety of stability notions found in the literature, such as stability of a prescribed motion, partial stability, stability of an invariant set and conditional stability, to name a few.

Let v_0 denote the class of functions $V : R_+ \times R^n \to R_+$, where V is locally Lipschitz in x, continuous everywhere except t_ks at which V may have jump discontinuities. For $V \in v_0$, $(t, x) \in R_+ \times R^n$ and $t \neq t_k$, we define $D^+V(t, x)$ by

$$D^+V(t,x) = \limsup_{\delta \to 0^+} \frac{1}{\delta}[V(t+\delta, x+\delta f(t,x)) - V(t,x)]. \tag{2.3}$$

We denote by $\mathcal{K}$ the class of functions $\phi : R_+ \to R_+$ which are continuous, strictly increasing and $\phi(0) = 0$, $\mathcal{K}_0$ the class of continuous functions $\psi : R_+ \to R_+$ such that $\psi(s) = 0$ if and only if $s = 0$, and PC the class of functions $\lambda : R_+ \to R_+$ where λ is continuous everywhere except t_ks at which λ may have jump discontinuities.

Definition 2.2. Let $V \in v_0$ and $h \in \Gamma$. Then V is said to be

(i) *h-positive definite* if there exist a constant $\rho > 0$ and a function $b \in \mathcal{K}$ such that $b(h(t,x)) \leq V(t,x)$ if $h(t,x) < \rho$;

(ii) *h-decrescent* if there exist a constant $\delta > 0$ and a function $a \in \mathcal{K}$ such that $V(t,x) \leq a(h(t,x))$, whenever $h(t,x) \leq \rho$.

Definition 2.3. Let $h, h_0 \in \Gamma$. Then we say that h_0 is *finer* than h if h is h_0-decrescent.

3. Main results. We shall state and prove our main results in this section. To motivate our first theorem, let us begin by discussing a simple example.

Example 3.1. Consider the linear impulsive system

$$\tag{3.1} \begin{cases} x' = Ax, & t \neq k, \\ \Delta x = Bx, & t = k, \\ x(0) = x_0, & k = 1, 2, \dots, \end{cases}$$

where $x = (x_1, x_2)^T$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -b_1 & 0 \\ 0 & -b_2 \end{bmatrix}$, $0 < b_i < 1$ and $(1-b_i)e < 1$. It should be noted that $x = 0$ is an unstable saddle point of the underlying system $x' = Ax$. But it is asymptotically stable with respect to system (3.1). In fact, letting $V(x) = (x_1^2 + x_2^2)/2$, we have

$$\begin{aligned} D^+V(x) &\leq 2V(x), \qquad t \neq k, \\ V(x(k^+)) &\leq (1-m)^2 V(x(k)), \qquad m = \min_{1 \leq i \leq 2} b_i, \end{aligned}$$

which implies that

$$V(x(t)) \leq \begin{cases} V(x(k^+))e^2(t-k), & k < t \leq k+1, \ k = 0, 1, 2, \dots, \\ V(x_0), & t = 0. \end{cases}$$

Thus, the conclusion follows from the fact

$$V(x(k^+)) \leq V(x_0)[(1-m)e]^{2k} \to 0 \quad \text{as } k \to \infty.$$

This example shows that an unstable system may be stabilized by impulses. Let $s(h, \rho) = [(t, x) \in R_+ \times R^n; h(t, x) < \rho]$. We have the following general result.

Theorem 3.1. *Assume that*

(i) $h_0, h \in \Gamma$, *h_0 is finer than h, and there exist constants ρ and ρ_0 with $0 < \rho_0 < \rho$ such that $(t_k, x) \in s(h, \rho_0)$ implies $(t_k, x + I_k(x)) \in s(h, \rho)$ for all $k = 1, 2, \dots$;*

(ii) $V \in v_0$, *$V(t, x)$ is h-positive definite, h_0-decrescent and there exists $\psi_k \in \mathcal{K}_0$ such that*

$$\tag{3.2} V(t_k^+, x + I_k(x)) \leq \psi_k(V(t_k, x)), \qquad k = 1, 2, \dots;$$

(iii) *there exist* $c \in \mathcal{K}$ *and* $p \in PC$ *such that*

$$D^+V(t,x) \le p(t)c(V(t,x)), \qquad (t,x) \in s(h,p),\ t \ne t_k; \tag{3.3}$$

(iv) *there exists a constant* $\sigma > 0$ *such that, for all* $z \in (0,\sigma)$,

$$\int_{t_k}^{t_{k+1}} p(s)\,ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \le -\gamma_k, \tag{3.4}$$

for some constant γ_k *and* $k = 1, 2, \dots$.

Then the system (2.2) *is* (h_0,h)*-stable if* $\gamma_k \ge 0$ *for all* $k = 1, 2, \dots,$ *and* (h_0,h)*-asymptotically stable if, in addition,* $\sum_{k=1}^{\infty} \gamma_k = \infty$.

Proof. Since $V(t,x)$ is h_0-decrescent and h-positive definite, there exist $a, b \in \mathcal{K}$ and $\delta_0, \alpha_0 > 0$ such that

$$V(t,x) \le a(h_0(t,x)), \quad \text{if } h_0(t,x) < \delta_0 \tag{3.5}$$

and

$$V(t,x) \ge b(h(t,x)), \quad \text{if } h(t,x) < \alpha_0. \tag{3.6}$$

By condition (i), there exist $\delta_1 > 0$ and $\phi \in \mathcal{K}$ such that

$$h(t,x) \le \phi(h_0(t,x)) < \alpha_0, \quad \text{whenever } h_0(t,x) < \delta_1. \tag{3.7}$$

Let $\varepsilon > 0$ with $0 < \varepsilon < \rho^* = \min(\rho_0, \alpha_0)$ and $t_0 \in R_+$ given. We may assume that $t_1 < t_0 \le t_2$, for if $t_j < t_0 \le t_{j+1}$ for some $j \ge 1$, we then set $t_k = t_{j+k-1}$. Choose $\eta = \min(b(\varepsilon), \sigma)$ and σ^* such that $0 < \sigma^* < \min(\eta, \psi_1(\eta))$. By the definition of a, there exists a $\delta_2 > 0$ such that

$$a(\delta_2) < \sigma^*. \tag{3.8}$$

Let $\delta = \min(\delta_0, \delta_1, \delta_2)$, $x_0 \in R^n$ such that $h_0(t_0, x_0) < \delta$. It is clear from (3.5)–(3.8) that

$$b(h(t_0,x_0)) \le V(t_0,x_0) \le a(h_0(t_0,x_0)) < \sigma^*,$$

which implies $h(t_0, x_0) < \varepsilon$. Let $x(t) = x(t, t_0, x_0)$ be any solution of system (2.2) with $h_0(t_0, x_0) < \sigma$. We claim that

$$h(t, x(t)) < \varepsilon, \qquad t \geq t_0. \tag{3.9}$$

If this is false, then there exist a solution $x(t) = x(t, t_0, x_0)$ of (2.2) with $h_0(t_0, x_0) < \delta$ and a $t^* > t_0$ such that $t_k < t^* \leq t_{k+1}$ for some k, satisfying

$$\varepsilon \leq h(t^*, x(t^*)) \quad \text{and} \quad h(t, x(t)) < \varepsilon, \quad \text{for } t_0 \leq t \leq t_k.$$

By the choice of ε and conditions (i), we see

$$h(t_k^+, x_k^+) = h(t_{k,k} + I_k(x_k)) < \rho,$$

where $x_k = x(t_k)$. Hence, we can find a $\bar{t}$ such that

$$\begin{aligned} \varepsilon \leq h(\tilde{t}, x(\tilde{t})) < \rho \quad & \text{and} \\ h(t, x(t)) < \rho \quad & \text{for } t \in [t_0, \tilde{t}]. \end{aligned} \tag{3.10}$$

Defining $m(t) = V(t, x(t))$ for $t \in [t_0, \tilde{t}]$ and using (3.2) and (3.3), we get

$$\begin{cases} D^+m(t) \leq p(t)c(m(t)), & t \neq t_i, \\ m(t_i^+) \leq \psi_i(m(t_i)), & i = 1, 2, \dots, k. \end{cases} \tag{3.11}$$

If we suppose that $\tilde{t} \in (t_0, t_2]$, then we get from (3.11)

$$\int_{m(t_0)}^{m(\tilde{t})} \frac{ds}{c(s)} \leq \int_{t_0}^{\tilde{t}} p(s)\, ds. \tag{3.12}$$

Since

$$\int_{m(t_0)}^{m(\tilde{t})} \frac{ds}{c(s)} > \int_{\psi_1(\eta)}^{\eta} \frac{ds}{c(s)},$$

and

$$\int_{t_0}^{\tilde{t}} p(s)\, ds \leq \int_{t_1}^{t_2} p(s)\, ds,$$

it follows from (3.12) that

$$\int_{t_1}^{t_2} p(s)\,ds + \int_{\eta}^{\psi_1(\eta)} \frac{ds}{c(s)} > 0,$$

which contradicts (3.4).

Now suppose for $t \in [t_0, t_i]$ that $m(t) < \eta$. Then, from (3.11), we obtain

$$\int_{m(t_i^+)}^{m(t)} \frac{ds}{c(s)} \le \int_{t_i}^{t} p(s)\,ds \le \int_{t_i}^{t_{i+1}} p(s)\,ds, \qquad t \in (t_i, t_{i+1}]. \tag{3.13}$$

It then follows from (3.4) and (3.13) that, for $t \in (t_i, t_{i+1}]$,

$$\int_{m(t_i)}^{m(t)} \frac{ds}{c(s)} \le \int_{t_i}^{t_{i+1}} p(s)\,ds + \int_{m(t_i)}^{\psi_i(m(t_i))} \frac{ds}{c(s)} \le -\gamma_i, \tag{3.14}$$

which implies, in view of the fact that $\gamma_i \ge 0$ and $1/c(s) > 0$,

$$m(t) \le m(t_i) < \eta, \quad \text{for } t \in (t_i, t_{i+1}].$$

It follows by induction that

$$m(t) < \eta, \qquad t \in [t_0, \tilde{t}],$$

which leads to the following contradiction

$$b(\varepsilon) \le b(h(\tilde{t}, x(\tilde{t}))) \le m(\tilde{t}) < b(\varepsilon).$$

Thus (3.9) is true and the system (2.2) is (h_0, h)-stable. We shall next prove (h_0, h)-asymptotic stability under the assumption that $\sum_{k=1}^{\infty} \gamma_k = \infty$.

From (h_0, h)-stability, we set $\varepsilon = \rho$ so that $\tilde{\delta} = \delta(\rho)$ and $h_0(t_0, x_0) < \tilde{\delta}$ implies

$$h(t, x(t)) < \rho, \qquad t \ge t_0.$$

Clearly, (3.11) and (3.14) remain true for all $t \ge t_0$. From (3.14) we see that $m(t_k)$ is nonincreasing in k and thus $\lim_{k\to\infty} m(t_k) = \beta$ exists. If $\beta > 0$, then it follows from (3.14) that

$$m(t_{k+q}) \le m(t_k) - c(\beta) \sum_{j=k}^{k+q-1} \gamma_j \to -\infty \quad \text{as } q \to \infty.$$

This contradiction shows $\beta = 0$.

Since $m(t) \le m(t_k)$ for $t \in (t_k, t_{k+1}]$ and $V(t,x)$ is h-positive definite, it follows that $\lim_{t\to\infty} h(t, x(t)) = 0$. Thus, system (2.2) is (h_0, h)-asymptotically stable and the proof is complete. □

Remark. It should be noted that assumption (3.3) includes the case when

$$D^+V(t,x) > 0, \qquad (t,x) \in s(h,\rho),\ t \neq t_k,$$

which indicates unstable behavior of the underlying system. That is, an unstable system has been stabilized by impulses.

The following example shows that impulses may destroy a stable system.

Example 3.2. Consider the impulsive differential system

$$\begin{cases} x' = -x, & t \neq k, \\ \Delta x = \sqrt{|x|} - x, & t = k, \\ x(0) = x_0, & k = 1, 2, \dots . \end{cases} \tag{3.15}$$

Clearly, $x = 0$ is an asymptotically stable equilibrium point of the underlying equation $x' = -x$, but it is unstable with respect to (3.15). Solving (3.15) directly we get

$$x(t) = \begin{cases} x_{k-1}^+ e^{-(t-k)}, & (k-1) < t \le k,\ k = 1, 2, \dots, \\ x_0, & t = 0, \end{cases}$$

where $x_k^+ = |x_0|^{1/2^k} \exp(-\sum_{j=1}^k 1/2^j) \to e^{-1}$ as $k \to \infty$. Thus, $\lim_{t\to\infty} x(t) = e^{-2}$ and $x = 0$ is unstable.

The next result provides sufficient conditions under which stability behavior is preserved under impulsive perturbations.

Theorem 3.2. *Assume that conditions* (i) *and* (ii) *of Theorem* 3.1 *hold. Suppose further that*

(iii*) *there exist functions* $c \in \mathcal{K}$ *and* $\lambda \in PC$ *such that*

$$D^+V(t,x) \le -\lambda(t)c(V(t,x)), \qquad (t,x) \in s(h,\rho),\ t \neq t_k; \tag{3.16}$$

(iv*) *there exists a constant* $\sigma > 0$ *such that for all* $z \in (0, \sigma)$

$$-\int_{t_{k-1}}^{t_k} \lambda(s)\,ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \le -\gamma_k, \tag{3.17}$$

for some constant γ_k *and* $k = 1, 2, \dots$.

Then the system (2.2) *is* (h_0, h)*-stable if* $\gamma_k \ge 0$ *for all* $k = 1, 2, \dots$, *and* (h_0, h)*-asymptotically stable if, in addition,* $\sum_{k=1}^{\infty} \gamma_k = \infty$.

The proof of Theorem 3.2 is similar to that of Theorem 3.3 in [**4**].

The next two theorems are on (h_0, h)-instability.

Theorem 3.3. *Assume that*

(i) $h_0, h \in \Gamma$, h_0 *is finer than* h, *and for* $\rho > 0$

$$\inf h_0(t, x) = 0; \qquad (t, x) \in s(h, \rho) \tag{3.18}$$

(ii) $V \in v_0$, $V(t, x)$ *is bounded on* $s(h, \rho)$ *and there exists* $\psi_k \in \mathcal{K}_0$ *such that*

$$V(t_k^+, x + I_k(x)) \ge \psi_k(V(t_k, x)), \qquad k = 1, 2, \dots ; \tag{3.19}$$

(iii) *there exist* $c \in \mathcal{K}$ *and* $\lambda \in PC$ *such that*

$$-\lambda(t) c(V(t, x)) \le D^+ V(t, x), \qquad (t, x) \in s(h, \rho),\ t \ne t_k; \tag{3.20}$$

(iv) *there exists a sequence* $\{\gamma_k\}$ *with* $\gamma_k \ge 0$ *for* $k = 1, 2, \dots$, *and* $\sum_{k=1}^{\infty} \gamma_k = \infty$ *such that*

$$-\int_{t_{k=1}}^{t_k} \lambda(s)\,ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \ge \gamma_k, \qquad z \in (0, \infty). \tag{3.21}$$

Then system (2.2) *is* (h_0, h)*-unstable.*

Proof. Let $\delta > 0$ be sufficiently small. Then (3.18) implies that there exists $(t_0, x_0) \in s(h, \rho)$ such that $h_0(t_0, x_0) < \delta$. Let $x(t) = x(t, t_0, x_0)$ be a solution of (2.2) starting at (t_0, x_0). We claim that $(t, x(t))$ will leave the set $s(h, \rho)$ in a finite time. Suppose, for the sake of contradiction, that $(t, x(t))$ stays in the set $s(h, \rho)$ for all $t \geq t_0$. Setting $m(t) = V(t, x(t))$, $t \geq t_0$, it follows from (3.19) and (3.20) that

$$\begin{cases} D^+ m(t) \geq -\lambda(t) c(m(t)), & t \geq t_0,\ t \neq t_k, \\ m(t_k^+) \geq \psi_k(m(t_k)), & k = 1, 2, \dots, \end{cases} \tag{3.22}$$

which implies

$$\int_{m(t_{k-1}^+)}^{m(t_k)} \frac{ds}{c(s)} \geq -\int_{t_{k-1}}^{t_k} \lambda(s)\, ds.$$

This, together with (3.21), yields

$$\int_{m(t_{k-1}^+)}^{m(t_k^+)} \frac{ds}{c(s)} \geq -\int_{t_{k-1}}^{t_k} \lambda(s)\, ds + \int_{m(t_k)}^{\psi_k(m(t_k))} \frac{ds}{c(s)} \geq \gamma_k,$$

which shows

$$m(t_k^+) \geq m(t_{k-1}^+), \qquad k = 2, 3, \dots,$$

and

$$m(t_k^+) \geq m(t_{k-1}^+) - c(m(t_1^+))\gamma_k, \qquad k = 2, 3, \dots .$$

Thus

$$m(t_{k+m}^+) \geq m(t_{k-1}^+) - c(m(t_1^+)) \sum_{j=k}^{k+m} \gamma_j \to -\infty \quad \text{as } m \to \infty,$$

which contradicts the boundedness of $m(t)$. Hence the point $(t, x(t))$ must leave the set $s(h, \rho)$ in a finite time and therefore the system (2.2) is (h_0, h)-unstable. □

Remark. Condition (3.20) includes the case when

$$D^+ V(t, x) < 0, \qquad (t, x) \in s(h, \rho),\ t \neq t_k$$

which indicates stable behavior of the underlying system.

Theorem 3.4. *Assume that conditions* (i) *and* (ii) *of Theorem* 3.3 *hold. Suppose further than*

(iii*) *there exist* $c \in \mathcal{K}$ *and* $p \in PC$ *such that*

$$D^+V(t,x) \geq p(t)c(V(t,x)), \qquad (t,x) \in s(h,\rho),\ t \neq t_k; \tag{3.23}$$

(iv*) *there exists a sequence* $\{\gamma_k\}$ *with* $\gamma_k \geq 0$ *for* $k = 1, 2, \dots$, *and* $\sum_{k=1}^{\infty} \gamma_k = \infty$ *such that*

$$\int_{t_k}^{t_{k+1}} p(s)\,ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \geq \gamma_k, \qquad z \in (0,\infty). \tag{3.24}$$

Then the system (2.2) *is* (h_0, h)*-unstable.*

The proof of Theorem 3.4 is similar to that of Theorem 3.3; we omit the details.

4. Applications. In this section we present a fish population model which exhibits impulsive behavior in its state variable and may have applications in fisheries management.

Consider a fish population in a lake which connects the upper and lower streams of a creek. Suppose that all members of the fish population have identical ecological properties. This means that age differences among members of the population are not important. Under this assumption the population can be modeled by the nonlinear differential equation

$$N' = NF(N) + u, \tag{4.1}$$

where $N(t)$ is the population at time t, $N(t)F(N(t))$ is the natural growth rate of the fish population and $u \geq 0$ represents a constant influx rate of the population into the lake from the creek.

Suppose that the natural growth of the fish population is disturbed by making catches and adding fish brood, i.e., at times $t_1, t_2, \dots$, a part of the fish population with amount $E_1(N), E_2(N), \dots$, are removed from the lake and simultaneously a new brood of fish with amount $D_1(N), D_2(N), \dots$, are released. Then the growth of the fish population is impulsive and can be described by the following impulsive differential equation

$$(4.2)\qquad \begin{cases} N' = NF(N) + u, & t \neq t_k, \\ \Delta N = I_k(N), & t = t_k, \ k = 1, 2, \ldots . \end{cases}$$

We shall first consider the case $u > 0$ and obtain some results on stability properties of system (4.2). To motivate appropriate assumptions, let us consider the logistic equation

$$(4.3)\qquad N' = \alpha N(C - N) + u,$$

where $\alpha, C > 0$ are constants. It is easy to verify that

$$\alpha N(C - N) + u = -\alpha(N - L)(N + M),$$

where $L = (\sqrt{C^2 + 4u/\alpha} + C)/2 > 0$ and $M = (\sqrt{C^2 + 4u/\alpha} - C)/2 > 0$. Thus, we have

$$(4.4)\qquad (N - L)[\alpha N(C - N) + u] \leq -\alpha M(N - L)^2.$$

Motivated by this observation, we make the assumption on the right-hand side of (4.1) that there exist constants $L, \sigma > 0$ such that

$$(4.5)\qquad (N - L)[NF(N) + u] \leq -\sigma(N - L)^2, \qquad N \geq 0.$$

Under this assumption and applying Theorem 3.2 to system (4.2), we get the following result.

Theorem 4.1. *Assume that*

(i) *$F(N)$ is continuously differentiable and there exist constants $L, \sigma > 0$ such that* (4.5) *holds*;

(ii) *$I_k(N)$ is continuous and $I_k(L) = 0$ for all $k = 1, 2, \ldots$;*

(iii) *for any $z \in (0, L)$, there exists a sequence $\{\gamma_k\}$, $\gamma_k \geq 0$ for all $k = 1, 2, \ldots$, and $\sum_{k=1}^{\infty} \gamma_k = \infty$, such that*

$$(4.6)\qquad -\sigma(t_k - t_{k-1}) + \max\left(\ln\left[1 + \frac{I_k(L + z)}{z}\right], \ln\left[1 - \frac{I_k(L - z)}{z}\right]\right) \leq -\gamma_k.$$

Then the steady state $N = L$ of (4.2) *is asymptotically stable.*

Proof. Condition (4.5) implies that $LF(L) + u = 0$. This, together with condition (ii) shows that $N = L$ is a solution of (4.2). Let $h = h_0 = |N - L|$ and $V(N) = (N - L)^2$. Then it follows from (4.2) and (4.5)

$$D^+V(N) \le -2\sigma V(N), \qquad t \ne t_k, \tag{4.7}$$

$$V(N(t_k^+)) = \begin{cases} [\sqrt{V(N(t_k))} + I_k(L + \sqrt{V(N(t_k))}\,)]^2, & \text{if } N \ge L, \\ [\sqrt{V(N(t_k))} - I_k(L - \sqrt{V(N(t_k))}\,)]^2, & \text{if } N < L, \end{cases} \tag{4.8}$$

for all $k = 1, 2, \dots$.

Set $\lambda(t) = \sigma$, $c(V) = 2V$ and $\psi_k(V) = (\sqrt{V} + I_k(L + \sqrt{V}))^2$, $\psi_k(V) = (\sqrt{V} - I_k(L - \sqrt{V}))^2$. Then application of (3.17) yields (4.6). Thus, all conditions of Theorem 3.2 are satisfied and the conclusion of Theorem 4.1 follows.

In case the inequality (4.5) is reversed or equality holds, then we get, in view of (4.8) and Theorem 3.4, the following result on instability. □

Theorem 4.2. *Assume that conditions* (i) *and* (ii) *of Theorem* 4.1 *hold except that the equality* (4.5) *is reversed or equality holds. Suppose further that*

(iii*) *for any $z \in (0, L)$, there exists a sequence $\{\gamma_k\}$, $\gamma_k \ge 0$ for all $k = 1, 2, \dots$, and $\sum_{k=1}^{\infty} \gamma_k = \infty$, such that*

$$-\sigma(t_k - t_{k-1}) + \min\left(\ln\left[1 + \frac{I_k(L+z)}{z}\right], \ln\left[1 - \frac{I_k(L-z)}{z}\right]\right) \ge \gamma_k. \tag{4.9}$$

Then the solution $N = L$ of (4.2) *is unstable.*

In the case when $u = 0$, which means biologically, that the environment is closed, it can be seen from (4.3) that we will have, instead of

(4.4),

$$(N - C)\alpha N(C - N) = -\alpha N(N - C)^2 \tag{4.10}$$

since $L = C$ and $M = 0$.

Thus, assumption (4.5) has to be modified. One possibility is that we set a threshold $N_0 > 0$ such that

$$E_k(N) = 0 \quad \text{if } N \leq N_0, \tag{4.11}$$

for all $k = 1, 2, \ldots$, which means that harvesting is not allowed when the fish population stays below the threshold value N_0. Under assumption (4.11), we can obtain the same conclusions as in Theorem 4.1 and Theorem 4.2 if we revise (4.5) to

$$(N - C)NF(N) \leq \begin{cases} -\sigma(N - C)^2, & \text{if } N > N_0, \\ 0, & \text{if } 0 < N \leq N_0. \end{cases} \tag{4.12}$$

The details for this case are left to the interested reader.

Finally, we consider a special case for the logistic equation (4.3) with $u = 0$. We set $t_k = kd$, $d > 0$ and

$$I_k(N) = \begin{cases} \delta(N - C), & \text{if } N > C, \\ 0, & \text{otherwise,} \end{cases} \tag{4.13}$$

where $\delta \in (0, 1)$ is a constant. For this case, the condition (3.17) in Theorem 3.2 reduces to, in view of (4.12),

$$e^{\sigma d} > 1 + \delta. \tag{4.14}$$

If $e^{\sigma d} < 1 + \delta$, then it can be verified by direct computation that

$$N(t) = \begin{cases} \frac{C\beta}{(C-\beta)e^{-\alpha C(t-kd)}+\beta}, & t \in (kd, (k+1)d], \\ & k = 0, 1, 2, \ldots, \\ \beta = \frac{\alpha C\delta}{(1+\delta)(1-e^{\alpha Cd})}, & t = 0, \end{cases} \tag{4.15}$$

is a periodic solution of the impulsive system

$$\begin{cases} N' = \alpha N(C - N), & t \neq kd, \\ \Delta N = I_k(N), & t = kd. \end{cases} \tag{4.16}$$

Thus, $N = C$ is unstable which shows the sharpness of condition (3.17).

REFERENCES

1. L.H. Erbe, H.I. Freedman, X.Z. Liu and J.H. Wu, *Comparison principles for impulsive parabolic equations with applications to models of single species growth*, J. Austral. Math. Soc. **32** (1991), 382–400.

2. H.I. Freedman, *Deterministic mathematical models in population ecology*, HIFR Consulting Ltd., Edmonton, 1987.

3. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

4. V. Lakshmikantham and X.Z. Liu, *Stability criteria for impulsive differential equations in terms of two measures*, J. Math. Anal. Appl. **137** (1989), 591–604.

5. X.Z. Liu, *Stability theory for impulsive differential equations in terms of two measures*, in *Differential equations stability and control* (S. Elyadi, ed.), Marcel Dekker, New York, 1990, 61–70.

Department of Applied Mathematics, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1