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# A CURIOUS PROPERTY OF THE ELEVENTH FIBONACCI NUMBER

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**1. Introduction.** As usual we denote by $F_n$ the $n$th Fibonacci number, defined recursively by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

The decimal expansion of the reciprocal of the eleventh Fibonacci number $F_{11} = 89$ has a remarkable shape: its six leading digits are the first 6 terms of the Fibonacci sequence, viz.,

$$\frac{1}{89} = 0.011235955\dots .$$

Looking more closely, it becomes apparent that the relation goes even beyond the sixth decimal place:

$$\frac{1}{89} = \frac{0}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{5}{10^6} + \frac{8}{10^7} + \frac{13}{10^8} + \frac{21}{10^9} + \frac{34}{10^{10}} + \frac{55}{10^{11}} + \cdots$$

seems even to hold, and it is not difficult to show that, indeed,

$$\frac{1}{89} = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

This raises the question, posed to me by Ray Steiner, whether a similar phenomenon occurs for expansions in the base $y$ number system of reciprocals of Fibonacci numbers for values of $y$ other than 10. A quick inspection shows that it happens also for $y = 2, 3, 8$, viz.,

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=0}^{\infty} \frac{F_k}{2^{k+1}},$$

$$\frac{1}{F_5} = \frac{1}{5} = \sum_{k=0}^{\infty} \frac{F_k}{3^{k+1}},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=0}^{\infty} \frac{F_k}{8^{k+1}}.$$

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It is the purpose of this note to show that these are all the Fibonacci numbers with this property, and thus that $F_{11} = 89$ is the largest one. Using the fact that

$$\sum_{k=0}^{\infty} \frac{F_k}{y^{k+1}} = \frac{1}{y^2 - y - 1}$$

(the proof of which is left as an exercise for the reader), we see that the problem is equivalent to solving the diophantine equation

$$F_n = y^2 - y - 1$$

in $n,\ y \in \mathbf{Z}$, with $n \geq 0,\ y \geq 2$. The main result is therefore that the largest solution of this equation is $F_{11} = 10^2 - 10 - 1$.

Using the well-known relation $L_n^2 - 5F_n^2 = (-1)^n 4$ (where $L_n$ is the $n$th Lucas number, defined recursively by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$) we see that this main result follows at once from Theorem 1 below.

**Theorem 1.** *The diophantine equation*

$$x^2 - 5(y^2 - y - 1)^2 = \pm 4 \tag{1}$$

*in* $x, y \in \mathbf{Z}$ *with* $x \geq 0$, $y \geq 1$, *has only the solutions* $(x, y) = (1, 1)$, (3,1), (1,2), (3,2), (11,3), (123,8), (199,10).

Since $y^2 - y - 1$ is symmetric about $y = 1/2$, the restriction $y \geq 1$ implies no loss of generality.

Note that equation (1) defines two elliptic curves, so that the problem can be restated as finding the integral points on these curves. The elliptic curve given by $x^2 - 5(y^2 - y - 1)^2 = 4$ has rank 1 and the elliptic curve given by $x^2 - 5(y^2 - y - 1)^2 = -4$ has rank 2.

**2. Deriving the first Thue equation.** Equation (1) with $-4$ on the righthand side leads to

$$4 - 5(y^2 - y - 1)^2 = -x^2,$$

of which the lefthand side factors over $\mathbf{Q}(\sqrt{5})$. This field has trivial class group, a fundamental unit is $(1+\sqrt{5})/2$, and the prime 2 remains prime.

A common prime divisor of $2+(y^2-y-1)\sqrt{5}$ and $2-(y^2-y-1)\sqrt{5}$ can only be 2. The factorization thus gives

$$2+(y^2-y-1)\sqrt{5}=(-1)^a 2^b\left(\frac{1+\sqrt{5}}{2}\right)^c\alpha^2,$$

where $a,b,c\in\{0,1\}$, and $\alpha\in\mathbf{Q}(\sqrt{5})$ integral. Without loss of generality, we may assume that $y\geq 2$. This implies $a=0$. Further, since $x$ clearly has to be odd, we have $b=0$. Finally, since $N(2+(y^2-y-1)\sqrt{5})=(-1)^c N(\alpha)^2=-x^2$, we have $c=1$. Now write $\alpha=(A+B\sqrt{5})/2$, where $A,B\in\mathbf{Z}$ have the same parity. Then we obtain

$$\begin{aligned} A^2+10AB+5B^2&=16,\\ A^2+2AB+5B^2&=8(y^2-y-1). \end{aligned}\tag{2}$$

We take a linear combination of these two equations such that the righthand side becomes the square of an integer. Namely, five times the first equation plus eight times the second equation yields

$$13A^2+66AB+65B^2=16(2y-1)^2.$$

The lefthand side of this equation factors over $\mathbf{Q}(\sqrt{61})$. Note that this field has trivial class group, a fundamental unit is $(39+5\sqrt{61})/2$, the prime 2 remains prime, and the prime 13 splits: $13=-((3+\sqrt{61})/2)(3-\sqrt{61})/2$. A common prime divisor of $13A+33B+2B\sqrt{61}$ and $13A+33B-2B\sqrt{61}$ will be a divisor of 2, 13 or 61 (note that $\gcd(A,B)|16$). Hence,

$$\begin{aligned} &13A+33B+2B\sqrt{61}\\ &\quad=\pm 2^a\left(\frac{3+\sqrt{61}}{2}\right)^b\left(\frac{3-\sqrt{61}}{2}\right)^c(\sqrt{61})^d\left(\frac{39+5\sqrt{61}}{2}\right)^e\alpha^2, \end{aligned}\tag{3}$$

where $a,b,c,d,|e|\in\{0,1\}$ and $\alpha\in\mathbf{Q}(\sqrt{61})$ integral (when $e$ is odd we have the freedom to choose either $e=1$ or $e=-1$). Taking the norm we obtain

$$208(2y-1)^2=(-1)^{b+c+d+e}2^{2a}13^{b+c}61^d N(\alpha)^2,$$

so that $a=0$, $(b,c)=(0,1)$ or (1,0), $d=0$, $e=\pm 1$. We prefer to take $e=1$ if $(b,c)=(0,1)$, and $e=-1$ if $(b,c)=(1,0)$. Because solutions come in pairs $\pm(A,B)$, the $\pm$-sign in (3) is irrelevant. Note that $N(\alpha)$ is even, hence there are $u,v\in \mathbf{Z}$ such that $\alpha = u+v\sqrt{61}$. Thus we find

$$13A+33B+2B\sqrt{61} = (47\pm 6\sqrt{61})(u+v\sqrt{61})^2. \tag{4}$$

The first case, $(b,c,e)=(1,0,-1)$, corresponding to $(47-6\sqrt{61})$ in the righthand side of (4), leads to

$$\begin{aligned} 13A+33B &= 47u^2-732uv+2867v^2,\\ 2B &= -6u^2+94uv-366v^2. \end{aligned}$$

We deduce

$$\begin{aligned} 13A &= 146u^2-2283uv+8906v^2,\\ B &= -3u^2+47uv-183v^2. \end{aligned}$$

The first equation implies $v\equiv 4u \pmod{13}$, and then we obtain by the first equation that $13|A$, and by the second equation that $13|B$, which contradicts the first equation of the system (2).

The second case, $(b,c,e)=(0,1,1)$, corresponding to $(47+6\sqrt{61})$ in the right hand side of (4), leads to

$$\begin{aligned} 13A+33B &= 47u^2+732uv+2867v^2,\\ 2B &= 6u^2+94uv+366v^2. \end{aligned}$$

We deduce

$$\begin{aligned} A &= -4u^2-63uv-244v^2,\\ B &= 3u^2+47uv+183v^2. \end{aligned}$$

We substitute this into the first equation of the system (2) and obtain

$$59u^4+1856u^3v+21794u^2v^2+113216uv^3+219539v^4=-16.$$

On putting $E=v$, $F=(u+7v)/2$ (note that $u+7v$ is even) we find the Thue equation

$$E^4+2E^3F-41E^2F^2-102EF^3-59F^4=1. \tag{5}$$

In a following section we will show that this equation has only the solutions $\pm(E,F)=(1,0)$, $(1,-1)$, $(3,-2)$. They lead to, respectively,

$\pm(u,v) = (7,-1)$, $(9,-1)$, $(25,-3)$, and further to $\pm(A,B) = (1,1)$, $(1,-3)$, $(29,-3)$, corresponding to $y = 2, 3, 10$, respectively.

**3. Deriving the second Thue equation.** Equation (1) with $+4$ on the righthand side leads to

$$4 + 5(y^2 - y - 1)^2 = x^2,$$

of which the lefthand side factors over $\mathbf{Q}(\sqrt{-5})$. This field has class number 2, and the prime 2 ramifies: $(2) = \mathfrak{p}^2$, where $\mathfrak{p}$ is a prime ideal of norm 2, which is not principal. A common prime divisor of $(2 + (y^2 - y - 1)\sqrt{-5})$ and $(2 - (y^2 - y - 1)\sqrt{-5})$ can only be $\mathfrak{p}$. The factorization thus gives the ideal equation

$$(2 + (y^2 - y - 1)\sqrt{-5}) = \mathfrak{p}^a \mathfrak{a}^2,$$

where $a \in \{0, 1\}$, and $\mathfrak{a}$ is some integral ideal. Since the class number is 2, $\mathfrak{a}^2$ is principal, and we must have $a = 0$. If $\mathfrak{a}$ is principal, we can write $\mathfrak{a} = (A + B\sqrt{-5})$, and this leads to $A^2 - 5B^2 = \pm 2$, which has no solutions $\pmod 5$. Hence $\mathfrak{a}$ is not principal, and it follows that $\mathfrak{pa}$ is principal, so that we can write $\mathfrak{pa} = (A + B\sqrt{-5})$. The ideal equation thus leads to

$$\begin{aligned} A^2 - 5B^2 &= \pm 4, \\ AB &= \pm(y^2 - y - 1). \end{aligned} \tag{6}$$

Again, we take a linear combination of these two equations such that the righthand side becomes the square of an integer. Namely, 5 times the first equation plus 16 times the second equation yields

$$5A^2 + 16AB - 25B^2 = \pm 4(2y - 1)^2.$$

The lefthand side of this equation factors over $\mathbf{Q}(\sqrt{21})$. Note that this field has trivial class group, a fundamental unit is $(5+\sqrt{21})/2$, the prime 2 remains prime, the prime 3 ramifies, namely $3 = ((3 + \sqrt{21})/2)^2((5 + \sqrt{21})/2)^{-1}$, the prime 5 splits: $5 = -((1 + \sqrt{21})/2)(1 - \sqrt{21})/2$, and the prime 7 ramifies: $7 = ((7 + \sqrt{21})/2)^2((5 + \sqrt{21})/2)^{-1}$. A common prime divisor of the factors of $5A^2 + 16AB - 25B^2$, which are $(5A + 8B + 3B\sqrt{21})/2$ and $(5A + 8B - 3B\sqrt{21})/2$ will be a divisor of

3, 5 or 7 (notice that the second equation of (6) implies that both $A$ and $B$ are odd). Hence,

$$\begin{aligned}\frac{5A+8B+3B\sqrt{21}}{2} &= \pm\left(\frac{3+\sqrt{21}}{2}\right)^a\left(\frac{1+\sqrt{21}}{2}\right)^b\left(\frac{1-\sqrt{21}}{2}\right)^c \\ &\quad\cdot\left(\frac{7+\sqrt{21}}{2}\right)^d\left(\frac{5+\sqrt{21}}{2}\right)^e\alpha^2,\end{aligned} \tag{7}$$

where $a,b,c,d,|e|\in\{0,1\}$, and $\alpha\in\mathbf{Q}(\sqrt{21})$ integral (when $e$ is odd we have the freedom to choose either $e=1$ or $e=-1$). Taking the norm we obtain

$$\pm 5(2y-1)^2 = (-1)^{a+b+c}3^a5^{b+c}7^dN(\alpha)^2,$$

so that $a=0$, $(b,c)=(0,1)$ or (1,0), $d=0$. If $e$ is odd, we prefer $e=1$ if $(b,c)=(0,1)$, and $e=-1$ if $(b,c)=(1,0)$. Because solutions come in pairs $\pm(A,B)$ the $\pm$ sign in (7) is irrelevant. There are $u,v\in\mathbf{Z}$ of equal parity such that $\alpha=(u+v\sqrt{21})/2$. Hence, equation (7) leads to

$$\frac{5A+8B+3B\sqrt{21}}{2} = \left\{\begin{matrix}(1\pm\sqrt{21})/2\\ -4\pm\sqrt{21}\end{matrix}\right\}\left(\frac{u+v\sqrt{21}}{2}\right)^2. \tag{8}$$

In the case $(b,c,e)=(0,1,0)$, corresponding to $(1-\sqrt{21})/2$ in the righthand side of (8), we find

$$\begin{aligned}20A+32B &= u^2-42uv+21v^2,\\ 12B &= -u^2+2uv-21v^2.\end{aligned}$$

We deduce

$$\begin{aligned}60A &= 11u^2-142uv+231v^2,\\ 12B &= -u^2+2uv-21v^2.\end{aligned}$$

The first equation implies $u\equiv v \pmod 5$. Then the first equation again implies $5|A$, and the second equation implies $5|B$, contradicting the first equation of (6).

In the case $(b,c,e)=(1,0,0)$, corresponding to $(1+\sqrt{21})/2$ in the righthand side of (8), we find

$$\begin{aligned}20A+32B &= u^2+42uv+21v^2,\\ 12B &= u^2+2uv+21v^2.\end{aligned}$$

We deduce

$$12A = -u^2 + 22uv - 21v^2,$$
$$12B = u^2 + 2uv + 21v^2.$$

Since $u$ and $v$ have the same parity, we may write $v = u + 2w$. Substituting the above expressions for $A$ and $B$ into the first equation of (6), we obtain

$$45u^4 + 330u^3w + 895u^2w^2 + 1050uw^3 + 441w^4 = \pm 36,$$

which clearly is impossible $\pmod 2$ if at least one of $u$ and $w$ is odd, and $(\bmod 16)$ if both $u, w$ are even.

In the case $(b, c, e) = (0, 1, 1)$, corresponding to $-4 - \sqrt{21}$ in the righthand side of (8), we find

$$10A + 16B = -4u^2 - 42uv - 84v^2,$$
$$6B = -u^2 - 8uv - 21v^2.$$

We deduce

$$15A = -2u^2 - 31uv - 42v^2,$$
$$6B = -u^2 - 8uv - 21v^2.$$

The first equation implies $u \equiv v \pmod 5$. Then the first equation again implies $5|A$, and the second equation implies $5|B$, contradicting the first equation of (6).

In the case $(b, c, e) = (1, 0, -1)$, corresponding to $-4 + \sqrt{21}$ in the righthand side of (8), we find

$$10A + 16B = -4u^2 + 42uv - 84v^2,$$
$$6B = u^2 - 8uv + 21v^2.$$

We deduce

$$3A = -2u^2 + 19uv - 42v^2,$$
$$6B = u^2 - 8uv + 21v^2.$$

Substituting this into the first equation of system (6), we obtain the Thue equation

$$11u^4 - 224u^3v + 1586u^2v^2 - 4704uv^3 + 4851v^4 = \pm 144.$$

On putting $E = v$, $F = (u - 5v)/2$ (note that $u - 5v$ is even), we obtain

$$9E^4 + 18E^3F + 31E^2F^2 + 2EF^3 - 11F^4 = \pm 9. \tag{9}$$

Reasoning $\pmod 4$ we see that the righthand side cannot be $-9$. In a following section we will show that this Thue equation has only the solutions $\pm(E, F) = (1, 0)$, $(1, -1)$, (1,2). They lead, respectively, to $\pm(u, v) = (5, 1)$, (3,1), (9,1) and further to $\pm(A, B) = (1, 1)$, $(1, -1)$, $(11, -5)$. Finally, we find, respectively, $y = 1, 2$, and 8, and Theorem 1 follows.

**4. Solving the first Thue equation.** In this section we prove the following result. This is exactly the announced result on equation (5), with $X = E$, $Y = F$.

**Theorem 2.** *The Thue equation*

$$X^4 + 2X^3Y - 41X^2Y^2 - 102XY^3 - 59Y^4 = 1 \tag{10}$$

*has only the solutions* $\pm(X, Y) = (1, 0)$, $(1, -1)$, $(3, -2)$.

*Proof.* Let $K = \mathbf{Q}(\theta)$ with $\theta = \sqrt{(5 + \sqrt{5})/2}$. This is a totally real quadratic field extension of $\mathbf{Q}(\sqrt{5})$. A basis for the ring of integers is $\{1, \theta, \theta^2, \theta^3\}$, so the field discriminant is 2000. A set of fundamental units is

$$\varepsilon = 3 - \theta^2 = \frac{1 \pm \sqrt{5}}{2}, \qquad \eta = 1 + \theta, \qquad \zeta = 2 + \theta.$$

We put

$$\xi = -8 + 2\theta + 3\theta^2.$$

Note that it satisfies

$$\xi^4 + 2\xi^3 - 41\xi^2 - 102\xi - 59 = 0,$$

so that the Thue equation (13) is equivalent to

$$X - Y\xi = \pm\varepsilon^a\eta^b\zeta^c \tag{11}$$

for unknowns $a, b, c \in \mathbf{Z}$.

We number the conjugates as follows:

$$\theta_1 = \sqrt{\frac{5+\sqrt{5}}{2}} = 1.9021130325\dots,$$
$$\theta_2 = -\sqrt{\frac{5+\sqrt{5}}{2}} = -1.9021130325\dots,$$
$$\theta_3 = \sqrt{\frac{5-\sqrt{5}}{2}} = 1.1755705045\dots,$$
$$\theta_4 = -\sqrt{\frac{5-\sqrt{5}}{2}} = -1.1755705045\dots,$$

and it then follows that

$$\xi_1 = \frac{-1+3\sqrt{5}}{2} + \sqrt{10+2\sqrt{5}} = 6.6583280314\dots,$$
$$\xi_2 = \frac{-1+3\sqrt{5}}{2} - \sqrt{10+2\sqrt{5}} = -0.9501240989\dots,$$
$$\xi_3 = \frac{-1-3\sqrt{5}}{2} + \sqrt{10-2\sqrt{5}} = -1.5029609570\dots,$$
$$\xi_4 = \frac{-1-3\sqrt{5}}{2} - \sqrt{10-2\sqrt{5}} = -6.2052429754\dots,$$

so that $\xi_1, \xi_2$ satisfy

$$x^2 + (1-3\sqrt{5})x + \frac{3-7\sqrt{5}}{2} = 0,$$

and $\xi_3, \xi_4$ satisfy the conjugate equation.

We treat equation (11) as in [**2**]. Let $i_0 \in \{1,2,3,4\}$ be the index, depending on $X, Y$, for which

$$|X - Y\xi_{i_0}| = \min_{i \in \{1,2,3,4\}} |X - Y\xi_i|.$$

When $i_0 = 1, 2$, then we take $j = 3$, $k = 4$, and when $i_0 = 3, 4$, then we take $j = 1$, $k = 2$. The Siegel identity

$$(\xi_j - \xi_k)(X - Y\xi_{i_0}) + (\xi_k - \xi_{i_0})(X - Y\xi_j) + (\xi_{i_0} - \xi_j)(X - Y\xi_k) = 0$$

now leads to

$$\frac{\xi_{i_0}-\xi_k}{\xi_{i_0}-\xi_j}\left(\frac{\varepsilon_j}{\varepsilon_k}\right)^a\left(\frac{\eta_j}{\eta_k}\right)^b\left(\frac{\zeta_j}{\zeta_k}\right)^c-1=\frac{\xi_j-\xi_k}{\xi_{i_0}-\xi_j}\,\frac{X-Y\xi_{i_0}}{X-Y\xi_k}.$$

Now notice that, due to the choice of $j,k$ and the fact that $\varepsilon$ is a special unit, namely, satisfying $\varepsilon_j=\varepsilon_k$, in the lefthand side the term with the $a$ in the exponent disappears. Hence, the number of variables is reduced by 1.

We introduce the linear form in logarithms

$$\Lambda=\log\left|\frac{\xi_{i_0}-\xi_k}{\xi_{i_0}-\xi_j}\right|+b\log\left|\frac{\eta_j}{\eta_k}\right|+c\log\left|\frac{\zeta_j}{\zeta_k}\right|,$$

and we write

$$A=\max\{|a|,|b|,|c|\},\qquad B=\max\{|b|,|c|\}.$$

By the methods of [**2**] (we omit details) we found, subject to the condition $|Y|\geq 4$, that

$$|\Lambda|<8.32280\times 10^5\exp(-0.869677A). \tag{12}$$

On the other hand, $\Lambda$ is a linear form in logarithms of algebraic numbers, which is nonzero. Hence, transcendence theory provides us with a lower bound for its absolute value, namely, we have (see [**1**], again we omit details), with the condition that $B\geq 3$, that

$$|\Lambda|>\exp(-8.03945\times 10^{15}\log B).$$

Combining upper and lower bounds, and noting that $B\leq A$, we find

$$B<3.74049\times 10^{17}.$$

We have four inhomogeneous linear forms $\Lambda$ (namely, for $i_0=1,2,3,4$) but they have only two different homogeneous parts

$$b\log\left|\frac{\eta_j}{\eta_k}\right|+c\log\left|\frac{\zeta_j}{\zeta_k}\right|.$$

For a given large positive number $C$ we define for each of these two homogeneous parts the lattice

$$\Gamma = \{\mathcal{A}x \mid x \in \mathbf{Z}^2\} \subset \mathbf{Z}^2$$

by the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ \left[C \log \left|\frac{\eta_j}{\eta_k}\right|\right] & \left[C \log \left|\frac{\zeta_j}{\zeta_k}\right|\right] \end{pmatrix},$$

where $[\cdot]$ stands for rounding towards zero to an integer. Further, for each of the four inhomogeneous parts we define the point

$$y = \begin{pmatrix} 0 \\ -[C \log |(\xi_{i_0} - \xi_k)|/|(\xi_{i_0} - \xi_j)|] \end{pmatrix} \in \mathbf{Z}^2.$$

At first we take $C = 10^{40}$, which is somewhat larger than the square of the upper bound $3.74049 \times 10^{17}$ for $B$. For each of the two lattices, we computed a reduced basis, which enables us to compute a lower bound for the distance from each of the points $y$ to the corresponding lattice, i.e.,

$$l(\Gamma, y) = \min_{x \in \mathbf{Z}^2} |y - \mathcal{A}x|.$$

In all four cases we found

$$l(\Gamma, y) > 5.39877 \times 10^{18}.$$

Define $\lambda \in \mathbf{Z}$ by

$$\mathcal{A} \begin{pmatrix} b \\ c \end{pmatrix} - y = \begin{pmatrix} b \\ \lambda \end{pmatrix}.$$

Then

$$\begin{aligned} |\lambda| &\geq \sqrt{l(\Gamma, y)^2 - b^2} > \sqrt{(5.39877 \times 10^{18})^2 - (3.74049 \times 10^{17})^2} \\ &> 5.38579 \times 10^{18}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \lambda &= \left[10^{40} \log \left|\frac{\xi_{i_0} - \xi_k}{\xi_{i_0} - \xi_j}\right|\right] + b \left[10^{40} \log \left|\frac{\eta_j}{\eta_k}\right|\right] \\ &\quad + c \left[10^{40} \log \left|\frac{\zeta_j}{\zeta_k}\right|\right] \\ &\approx 10^{40} \Lambda. \end{aligned}$$

To be precise, by the definition of $[\cdot]$,

$$|\lambda - 10^{40}\Lambda| \leq 1 + |b| + |c| < 1 + 2 \times 3.74049 \times 10^{17} < 7.48099 \times 10^{17},$$

and thus we find

$$\begin{aligned} |\Lambda| &\geq 10^{-40}(|\lambda| - |\lambda - 10^{40}\Lambda|) \\ &> 10^{-40}(5.38579 \times 10^{18} - 7.48099 \times 10^{17}) \\ &> 4.63769 \times 10^{-22}. \end{aligned}$$

We combine this with (12) and obtain $B \leq A \leq 72$.

We repeat the reduction procedure with $C = 10^8$. We found in all cases $l(\Gamma, y) > 5105.14$, leading as above to

$$|\Lambda| > 4.95963 \times 10^{-5},$$

where we used $B \leq 72$. Combining this with (12) leads to $A \leq 27$.

Notice that the above reduction is valid only under the conditions $B \geq 3$ and $|Y| \geq 4$. The solutions with $A \leq 27$, $B < 3$ or $|Y| < 4$ are very easy to find. In fact, from the Siegel identity we easily see that we can find $a$ from $b, c$, namely by noting that $\varepsilon_3 = \varepsilon_4 = \varepsilon_1^{-1}$ we find

$$\varepsilon_1^{2a} = \frac{(\xi_1 - \xi_4)\eta_3^b\,\zeta_3^c + (\xi_3 - \xi_1)\eta_4^b\,\zeta_4^c}{(\xi_3 - \xi_4)\eta_1^b\,\xi_1^c}.$$

We checked this for all $|b|$, $|c| \leq 30$, and found only the solutions $(a, b, c) = (0, 0, 0)$, $(-3, -2, 2)$, $(-9, -4, 2)$, leading respectively to $\pm(X, Y) = (1, 0), (1, -1), (3, -2)$.

This completes the proof of Theorem 2. □

**5. Solving the second Thue equation.** In this section we prove the following result. Notice that the announced result on equation (9) is equivalent to this, because equation (9) is easily seen to imply either $3 \mid F$, leading to the first Thue equation below with $X = E$, $Y = F/3$, or $3|E + F$, leading to the second Thue equation below with $X = E$, $Y = (E + F)/3$.

**Theorem 3.** (i) *The Thue equation*

$$X^4 + 6X^3Y + 31X^2Y^2 + 6XY^3 - 99Y^4 = 1 \tag{13}$$

*has only the solutions* $\pm(X,Y)=(1,0)$.

(ii) *The Thue equation*

$$X^4+2X^3Y-41X^2Y^2+138XY^3-99Y^4=1 \tag{14}$$

*has only the solutions* $\pm(X,Y)=(1,0),(1,1)$.

*Proof.* Let $K=\mathbf{Q}(\theta)$ with $\theta=\sqrt{(5+3\sqrt{5})/2}$. This is a nonreal quadratic field extension of $\mathbf{Q}(\sqrt{5})$. A basis for the ring of integers is $\{1,\theta,(-1+\theta^2)/3,(-\theta+\theta^3)/3\}$, so the field discriminant is $-2000$. A set of fundamental units is

$$\varepsilon=\frac{1}{3}(-1+\theta^2)=\frac{1\pm\sqrt{5}}{2},\qquad \eta=1-\theta+\frac{1}{3}(-1+\theta^2).$$

We put

$$\xi'=-2+4\theta+\frac{1}{3}(-1+\theta^2)-2\frac{1}{3}(-\theta+\theta^3).$$

Note that it satisfies

$$\xi'^4+6\xi'^3+31\xi'^2+6\xi'-99=0,$$

so that the Thue equation (13) is equivalent to

$$X-Y\xi'=\pm\varepsilon^a\eta^b$$

for unknowns $a,b\in\mathbf{Z}$.

Next, we put

$$\xi''=2+2\theta-\theta^2.$$

Note that it satisfies

$$\xi''^4+2\xi''^3-41\xi''^2+138\xi''-99=0,$$

so that the Thue equation (14) is equivalent to

$$X-Y\xi''=\pm\varepsilon^a\eta^b$$

for unknowns $a, b \in \mathbf{Z}$.

Further, note that

$$\xi'' = \frac{3\xi'}{3+\xi'}, \qquad \xi' = \frac{3\xi''}{3-\xi''}.$$

We number the conjugates as follows:

$$\begin{aligned}
\theta_1 &= \sqrt{\frac{5+3\sqrt{5}}{2}} = 2.4195251530\ldots\,, \\
\theta_2 &= -\sqrt{\frac{5+3\sqrt{5}}{2}} = -2.4195251530\ldots\,, \\
\theta_3 &= \sqrt{\frac{5-3\sqrt{5}}{2}} = 0.9241763718\ldots i, \\
\theta_4 &= -\sqrt{\frac{5-3\sqrt{5}}{2}} = -0.9241763718\ldots i,
\end{aligned}$$

and it then follows that

$$\begin{aligned}
\xi_1' &= \frac{-3+\sqrt{5}}{2} + \sqrt{-10+6\sqrt{5}} = 1.4663867324\ldots\,, \\
\xi_2' &= \frac{-3+\sqrt{5}}{2} - \sqrt{-10+6\sqrt{5}} = -2.2303187549\ldots\,, \\
\xi_3' &= \frac{-3-\sqrt{5}}{2} + \sqrt{-10-6\sqrt{5}} \\
&= -2.6180339887\ldots + 4.8390503061\ldots i, \\
\xi_4' &= \frac{-3-\sqrt{5}}{2} - \sqrt{-10-6\sqrt{5}} \\
&= -2.6180339887\ldots - 4.8390503061\ldots i,
\end{aligned}$$

so that $\xi_1', \xi_2'$ satisfy

$$x^2 + (3-\sqrt{5})x + \frac{27-15\sqrt{5}}{2} = 0,$$

and $\xi_3', \xi_4'$ satisfy the conjugate equation, and

$$\begin{aligned}
\xi_1'' &= \frac{-1-3\sqrt{5}}{2} + \sqrt{10+6\sqrt{5}} = 0.9849483398\dots\,,\\
\xi_2'' &= \frac{-1-3\sqrt{5}}{2} - \sqrt{10+6\sqrt{5}} = -8.6931522723\dots\,,\\
\xi_3'' &= \frac{-1+3\sqrt{5}}{2} + \sqrt{10-6\sqrt{5}}\\
&= 2.8541019662\dots + 1.8483527436\dots i,\\
\xi_4'' &= \frac{-1+3\sqrt{5}}{2} - \sqrt{10-6\sqrt{5}}\\
&= 2.8541019662\dots - 1.8483527436\dots i,
\end{aligned}$$

so that $\xi_1'', \xi_2''$ satisfy

$$x^2 + (1+3\sqrt{5})x + \frac{3-9\sqrt{5}}{2} = 0,$$

and $\xi_3'', \xi_4''$ satisfy the conjugate equation.

In the sequel, let either $\xi = \xi'$ or $\xi = \xi''$. Again, we follow the procedure outlined in [**2**]. Since we now have only two real conjugates, and two nonreal ones, we take $i_0 \in \{1,2\}$ to be the index, depending on $X, Y$, for which

$$|X - Y\xi_{i_0}| = \min_{i\in\{1,2\}} |X - Y\xi_i|.$$

The Siegel identity now yields

$$\frac{\xi_{i_0}-\xi_4}{\xi_{i_0}-\xi_3}\left(\frac{\varepsilon_3}{\varepsilon_4}\right)^a \left(\frac{\eta_3}{\eta_4}\right)^b - 1 = \frac{\xi_3-\xi_4}{\xi_{i_0}-\xi_3}\,\frac{X-Y\xi_{i_0}}{X-Y\xi_4}.$$

Again we notice that $\varepsilon_3 = \varepsilon_4$, so that in the lefthand side the term with the $a$ in the exponent disappears.

We fix the branches of the complex logarithm Log and of the argument function Arg such that $-\pi < \operatorname{Arg} = \operatorname{Im}\operatorname{Log} \le \pi$. Note that $(\xi_{i_0}-\xi_4)/(\xi_{i_0}-\xi_3)$ and $\eta_3/\eta_4$ are quotients of complex conjugated numbers, hence for these numbers $\operatorname{Log} = i \operatorname{Arg}$.

We introduce the linear form in logarithms

$$\Lambda = \mathrm{Log}\,\frac{\xi_{i_0}-\xi_4}{\xi_{i_0}-\xi_3} + b\mathrm{Log}\,\frac{\eta_3}{\eta_4} + k2\pi i,$$

with $k \in \mathbf{Z}$ such that $|\Lambda| \le \pi$. Note that

$$\frac{1}{i}\Lambda = \mathrm{Arg}\,\frac{\xi_{i_0}-\xi_4}{\xi_{i_0}-\xi_3} + b\mathrm{Arg}\,\frac{\eta_3}{\eta_4} + k2\pi \in \mathbf{R}.$$

Put

$$A = \max\{|a|,|b|\}, \qquad B = \max\{|b|,|k|\}.$$

By the methods of [**2**] (again we omit details), we found, for both the cases $\xi = \xi', \xi''$, subject to the condition $|Y| \ge 3$, that

$$|\Lambda| < 2797.35\exp(-1.92485A). \tag{15}$$

On the other hand, $\Lambda$ is a linear form in logarithms of algebraic numbers, which is nonzero. Hence, transcendence theory provides us with a lower bound for its absolute value, namely we have (see [**1**], again we omit details), with the condition that $B \ge 3$, that

$$|\Lambda| > \exp(-2.53408 \times 10^{16}\log B).$$

Note that, under the condition $B \ge 3$, we have

$$\begin{aligned}|k| &\le \frac{1}{2\pi}\left(|\Lambda| + \left|\mathrm{Arg}\,\frac{\xi_{i_0}-\xi_4}{\xi_{i_0}-\xi_3}\right| + |b|\left|\mathrm{Arg}\,\frac{\eta_3}{\eta_4}\right|\right)\\ &\le \frac{1}{2}|b| + 1 < |b|,\end{aligned}$$

so that $B = |b|$, and $B \le A$. It thus follows by combining upper and lower bound, that

$$B < 5.37474 \times 10^{17}.$$

We have four inhomogeneous linear forms $(1/i)\Lambda$ (namely for $i_0 = 1, 2$ and $\xi = \xi', \xi''$), but they all have the same homogeneous part

$$b\mathrm{Arg}\,\frac{\eta_3}{\eta_4} + k2\pi.$$

For a given large positive number $C$ we define the lattice

$$\Gamma = \{\mathcal{A}x \mid x \in \mathbf{Z}^2\} \subset \mathbf{Z}^2$$

by the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ [C\operatorname{Arg}(\eta_3/\eta_4)] & [C2\pi] \end{pmatrix}.$$

For each of the four inhomogeneous parts, we define the point

$$y = \begin{pmatrix} 0 \\ -[C\operatorname{Arg}(\xi_{i_0} - \xi_4)/(\xi_{i_0} - \xi_3)] \end{pmatrix} \in \mathbf{Z}^2.$$

At first, we take $C = 10^{40}$, and we compute a reduced basis of the lattice. From this we found in all four cases,

$$l(\Gamma, y) > 8.49056 \times 10^{19}.$$

Define $\lambda \in \mathbf{Z}$ by

$$\mathcal{A}\begin{pmatrix} b \\ k \end{pmatrix} - y = \begin{pmatrix} b \\ \lambda \end{pmatrix}.$$

Then

$$\begin{aligned} |\lambda| &\geq \sqrt{l(\Gamma, y)^2 - b^2} \\ &> \sqrt{(8.49056 \times 10^{19})^2 - (5.37474 \times 10^{17})^2} \\ &> 8.49038 \times 10^{19}. \end{aligned}$$

On the other hand,

$$\left|\lambda - 10^{40}\frac{1}{i}\Lambda\right| \leq 1 + |b| + |k| < 2 + \frac{3}{2}|b| < 8.06212 \times 10^{17},$$

so that

$$\begin{aligned} |\Lambda| &\geq 10^{-40}\left(|\lambda| - \left|\lambda - 10^{40}\frac{1}{i}\Lambda\right|\right) \\ &> 10^{-40}(8.49038 \times 10^{19} - 8.06212 \times 10^{17}) \\ &> 8.40976 \times 10^{-21}. \end{aligned}$$

We combine this with (15), and obtain $B \leq A \leq 28$. This bound is already so good that we do not perform a second reduction.

Notice that the above reduction is valid only under the conditions $B \geq 3$ and $|Y| \geq 3$. The solutions with $A \leq 28$, $B < 3$ or $|Y| < 3$ are very easy to find. In fact, from the Siegel identity we easily see that we can find $a$ from $b$, namely, by noting that $\varepsilon_3 = \varepsilon_4 = \varepsilon_1^{-1}$ we find

$$\varepsilon_1^{2a} = \frac{(\xi_1 - \xi_4)\eta_3^b + (\xi_3 - \xi_1)\eta_4^b}{(\xi_3 - \xi_4)\eta_1^b}.$$

We checked this for all $|b| \leq 30$, and found only the following solutions:

i) in the case $\xi = \xi'$ only $(a, b) = (0, 0)$, leading to $\pm(X, Y) = (1, 0)$,

ii) in the case $\xi = \xi''$ only $(a, b) = (0, 0)$, $(-2, 2)$, leading respectively to $\pm(X, Y) = (1, 0)$, (1,1).

This completes the proof of Theorem 3. □

## REFERENCES

**1.** A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. für die reine und angewandte Mathematik **442** (1993), 19–62.

**2.** N. Tzanakis and B.M.M. de Weger, *On the practical solution of the Thue equation*, J. Number Theory **31** (1989), 99–132.

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