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PIECEWISE MONOTONIC DOUBLING MEASURES

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1. Introduction. A positive Borel measure μ, defined on $\mathbf{R}$ or on some interval, is a doubling measure if there exists a constant C such that for each interval I, $\mu(2I) \leq C\mu(I)$, where $2I$ is the interval with the same center as I and twice the length. Somewhat surprisingly, doubling measures are not necessarily absolutely continuous–Beurling and Ahlfors [**2**] constructed a singular doubling measure.

If a doubling measure is absolutely continuous, its Radon-Nikodym derivative is called a doubling weight. An important class of doubling weights is (A_∞). For $p > 1$ a nonnegative function w is an (A_p) weight if

$$\sup_I \left(\frac{1}{|I|}\int_I w\,dx\right)\left(\frac{1}{|I|}\int_I w^{1-p'}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I and p' is the conjugate exponent of p. If $Mw(t) \leq Cw(t)$ almost everywhere, where Mw is the Hardy-Littlewood maximal function of w, then w is an (A_1) weight. The union of the (A_p) classes is denoted by (A_∞). Not every doubling weight is an (A_∞) weight; C. Fefferman and Muckenhoupt [**4**] and more recently Wik [**14**] have given counter-examples.

In this paper we study those doubling measures which are piecewise monotonic. A measure μ is *monotonic* if the measure of a right translate of a set is always larger (or smaller) than the measure of the set itself, and μ is *piecewise monotonic* if its support is the union of a (finite) number of intervals on which μ is monotonic. We show that piecewise monotonic doubling measures are absolutely continuous and their Radon-Nikodym derivatives are (A_∞) weights.

The paper is organized as follows. In Section 2 we determine the singular parts of monotonic measures and show that piecewise mono-

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tonic doubling measures are absolutely continuous. In Sections 3 and 4 we characterize monotonic doubling weights defined on $[0,\infty)$. In Section 3 we give several characterizations of the increasing and decreasing doubling weights and construct two examples which highlight the differences between these two classes. Using these characterizations, in Section 4 we show that monotonic doubling weights are (A_∞) weights. Further, we show that every decreasing doubling weight is an (A_1) weight, and give necessary and sufficient conditions for an increasing doubling weight to be in (A_p) for $p \geq 1$.

In Section 5 we consider the general case of a function on **R** which is piecewise monotonic on a finite number of intervals, and give necessary and sufficient conditions for such a function to be a doubling weight. We also briefly consider the case of a function which is piecewise monotonic on an infinite number of intervals.

The last three sections are applications. In Section 6 we apply our results to earlier work on monotonic (A_∞) weights. We give a new and simpler proof of a characterization of monotonic (A_p) weights found independently by Guseinov [**6**] and by Benedetto, Heinig and Johnson [**1**], and then derive from it a more elegant characterization of increasing (A_p) weights. We also show a connection between monotonic doubling weights and arbitrary doubling measures by generalizing a theorem of Johnson and Neugebauer [**8**] on the integrals of (A_p) weights.

In Section 7 we determine the action of the Hardy-Littlewood maximal operator on monotonic doubling weights. As applications we extend an example given in Section 3 and give another characterization of decreasing doubling weights.

Finally, in Section 8 we characterize the multipliers of monotonic doubling weights. As a corollary to these characterizations we show that multipliers of the monotonic doubling weights are precisely those multipliers of all the (A_p) weights which are themselves monotonic. This material builds upon two theorems of Johnson and Neugebauer [**7**] on multipliers, for one of which we give a new proof.

Other applications of our results might be gotten by combining them with the work of Wik [**14**], who showed that the increasing rearrangement of an (A_∞) weight is again an (A_∞) weight.

Throughout this paper all notation is standard or will be defined as needed. All measures are assumed to be positive Borel measures which are finite on compact sets. All functions are assumed to be nonnegative. For an interval I and a function w, let $|I|$ be the Lebesgue measure of I and define $w(I) = \int_I w\,dx$ and $I(w) = w(I)/|I|$. The letter C denotes a positive constant whose value may change at each appearance. Given $p > 1$, $p' = p/(p-1)$ is the conjugate exponent of p.

2. Singular parts of monotonic measures. We begin with a precise definition of monotonic measures.

Definition 2.1. A measure μ, defined on $\mathbf{R}$ or some interval, is *decreasing* if, given two intervals I and J, J a right translate of I, then $\mu(I) \geq \mu(J)$. If $\mu(I) \leq \mu(J)$, then μ is *increasing*. A measure is piecewise monotonic if its support is the union of intervals on which the measure is monotonic.

This definition generalizes in a natural way the idea of measures of the form $w\,dx$, where w is a piecewise monotonic function. It turns out, however, that almost nothing is gained from this generalization.

Lemma 2.2. *Every monotonic measure on* $\mathbf{R}$ *is absolutely continuous and its Radon-Nikodym derivative is also monotonic.*

Proof. We will prove this for decreasing measures; the proof for increasing measures is essentially the same.

Let μ be a decreasing measure. We will show that μ is dominated by some absolutely continuous measure ν. Since Borel measures are regular, and since open sets in $\mathbf{R}$ are unions of disjoint intervals, it will suffice to find ν such that $\mu(I) \leq \nu(I)$ for all intervals I.

Let $\nu = \chi_{(0,1)} * \mu$. Then ν is absolutely continuous, and for any interval I

$$\begin{aligned}\nu(I) &= \int_{\mathbf{R}} \int_{\mathbf{R}} \chi_I(x+y)\chi_{(0,1)}(x)\,d\mu(y)\,dx \\ &= \int_0^1 \mu(I-x)\,dx \geq \mu(I),\end{aligned}$$

the last inequality following since μ is decreasing.

Thus $d\mu = w\,dx$ for some function w. By the Lebesgue differentiation theorem,

$$w(x) = \lim_{t \to 0} \frac{\mu([x, x+t])}{t}$$

for almost every x. Hence, since μ is decreasing, for almost every $y > x$, $w(x) \geq w(y)$. Finally, to make w a decreasing function, we normalize it to be left continuous. $\square$

If μ is a monotonic measure on a subinterval of $\mathbf{R}$, then this proof goes through (possibly replacing $\chi_{(0,1)}$ by $(1/\delta)\chi_{(0,\delta)}$ with δ small), but the conclusion is that if μ is decreasing and the subinterval has a finite left endpoint then μ is absolutely continuous with the possible exception of an atom at the left endpoint. Similarly, an increasing measure is absolutely continuous with the possible exception of an atom at a finite right endpoint. However, it is well known that doubling measures do not have atoms. (See Garcia-Cuerva and Rubio de Francia [**5**, p. 403] or Sawyer [**12**]. Stronger results are proved by Wu [**15**].) Therefore we have proved the following theorem.

Theorem 2.3. *Every piecewise monotonic doubling measure is absolutely continuous.*

3. Characterizations of monotonic doubling weights.

Because of Theorem 2.3, we need only characterize piecewise monotonic doubling weights. In this section and in Section 4 we consider monotonic doubling weights on $[0, \infty)$; in Section 5 we will derive the general case of piecewise monotonic weights on $\mathbf{R}$ from this special case.

We first give a necessary and sufficient condition for a measure to be doubling, one which is often easier to apply to monotonic measures than the definition. (This result is sometimes used to define doubling measures–for example, see Wu [**15**].) Its proof is straightforward.

Lemma 3.1. *A measure μ is a doubling measure if and only if there exists a constant C such that, given two adjacent intervals I and J of equal length, $\mu(I) \leq C\mu(J)$.*

For monotonic doubling weights on $[0, \infty)$, this condition may be further simplified: our first result shows that a monotonic function is a doubling weight if and only if it satisfies Lemma 3.1 for intervals I and J where I is adjacent to the origin.

Lemma 3.2. *Let w be a locally integrable, monotonic function on $[0, \infty)$ and define*

$$W(t) = \int_0^t w\,dx.$$

If w is decreasing, then w is a doubling weight if and only if there exists a constant γ, $1 < \gamma < 2$, such that $\gamma W(t) \leq W(2t)$ for all t.

If w is increasing, then w is a doubling weight if and only if there exists a constant δ, $0 < \delta < 1/2$, such that $\delta W(2t) \leq W(t)$ for all t.

Proof. We prove this only for w decreasing; the proof for w increasing is essentially the same.

If w is a doubling weight, then by Lemma 3.1 there exists a constant C such that for all t, $W(t) - W(0) \leq C(W(2t) - W(t))$. Rearranging this we get $(1 + 1/C)W(t) \leq W(2t)$, which is the desired inequality with $\gamma = 1 + 1/C$.

Conversely, suppose that such a γ exists. Let $I = [x, y]$, $J = [y, z]$. Since w is decreasing, to apply Lemma 3.1 we only need to show that there is a constant C such that $w(I) \leq Cw(J)$. There are three cases which correspond to the relative distance of I from the origin.

Case 1. $x = 0$. If we reverse the above calculations then we see that $w(I) \leq Cw(J)$, where $C = (\gamma - 1)^{-1}$.

Case 2. $x \leq |I|$. Define the intervals $I' = [0, y]$ and $J' = [y, z + x]$. Then by Case 1 there is a constant C such that $w(I') \leq Cw(J')$. Since $x \leq |I|$, $|J'| \leq 2|J|$. Because w is decreasing, this implies that $w(J') \leq 2w(J)$. Since $I \subset I'$, $w(I) \leq w(I')$. Together these give us the desired inequality with constant $2C$.

Case 3. $x > |I|$. For an integer $n > 0$, let I' be the interval with right endpoint y and length $n|I|$; let J' be the interval with left endpoint y and length $n|J|$. Fix n so that I' and J' satisfy the conditions of Case 2. Then, since w is decreasing, $nw(I) \leq w(I') \leq 2Cw(J') \leq 2Cnw(J)$. $\square$

In the hypotheses of Lemma 3.2 the upper bounds on the constants

γ and δ are natural, since for any decreasing function $2W(t) \geq W(2t)$, and for any increasing function $W(2t) \geq 2W(t)$.

An immediate consequence of Lemma 3.2 is a very elegant characterization of increasing doubling weights.

Theorem 3.3. *An increasing function w on $[0, \infty)$ is a doubling weight if and only if there exists a constant β, $0 < \beta < 1$, such that $\beta w(2t) \leq w(t)$ for all t.*

Proof. If w is a doubling weight, then by Lemma 3.1 there exists a constant $C > 1$ such that for all t

$$\begin{aligned} tw(2t) &\leq \int_{2t}^{3t} w\,dx \leq C \int_{t}^{2t} w\,dx \\ &\leq C^2 \int_0^t w\,dx \leq C^2 tw(t). \end{aligned}$$

This is the desired inequality with $\beta = 1/C^2$.

Conversely, suppose such a β exists. We will apply Lemma 3.2. Since w is increasing it is locally integrable. If we integrate the given inequality and make a change of variables we get $W(t) \geq (\beta/2)W(2t)$. Since $\beta/2 < 1/2$, w is a doubling weight. □

We see immediately from Theorem 3.3 that the functions $w(t) = t^r$, $r \geq 0$, are increasing doubling weights on $[0, \infty)$. (Note that every possible value of β is already obtained by one of these functions.) We also see that functions which grow or decay exponentially, for example e^t or $e^{-1/t}$, cannot be doubling weights.

Functions that satisfy the inequality in Theorem 3.3 are sometimes referred to as moderately increasing functions. They play a role in the study of the so-called good-λ inequalities. (See, for example, Burkholder and Gundy [**3**].)

Since the inequality of Theorem 3.3 can be gotten by "differentiating" the inequality of Lemma 3.2, we originally speculated that this would also yield a characterization of decreasing doubling weights. This conjecture, however, is false. Rather, we discovered two very similar conditions, one necessary and the other sufficient, but neither a complete characterization.

Theorem 3.4. *If w is a decreasing doubling weight on $[0,\infty)$ then there exists a constant α, $0<\alpha<1$, such that $\alpha w(t)\le w(2t)$ for all t. Conversely, if a decreasing function w satisfies this inequality for some $\alpha>1/2$, then w is a doubling weight.*

Proof. If w is a decreasing doubling weight, then the proof that it satisfies the given inequality for some $\alpha<1$ is the same as the proof of the first half of Theorem 3.3.

Conversely, suppose that such an $\alpha>1/2$ exists. Fix t and partition $[0,t]$ into the intervals $[2^{-(n+1)}t, 2^{-n}t]$, $n\ge 0$. Then

$$\int_0^t w\,dx \le \sum_{n=1}^{\infty} w(2^{-n}t)2^{-n}t \le tw(t)\sum_{n=1}^{\infty}(2\alpha)^{-n} < \infty,$$

so w is locally integrable. The remainder of the proof that w is a doubling weight is the same as the proof of the second half of Theorem 3.3. □

One consequence of Theorems 3.3 and 3.4 is that increasing and decreasing doubling weights are related: if w is a decreasing doubling weight then so is w^{-1}; conversely, if w is an increasing doubling weight then there exists s, $0<s<1$, such that w^{-s} is a decreasing doubling weight. We will explore this relationship more closely in Section 4.

As an example of this relationship, Theorem 3.4 shows that $w(t)=t^{-r}$ is a doubling weight on $[0,\infty)$ for $0<r<1$. These functions, however, only attain the values $\alpha>1/2$. As the next two examples show, every possible value of $\alpha<1$ is attained, but the existence of such an α for $\alpha\le 1/2$ does not imply that a function is a doubling weight. Thus neither half of Theorem 3.4 characterizes decreasing doubling weights.

Example 3.5. There exists a decreasing doubling weight w on $[0,\infty)$ with the property that the largest α such that $\alpha w(t)\le w(2t)$ for all t is as small as desired.

Proof. Fix $\alpha < 1/2$, and let $a = 1/\alpha$ and $b = 2a$. Let w be the decreasing function

$$w(t) = \sum_{n=-\infty}^{\infty} a^n \chi_{I_n}(t), \qquad I_n = (b^{-(n+1)}, b^{-n}].$$

Given $t > 0$, if t is in I_n then $2t$ is in either I_{n-1} or I_n. Hence $\alpha w(t) \leq w(2t)$, and α is the largest value such that this inequality is true for all t.

We will apply Lemma 3.2 to show that w is a doubling weight. For all $k > 0$

$$\int_0^{b^k} w\,dx = \frac{b-1}{b} \sum_{n=-k}^{\infty} \left(\frac{a}{b}\right)^n < \infty,$$

so w is locally integrable. We will now show that (in the notation of Lemma 3.2) there exists a constant $\gamma > 1$ such that $\phi(t) = W(2t)/W(t) \geq \gamma$ for all t. Fix t, and suppose that t is in I_n. Then

$$W(t) = (1 - 1/b)(1/2)^n + a^n(t - b^{-(n+1)}). \tag{1}$$

There are two cases. If $2t$ is also in I_n, then we see from equation (1) that $W(2t) = W(t) + a^n t$, or equivalently,

$$\phi(t) = 1 + \frac{a^n t}{W(t)}.$$

If we differentiate ϕ we get

$$\phi'(t) = \frac{(a/2)^n(1 - 2/b)}{W(t)^2} > 0.$$

Hence ϕ is increasing, and so tends towards its minimum as t tends towards $b^{-(n+1)}$. If we substitute this value into (1) we see that $\phi(t)$ decreases to

$$\gamma = 1 + \frac{a^n b^{-(n+1)}}{(1 - 1/b)(1/2)^n} = \frac{b}{b-1}.$$

If, instead, $2t$ is in I_{n-1}, a similar but slightly more lengthy calculation shows that ϕ' is negative and so ϕ tends to its minimum as t tends toward b^{-n}. The minimum value of ϕ is again $\gamma = b/(b-1)$. Therefore w is a doubling weight. $\square$

I want to thank James Akao for an illuminating conversation which led to the next example.

Example 3.6. For any $\alpha < 1/2$, there exists a locally integrable, decreasing function w on $[0, \infty)$ which satisfies the inequality $\alpha w(t) \leq w(2t)$ for all t but which is not a doubling weight.

Proof. We will construct w on $[0, 1]$ and then extend it to all of $[0, \infty)$ by making it constant on $[1, \infty)$.

Fix $\alpha < 1/2$. We will construct a positive, increasing sequence $\{a_n\}$ such that $\alpha a_{n+1} \leq a_n$ and the function

$$w(t) = \sum_{n=0}^{\infty} a_n \chi_{I_n}, \qquad I_n = (2^{-(n+1)}, 2^{-n}],$$

is integrable but is not a doubling weight. To show this, we will show that for any constant C there exists an integer k such that (letting $J_k = [0, 2^{-(k+1)}]$)

$$\int_{J_k} w\,dx > C \int_{I_k} w\,dx,$$

or equivalently,

$$\sum_{n=k+1}^{\infty} a_n 2^{-n} > C a_k 2^{-k}. \tag{2}$$

Define $b_n = a_n 2^{-n}$. We will determine the sequence $\{a_n\}$ by constructing a sequence $\{b_n\}$ such that: $\sum b_n$ is finite; $2\alpha b_{n+1} \leq b_n$ and $b_n \leq 2b_{n+1}$ for all n; and, given any constant C, there exists a pair of integers $k' > k$ such that $b_{k'}/b_k > C$. (This implies that inequality (2) holds.)

We define the sequence $\{b_n\}$ inductively. Let $b_0 = 1$ and $N_0 = 0$. In general, for $i > 0$ odd, define b_n for $N_{i-1} \leq n \leq N_i$ by

$$b_n = b_{N_{i-1}} 2^{N_{i-1}-n},$$

where N_i is the least integer such that

$$\frac{(2/3)^{N_i}}{b_{N_i}} \geq 2^i.$$

Such an integer N_i exists, since for all $n \geq N_{i-1}$

$$\frac{(2/3)^n}{b_n} = \frac{(4/3)^n}{b_{N_{i-1}} 2^{N_{i-1}}},$$

and the righthand side can be made as large as desired.

For $N_i \leq n \leq N_{i+1}$ define

$$b_n = b_{N_i}(2\alpha)^{N_i - n},$$

where N_{i+1} is the largest integer such that

$$b_{N_{i+1}} \leq (2/3)^{N_{i+1}}.$$

Again such an integer N_{i+1} exists, since for all $n \geq N_i$

$$\frac{b_n}{(2/3)^n} = b_{N_i}(2\alpha)^{N_i}(3/4\alpha)^n;$$

because $3/4\alpha > 1$, the righthand side can be made as large as desired.

To see that $\{b_n\}$ has the desired properties: $b_n \leq (2/3)^n$ for all n, so $\sum b_n$ converges. Further, $2\alpha b_{n+1} \leq b_n$ and $b_n \leq 2b_{n+1}$; equality holds in the first if $N_i \leq n \leq N_{i+1}$ and in the second if $N_{i-1} \leq n \leq N_i$. To see that the last condition holds, let $k = N_i$ and $k' = N_{i+1}$ for i odd. Then

$$\frac{b_{k'}}{b_k} = (2\alpha)^{N_i - N_{i+1}},$$

so we need to show that $N_{i+1} - N_i$ gets arbitrarily large. But by our choice of N_i and N_{i+1},

$$\begin{aligned} 1 &\geq \frac{2\alpha(2/3)^{N_{i+1}+1}}{b_{N_{i+1}}} = \frac{(4\alpha/3)(2/3)^{N_{i+1}}}{b_{N_i}(2\alpha)^{N_i - N_{i+1}}} \\ &\geq 2^i(4\alpha/3)^{N_{i+1} - N_i + 1}. \end{aligned}$$

If we take the logarithm and rearrange terms we get

$$N_{i+1} - N_i \geq \frac{i \log 2 - \log(3/4\alpha)}{\log(3/4\alpha)}$$

and the righthand side tends to infinity as i does. □

Example 3.6 does not cover the case $\alpha = 1/2$; we will give such an example in Section 7. (See the remark after Theorem 7.3.)

The construction of Example 3.6 motivated the following characterization of decreasing doubling weights. I want to thank Donald Sarason for suggesting (in the context of Theorem 3.8 below) the technique used in the second half of the proof. It is an improvement over the original proof.

Theorem 3.7. *A decreasing function w on $[0,\infty)$ is a doubling weight if and only if there exists a constant C such that for all t*

$$\frac{1}{t}\int_0^t w\,dx \leq Cw(t). \tag{3}$$

Proof. If w is a doubling weight, then by Lemma 3.1 there exists a constant C such that for all t

$$\frac{1}{t}\int_0^t w\,dx \leq \frac{C}{t}\int_t^{2t} w\,dx \leq Cw(t).$$

To prove the converse we will apply Lemma 3.2. If inequality (3) holds, then

$$2Cw(2t) \geq \frac{1}{t}\int_0^t w\,dx + \frac{1}{t}\int_t^{2t} w\,dx \geq w(t) + w(2t),$$

so $(2C-1)w(2t) \geq w(t)$. If we combine this with inequality (3) we get

$$\frac{1}{t}\int_0^t w\,dx \leq \frac{2C^2 - C}{t}\int_t^{2t} w\,dx.$$

Hence $\gamma W(t) \leq W(2t)$ with $\gamma = 1 + (2C^2 - C)^{-1}$, and so w is a doubling weight. □

We conclude this section by proving the analogue of Theorem 3.7 for increasing functions. This result emphasizes both the similarities and the differences between increasing and decreasing doubling weights.

Theorem 3.8. *An increasing function w on $[0,\infty)$ is a doubling weight if and only if there exists a constant C such that for all t*

$$\frac{C}{t}\int_0^t w\,dx \geq w(t). \tag{4}$$

Proof. If w is a doubling weight then the proof that it satisfies inequality (4) is the same as the proof of the first half of Theorem (3.7).

To prove the converse we will apply Theorem 3.3. Fix δ, $1-1/C<\delta<1$; then inequality (4) becomes

$$\begin{aligned} w(t) &\leq \frac{C}{t}\int_0^{\delta t} w\,dx + \frac{C}{t}\int_{\delta t}^t w\,dx \\ &\leq C\delta w(\delta t) + C(1-\delta)w(t). \end{aligned}$$

Since $C(1-\delta)<1$, rearranging terms gives us $w(t)\leq \lambda w(\delta t)$, where

$$\lambda = \frac{C\delta}{1-C(1-\delta)} > 1.$$

Hence for all $k\geq 1$, $w(t)\leq \lambda^k w(\delta^k t)$. Fix n such that $\delta^n \leq 1/2$, and let $\beta=\lambda^{-n}$. Then $\beta w(t)\leq w(t/2)$ and so w is a doubling weight. □

4. Monotonic doubling weights and (A_∞) weights. In this section we show that monotonic doubling weights on $[0,\infty)$ are (A_∞) weights. For decreasing weights, we actually have a much stronger result as an immediate corollary to Theorem 3.7.

Theorem 4.1. *If w is a decreasing doubling weight on $[0,\infty)$ then w is in (A_1).*

Proof. For a decreasing function w, the Hardy-Littlewood maximal function is given explicitly by

$$Mw(t) = \frac{1}{t}\int_0^t w\,dx.$$

Hence, by Theorem 3.7 $Mw(t) \leq Cw(t)$ for all t, so w is an (A_1) weight. □

The proof that increasing doubling weights are in (A_∞) is equally direct. However, we first need a definition.

Definition 4.2. A function w satisfies the *reverse Hölder inequality* with exponent $s > 1$ if there exists a constant C such that, for every interval I, $I(w^s)^{1/s} \leq CI(w)$. We say that w belongs to the *reverse Hölder class* (RH_s).

It is well known that a function is in (A_∞) if and only if it is in (RH_s) for some s. (See, for example, Garcia-Cuerva [**5**, p. 400].) Using this we can prove the following result.

Theorem 4.3. *If w is an increasing doubling weight on $[0,\infty)$ then it is in (A_∞). Moreover, w is in (RH_s) for all $s > 1$ with a "reverse Hölder" constant independent of s.*

Proof. Fix $s > 1$, and take any interval I. Let J be the interval of equal length adjacent to I on the right, and let t be the point between I and J. Then by Lemma 3.1

$$I(w^s) \leq w(t)^s \leq J(w)^s \leq CI(w)^s.$$

Hence w is in (RH_s) and the constant is independent of s. □

Since (A_∞) weights are doubling weights (see, for example, Garcia-Cuerva [**5**, p. 396]) we can compactly summarize Theorems 4.1 and 4.3 in the following corollary.

Corollary 4.4. *A monotonic function on $[0,\infty)$ is a doubling weight if and only if it is an (A_∞) weight.*

While we have shown that increasing doubling weights are in (A_∞), our proof, in sharp contrast to our proof for decreasing weights, gives us no information about which particular (A_p) class a given weight belongs to. It is easy to see that the only increasing functions which are (A_1) weights are those which are bounded and bounded away from zero: if w is increasing then for all t

$$Mw(t) = \lim_{x\to\infty} w(x).$$

Further, for any $p > 1$ there exists an increasing doubling weight which is not in (A_p). An example is the function $w(t) = t^{p-1}$, which is in (A_q) for $q > p$ but is not in (A_p). Proving this is a straightforward computation. However, it is also an immediate consequence of the next result, which gives precise information about the (A_p) class of an increasing weight. If Theorems 3.3 and 3.4 are thought of as "ratio tests" for doubling weights, then Theorem 4.5 may be thought of as a sharper "root test."

Theorem 4.5. *A decreasing function w on $[0,\infty)$ is in (A_1) if and only if for all integers k*

$$\limsup_{n\to\infty} \left(\frac{w(2^{k-n})}{w(2^k)}\right)^{1/n} = L < 2, \tag{5}$$

and the limit supremum is uniform in k. An increasing function w on $[0,\infty)$ is in (A_p), $p > 1$, if and only if

$$\liminf_{n\to\infty} \left(\frac{w(2^{k-n})}{w(2^k)}\right)^{1/n} > (1/2)^{p-1}, \tag{6}$$

and the limit infimum is uniform in k.

Proof. Suppose first that w is decreasing and in (A_1). Then w is in (A_∞), so there exist constants C and $\delta < 1$ such that, given an interval I and a measurable subset E of I,

$$w(E) \le C\left(\frac{|E|}{|I|}\right)^{\delta} w(I). \tag{7}$$

Fix integers k and $n > 0$, and let $I = [0, 2^k]$ and $E = [0, 2^{k-n}]$. By Theorem 3.7, $w(t)t \le w([0,t]) \le Cw(t)t$ for all t. Combining this with inequality (7), we see that $w(2^{k-n})2^{k-n} \leq C2^{k-n\delta}w(2^k)$, or equivalently,

$$\left(\frac{w(2^{k-n})}{w(2^k)}\right)^{1/n} \leq C^{1/n}2^{1-\delta}. \tag{8}$$

Inequality (5) follows immediately. Since the righthand side of (8) is independent of k, the limit supremum is uniform in k.

Now suppose that w is an increasing (A_p) weight, $p > 1$. By the duality of (A_p) weights, $v = w^{1-p'}$ is a decreasing doubling weight, so inequality (5) holds for v. If we raise it to the power $1 - p$ we get inequality (6) for w, and it also holds uniformly in k.

To prove the converse, suppose first that w is decreasing and inequality (5) holds. Then for any integer k, the series

$$\sum_{n=1}^{\infty} \frac{w(2^{k-n})}{w(2^k)} 2^{-n} \tag{9}$$

converges by the root test. Further, since the limit supremum in (5) is uniform in k, for any constant L_1, $L < L_1 < 2$, there exists $N > 0$ such that

$$\sum_{n \geq N} \frac{w(2^{k-n})}{w(2^k)} 2^{-n} \leq \sum_{n \geq N} L_1^n 2^{-n} \leq C.$$

The first N terms of (9) are also uniformly bounded, since for $n \leq N$

$$\frac{w(2^{k-n})}{w(2^k)} \leq \frac{w(2^{k-N})}{w(2^k)} \leq L_1^N.$$

Therefore (9) is bounded by some constant independent of k; equivalently,

$$\sum_{n=1}^{\infty} w(2^{k-n})2^{-n} \leq Cw(2^k).$$

Now fix t, and let k be such that $2^{k-1} < t \le 2^k$. Then

$$\begin{aligned}\frac{1}{t}\int_0^t w\,dx &\le \frac{1}{2^{k-1}}\sum_{n=-k}^{\infty}\int_{2^{-(n+1)}}^{2^{-n}} w\,dx\\ &\le 2\sum_{n=1}^{\infty} w(2^{k-n})2^{-n}\\ &\le Cw(2^k)\\ &\le Cw(t).\end{aligned}$$

Therefore, by Theorems 3.7 and 4.1 w is in (A_1).

Finally, suppose that w is increasing and inequality (6) holds for some $p > 1$. Then inequality (5) holds for the decreasing function $v = w^{1-p}$. Hence, v is in (A_1) and so in $(A_{p'})$, and by duality w is an (A_p) weight. □

One consequence of Theorem 4.5 is that it gives a condition for determining which reverse Hölder class a decreasing doubling weight is in.

Corollary 4.6. *A decreasing function w on $[0,\infty)$ is in (RH_s) if and only if for all integers k*

$$\limsup_{n\to\infty}\left(\frac{w(2^{k-n})}{w(2^k)}\right)^{1/n} < 2^{1/s},$$

and the limit supremum is uniform in k.

Proof. This follows at once from Theorem 4.5 and a result of Strömberg and Wheeden [**13**]: a function w is in (RH_s) if and only if w^s is in (A_∞). (Another proof of this theorem is given by Johnson and Neugebauer [**7**].) □

From Corollary 4.6 we see that the doubling weights $w(t) = t^{-r}$, $0 < r < 1$, are in (RH_s) for $s < 1/r$. There exist decreasing weights which are in (RH_s) for all $s > 1$: for example, $w(t) = \max(\log(1/t), 1)$. However, in contrast to increasing weights, the only decreasing functions which are in (RH_s) for all s with a constant independent of s are those which are bounded and bounded away from zero. To see this, let $I = [0, t]$. Then by Theorem 3.7

$$I(w^s)^{1/s} \leq Cw(t),$$

where C is independent of s. As s tends to infinity, the lefthand side tends to the essential supremum of w on I. Hence w is bounded. Further, if w is not bounded below then we can make the righthand side arbitrarily small, so w is identically 0.

We conclude with several open questions inspired by our work on this section, all of which are true for monotonic weights.

Question 4.7. If w is in (A_∞) and, for some $s > 1$, w^s is a doubling weight, is w^s an (A_∞) weight?

Question 4.8. If w is a doubling weight, does there exist an $s > 1$ such that w^s is also a doubling weight? Or is this property equivalent to w being an (A_∞) weight?

Question 4.9. If w is a doubling weight, is w^r also a doubling weight for $0 < r < 1$?

Question 4.10. If w is a doubling weight, is it in L^p (locally) for some $p > 1$?

5. Piecewise monotonic weights. In this section we extend the results of Sections 3 and 4 to monotonic functions on arbitrary subintervals of **R** and use these results to characterize piecewise monotonic doubling weights and (A_∞) weights. To avoid obscuring the main ideas, many of the proofs (especially the later ones) are only sketched.

Trivially, all of the results in Sections 3 and 4 hold for monotonic functions on the intervals $[a, \infty)$, a in **R**. Since monotonic functions on $(-\infty, -a]$ can be thought of as reflections of monotonic functions on $[a, \infty)$, these results extend to such functions as well.

To apply our results to monotonic functions on finite intervals, we first consider the special case of monotonic functions which are either bounded or bounded away from zero. Call such functions restricted monotonic functions. These can be extended to monotonic functions on semi-infinite intervals. Since nonzero constant functions are in every (A_p) class, the extended function is an (A_p) weight if and only if the original function is. Thus our previous results apply to restricted monotonic functions with simple modifications.

In general, since no doubling weight is identically zero on an interval, we may assume that a monotonic function w decomposes into two restricted monotonic functions. Since w satisfies the (A_p) condition on any subinterval on which it is bounded and bounded away from zero, and since on finite intervals w is in (A_p) if it satisfies the (A_p) condition on sufficiently small intervals, w is an (A_p) weight if and only if its restricted monotonic components are. Hence our previous results extend to cover this case as well.

We will now characterize piecewise monotonic doubling weights and (A_∞) weights by giving necessary and sufficient conditions for assembling new weights out of monotonic weights on adjacent intervals. The simplest case is that of extending a function on $[0,1]$ to an even function on $[-1,1]$. In the next lemma we treat arbitrary doubling and (A_∞) weights.

Lemma 5.1. *Let w be an even function on $[-1,1]$. If w is a doubling weight on $[0,1]$, then it is a doubling weight on $[-1,1]$. The same is true if w is an (A_p) weight.*

Proof. Let w be a doubling weight on $[0,1]$. We will apply Lemma 3.1. It will suffice to consider only those adjacent intervals I and J such that J contains the origin. Let I' be the smallest interval symmetric about the origin containing I, and let J_0 be the subinterval of J between the origin and I. Then $w(J) \leq w(I') = 2w(I) + 2w(J_0)$. Let I_0 be the subinterval of I adjacent to J_0 of the same length. Then by Lemma 3.1 there exists a constant C such that $w(J_0) \leq Cw(I_0) \leq Cw(I)$, so $w(J) \leq Cw(I)$. A similar argument shows that $w(I) \leq Cw(J') \leq Cw(J)$, where J' is the smallest interval symmetric about the origin containing J.

Now let w be an (A_p) weight on $[0,1]$. For $p > 1$, it suffices to show that w satisfies the (A_p) condition on any interval I that contains the origin. Let I' be the smallest interval symmetric about the origin which contains I, and let I_0 be the right half of I'. Then

$$\begin{aligned} I(w)I(w^{1-p'})^{p-1} &\leq 2^p I'(w)I'(w^{1-p'})^{p-1} \\ &= 2^p I_0(w)I_0(w^{1-p'})^{p-1}, \end{aligned}$$

so w is in (A_p) on $[-1,1]$. Finally, if $p=1$, since $Mw(-t) = Mw(t)$ for all t, w is in (A_1) on $[-1,1]$. □

Lemma 5.1 is sufficient but by no means necessary. In the next two lemmas we give a weaker sufficient condition which is also necessary for piecewise monotonic functions.

Lemma 5.2. *Let w be a function on $[-1,1]$ which is a doubling weight on $[-1,0]$ and $[0,1]$. If there exists T, $0 < T \leq 1$, such that $w(t)/w(-t)$ is bounded and bounded away from zero for $0 < t \leq T$, then w is a doubling weight on $[-1,1]$. The same is true if w is an (A_p) weight.*

Proof. We will show this for w a doubling weight; the proof for w an (A_p) weight is identical. On finite intervals, a function is a doubling weight if it satisfies the doubling condition for sufficiently small intervals. Hence it will suffice to show that w is a doubling weight on $[-T,T]$. Define $v(t)$ to equal $w(t)$ on $[0,T]$ and $w(-t)$ on $[-T,0]$. Then by Lemma 5.1 v is a doubling weight on $[-T,T]$. Now define $\phi(t)$ to equal $w(t)/w(-t)$ on $[-T,0]$ and to equal one on $[0,T]$. Then ϕ is bounded and bounded away from zero on $[-T,T]$, so clearly $w = \phi v$ is also a doubling weight on $[-T,T]$. □

Lemma 5.3. *If w is a doubling weight on $[-1,1]$ that is monotonic on $[-1,0]$ and $[0,1]$, then for all T, $0 < T < 1$, $w(t)/w(-t)$ is bounded and bounded away from zero for $0 < t \leq T$.*

Proof. Fix T, $0 < T < 1$. Then w is a restricted monotonic function on $[-T,0]$ and $[0,T]$. For $0 < t \leq T$, let $I = [-t,0]$ and $J = [0,t]$. By Lemma 3.1, Theorems 3.7 and 3.8, and the remarks at the beginning of this section, there exist constants (depending only on w and T)

such that $w(-t) \leq CI(w) \leq CJ(w) \leq Cw(t)$. By symmetry we can conclude that $w(t)/w(-t)$ is bounded and bounded away from zero. $\square$

Though for simplicity we only gave Lemmas 5.2 and 5.3 for functions on $[-1,1]$, by scaling and translation we may replace $[-1,1]$ by any two finite, adjacent intervals. On the other hand, if I and J are two semi-infinite intervals with common point x, the doubling condition depends on the behavior of w on large intervals as well; more precisely, w is a doubling weight if and only if $w(x+t)/w(x-t)$ is bounded and bounded away from zero for both t large and t small.

Similar remarks hold for (A_p) weights and for a finite collection of intervals which partition $\mathbf{R}$. Together they prove:

Theorem 5.4. *Let $I_0, \dots, I_n$ be a partition of $\mathbf{R}$ into intervals with finite endpoints $x_1, \dots, x_n$, and let w be a function on $\mathbf{R}$ which is monotonic on each I_j. Then w is an (A_p) weight on $\mathbf{R}$ if and only if*

(1) *it is an (A_p) weight on each I_j;*

(2) *there exists a constant $T_0 > 0$ such that $w(x_j+t)/w(x_j-t)$ is bounded and bounded away from zero for $1 \leq j \leq n$ and $0 < t < T_0$;*

(3) *there exists a constant $T_1 > 0$ such that $w(t)/w(-t)$ is bounded and bounded away from zero for $t > T_1$.*

An immediate consequence of Theorem 5.4 and the preceding discussion is that any function which is a doubling weight and finitely piecewise monotonic is also an (A_∞) weight. This is no longer the case for functions which are piecewise monotonic on an infinite number of intervals: Wik [**14**] gives an example of a doubling weight which is piecewise linear but is not in (A_∞). In general, the problem of assembling an infinite number of monotonic weights into either a doubling weight or an (A_∞) weight is so complicated as to appear intractable. We give one result in this direction which is a generalization of Theorem 5.4.

Corollary 5.5. *Suppose that the intervals $I_n = [x_n, x_{n+1}]$ are an infinite partition of $\mathbf{R}$ with $|I_n| \geq \delta > 0$ for all n. Let w be a function on $\mathbf{R}$ such that*

(1) *w is a monotonic (A_p) weight on each I_n with uniformly bounded (A_p) constant;*

(2) *there exists a $T > 0$ such that if $t < T$ then $w(x_n + t)/w(x_n - t)$ is uniformly bounded and bounded away from zero for all n;*

(3) *$w(I)$ is uniformly bounded and bounded away from zero for all intervals I such that $|I| = \delta$.*

Then w is an (A_p) weight.

Proof. We first consider the case $p > 1$. By Theorem 5.4 and the discussion preceding it, conditions (1) and (2) imply that w satisfies the (A_p) condition for all intervals I such that $|I| \leq \delta$. If $|I| = \delta$ then from condition (3) and the (A_p) condition we see that $I(w)$ and $I(w^{1-p'})$ are uniformly bounded. Hence they are uniformly bounded for all I such that $|I| \geq \delta$, so w satisfies the (A_p) condition for such I and is therefore an (A_p) weight.

We now consider the case $p = 1$. By conditions (1) and (2) and Theorem 5.4 we need only consider intervals I such that $|I| \geq \delta$. Fix such an I. Then by condition (3), $I(w)$ is uniformly and bounded and bounded away from zero. Let t_n be the minimum of w on I_n. By condition (1), $I_n(w) \leq Cw(t_n)$ for some constant C independent of n. Therefore w is bounded away from zero on $\mathbf{R}$, so $I(w) \leq Cw(t)$ for all t in I. Hence w is in (A_1). □

6. Further results on monotonic (A_∞) weights. In this section we apply the results of Sections 3 and 4 to simplify and extend several known results on monotonic (A_∞) weights. Again we restrict ourselves to monotonic functions on $[0, \infty)$.

The first theorem characterizes monotonic (A_p) weights, $p > 1$. It was independently discovered by Guseinov [**6**] and by Benedetto, Heinig and Johnson [**1**]. Though Theorems 3.7 and 4.1 give a sharper result for decreasing functions, Theorem 6.1 and, more importantly, Corollary 6.3 provide alternatives to Theorem 4.5 for increasing weights. Here we give a new and simpler proof.

Theorem 6.1. *An increasing function w on $[0, \infty)$ is an (A_p) weight,*

$p > 1$, if and only if there exists a constant C such that for all t

$$\int_t^\infty x^{-p} w\,dx \left(\int_0^t w^{1-p'}\,dx \right)^{p-1} \leq C. \tag{10}$$

A decreasing function w on $[0,\infty)$ is an (A_p) weight, $p > 1$, if and only if there exists a constant C such that for all t

$$\int_0^t w\,dx \left(\int_t^\infty x^{-p'} w^{1-p'}\,dx \right)^{p-1} \leq C. \tag{11}$$

Proof. Let w be an increasing function and suppose that inequality (10) holds. Then

$$w(t) \int_t^\infty x^{-p}\,dx \left(\int_0^t w^{1-p'}\,dx \right)^{p-1} \leq C,$$

or equivalently,

$$\frac{1}{t} \int_0^t w^{1-p'}\,dx \leq C w(t)^{1-p'}.$$

By Theorems 3.7 and 4.1 $w^{1-p'}$ is in (A_1), and so in $(A_{p'})$. By the duality of (A_p) weights, w is in (A_p).

If w is a decreasing function and (11) holds, then raising both sides of this inequality to the power $p'-1$ shows that (10) holds for $w^{1-p'}$ with exponent p'. So by the previous argument $w^{1-p'}$ is in $(A_{p'})$ and by duality w is in (A_p).

Now suppose that w is an increasing (A_p) weight. Then the Hardy-Littlewood maximal operator is bounded on $L^p(w)$. (See Garcia-Cuerva [**5**, p. 400].) Fix $t > 0$ and let $f(x) = \chi_{[0,t]}(x)$. Then on $[t,\infty)$, $Mf(x) = t/x$, so

$$\int_t^\infty \left(\frac{t}{x}\right)^p w\,dx \leq C \int_0^t w\,dx,$$

where C is independent of t. Hence

$$\int_t^\infty x^{-p} w\,dx \left(\int_0^t w^{1-p'}\,dx \right)^{p-1} \\ \leq C \frac{1}{t} \int_0^t w\,dx \left(\frac{1}{t} \int_0^t w^{1-p'}\,dx \right)^{p-1},$$

which implies inequality (10).

If w is a decreasing (A_p) weight, then (11) follows from a duality argument like those above. $\square$

Note that the necessity of conditions (10) and (11) does not depend on w being monotonic.

Making the change of variables $x = y^{1-p'}$ in condition (10), we see that if w is an increasing (A_p) weight then $w(y^{1-p'})$ is a decreasing (A_p) weight. By a similar substitution in condition (11), we see that if w is a decreasing $(A_{p'})$ weight then $w(y^{1-p'})$ is an increasing $(A_{p'})$ weight. By Theorem 4.1, however, decreasing (A_p) and $(A_{p'})$ weights are the same. Therefore we have shown the following generalization of a result proved by Benedetto, Heinig and Johnson [**1**] for (A_2) weights.

Corollary 6.2. *Let w be a monotonic (A_p) weight on $[0,\infty)$, $p > 1$. Then $w(y^{1-p'})$ is in $(A_{p'})$.*

We can now recast inequality (10) in a more elegant form which is analogous to Theorem 3.7. The proof follows immediately from a change of variables, Theorem 3.7 and Corollary 6.2.

Corollary 6.3. *An increasing function w on $[0,\infty)$ is an (A_p) weight, $p > 1$, if and only if there exists a constant C such that for all t*

$$t^{p-1}\int_t^\infty x^{-p}w\,dx \le Cw(t). \tag{12}$$

As an application of Corollary 6.3, we give a new proof of a theorem of Johnson and Neugebauer [**8**] on the integrals of (A_p) weights.

Theorem 6.4. *Let w be an (A_p) weight on $[0,\infty)$, $p > 1$, and let*

$$W(t) = \int_0^t w\,dx.$$

Then W is in (A_{p+1}).

Proof. Central to the proof is the observation that W is increasing. Define

$$W_1(t) = \int_0^t w^{1-p'}\,dx.$$

The (A_p) condition implies that for all t, $t^p \le W(t)W_1(t)^{p-1} \le Ct^p$. Fix $q > p+1$. Then

$$\begin{aligned}\int_t^\infty x^{-q}W\,dx &\le C\int_t^\infty x^{p-q}W_1^{1-p}\,dx\\ &\le CW_1(t)^{1-p}\int_t^\infty x^{p-q}\,dx\\ &\le Ct^{1-q}W(t).\end{aligned}$$

Therefore, by Corollary 6.3, W is in (A_q). However, since w is in (A_p), it is in $(A_{p-\varepsilon})$ for some $\varepsilon > 0$. (See Garcia-Cuerva [**5**, p. 399].) Hence we can repeat the above argument, replacing p by $p-\varepsilon$. This shows that W is an (A_q) weight for all $q > p+1-\varepsilon$, and in particular that W is in (A_{p+1}). $\square$

Theorem 6.4 is sharp. Let $w(t) = t$: then w is in (A_p) for $p > 2$ but not in (A_2), and $W(t) = t^2/2$ is in (A_p) for $p > 3$ but not in (A_3).

A result which is much broader and only slightly weaker than Theorem 6.4 is also true. Its proof is extremely simple.

Theorem 6.5. *If μ is a doubling measure on $[0,\infty)$, then $W(t) = \mu([0,t])$ is an (A_p) weight for $p > 1 - \log\delta/\log 2$, where $\delta < 1/2$ depends only on the doubling constant of μ.*

Proof. Since μ is a doubling measure, the first part of the proof of Lemma 3.2 also shows that $\delta W(2t) \le W(t)$ for some $\delta < 1/2$ which depends only on the doubling constant of μ. Combining this with Theorem 4.5, we see that W is in (A_p) for p greater than the given bound. $\square$

The function W has several additional properties: since $\mu(I) > 0$ for any interval I, W is strictly increasing; since μ has no atoms, W is continuous. In fact, it can also be shown that W is Hölder continuous (see, for example, Lehto and Virtanen [**11**, p. 90]). It is tempting to conjecture that with these conditions as additional hypotheses, the converse of Theorem 6.5 is true.

Question 6.6. If W is a strictly increasing function on $[0, \infty)$ which is in (A_∞) and is Hölder continuous, then is the measure μ defined by $\mu([0, t]) = W(t)$ a doubling measure?

This question, if true, would show a very deep connection between (A_∞) weights and doubling measures.

7. The maximal operator. In this section we study the action of the Hardy-Littlewood maximal operator on monotonic doubling weights. As in previous sections, we restrict ourselves to functions on $[0, \infty)$.

First, as we noted in Section 4, if w is any bounded, increasing function then Mw is constant and so a doubling weight. (See the remarks prior to Theorem 4.5.) For decreasing functions almost the exact opposite is true: Mw is a doubling weight if and only if w is. This is surprising since the maximal operator is a smoothing and averaging operator. Furthermore, every positive function has the property that $W(t) \leq W(2t)$, which implies that for w decreasing, $(1/2)Mw(t) \leq Mw(2t)$. Thus Mw is "almost" a doubling weight: more precisely, for any δ, $0 < \delta < 1$, Theorems 3.4 and 4.1 show that $(Mw)^\delta$ is an (A_1) weight, a much stronger property. (This is a special case of a theorem of Coifman and Rochberg; see Garcia-Cuerva [**5**, pp. 158–160] for details.)

To show that Mw is a doubling weight if and only if w is, we need two lemmas. The first shows that monotonic functions are in (A_∞) if and only if they satisfy the reverse Hölder inequality on intervals adjacent to the origin.

Lemma 7.1. *If w is a monotonic function on $[0, \infty)$ then w is in*

(A_∞) if and only if there exist constants C and $s > 1$ such that for all t

$$\frac{1}{t}\int_0^t w^s\,dx \le C\left(\frac{1}{t}\int_0^t w\,dx\right)^s. \tag{13}$$

Proof. Since every (A_∞) weight is in (RH_s) for some $s > 1$, one direction is immediate.

To show the converse: suppose w is monotonic and satisfies (13). Then the same proof which shows that the reverse Hölder inequality implies the (A_∞) condition also shows that if w satisfies (13) then there exist constants C and $\delta > 0$ such that

$$\frac{w(E)}{w(I)} \le C\left(\frac{|E|}{|I|}\right)^\delta$$

whenever $E \subset I = [0,t]$ is measurable. (See Garcia-Cuerva [**5**, p. 401] for details.)

Let w be decreasing. If we set $E = [0, \lambda t]$, $\lambda < 1$ to be chosen below, then this inequality becomes

$$\int_0^{\lambda t} w\,dx \le C\lambda^\delta \int_0^t w\,dx,$$

or equivalently,

$$(1 - C\lambda^\delta)\int_0^{\lambda t} w\,dx \le C\lambda^\delta \int_{\lambda t}^t w\,dx \le C\lambda^\delta(1-\lambda)t\,w(\lambda t).$$

For λ sufficiently small, $(1 - C\lambda^\delta) > 0$; if we divide by this amount then we see from Theorem 3.7 that w is a doubling weight.

A similar argument using Theorem 3.8 holds for w increasing. □

The second lemma was originally shown to me by C.J. Neugebauer.

Lemma 7.2. *Given a function w, Mw is in (A_1) if and only if there exists $s > 1$ such that $(Mw^s)^{1/s} \le CMw$.*

Proof. If such an s exists, then by the theorem of Coifman and Rochberg referred to above, $(Mw^s)^{1/s}$ is in (A_1). (See Garcia-Cuerva [**5**, pp. 158–160].) Therefore $M(Mw) \le M((Mw^s)^{1/s}) \le C(Mw^s)^{1/s} \le CMw$ and so Mw is also an (A_1) weight.

To show the converse: if Mw is in (A_1), then it is in (RH_s) for some $s > 1$: that is, $I((Mw)^s)^{1/s} \le CI(Mw)$ for all intervals I. If we fix x and take the supremum over all I containing x, we see that $(Mw^s)^{1/s} \le (M(Mw)^s)^{1/s} \le CM(Mw) \le CMw$. □

Theorem 7.3. *If w is a decreasing function on $[0,\infty)$ then Mw is a doubling weight if and only if w is.*

Proof. Suppose that w is a doubling weight. Then $Mw(t) = W(t)/t$, where W is defined as in Lemma 3.2. By this result there exists a γ, $1 < \gamma < 2$, such that $\gamma Mw(t) \le 2Mw(2t)$. But then by Theorem 3.4 Mw is a doubling weight.

Now suppose that Mw is a doubling weight. Then by Lemma 7.2 there exist constants C and $s > 1$ such that $(Mw^s)^{1/s} \le CMw$, or equivalently,

$$\left(\frac{1}{t}\int_0^t w^s\,dx\right)^{1/s} \le \frac{C}{t}\int_0^t w\,dx.$$

By Lemma 7.1, w is a doubling weight. □

It is an open question whether Theorem 7.3 is a special case of a more general result.

Question 7.4. If w is a doubling weight, is Mw a doubling weight?

Theorem 7.3 lets us construct a decreasing function w which is not a doubling weight but such that $w(t) \le 2w(2t)$ for all t, thus extending Example 3.6 to the case $\alpha = 1/2$. Fix $\alpha < 1/2$ and let w be the function constructed in that example. Then w is not a doubling weight and so Mw is not one either, but $Mw(t) \le 2Mw(2t)$.

Finally, Theorem 7.3 has as an unexpected corollary another characterization of decreasing doubling weights.

Corollary 7.5. *A decreasing function w on $[0,\infty)$ is a doubling weight if and only if there exists a constant C such that for all t*

$$\int_0^t w\log(1/x)\,dx \le (C+\log(1/t))\int_0^t w\,dx. \tag{14}$$

Proof. If w is a doubling weight, then by Theorem 7.3 Mw is as well, so by Theorem 3.7 there exists a constant C such that

$$\frac{1}{t}\int_0^t Mw\,dx \le CMw(t) = \frac{C}{t}\int_0^t w\,dx.$$

If we substitute the definition of Mw into the lefthand side and apply Fubini's theorem we get

$$\begin{aligned}\frac{1}{t}\int_0^t \frac{1}{x}\int_0^x w(y)\,dy\,dx &= \frac{1}{t}\int_0^t w(y)\int_y^t \frac{1}{x}\,dx\,dy\\ &= \frac{1}{t}\int_0^t w(y)(\log t-\log y)\,dy.\end{aligned}$$

If we rearrange terms this becomes inequality (14).

Conversely, if (14) holds, then these calculations may be reversed; since Theorems 3.7 and 7.3 are necessary and sufficient conditions, w is a doubling weight. □

8. Multipliers of monotonic doubling weights. In this section we characterize the pointwise multipliers of the monotonic doubling weights: those functions ϕ such that ϕw is a doubling weight for every increasing (or decreasing) doubling weight w. As in previous sections, we restrict ourselves to monotonic functions on $[0,\infty)$.

As we noted in passing above (see Lemma 5.2), every function which is bounded and bounded away from zero is a multiplier of the doubling weights; beyond this fact very little is known. However, monotonic doubling weights are also (A_∞) weights, and the multipliers of (A_∞) have been completely characterized by Johnson and Neugebauer [**7**]. Their two main results are Theorems 8.1 and 8.6 below. Our work builds upon these theorems.

The multipliers of (A_∞) naturally split into two cases: those which preserve (A_∞) and those which preserve each (A_p) class. We will treat these two cases for monotonic weights in turn. Johnson and Neugebauer [**7**] characterized the multipliers of all of the (A_∞) weights in terms of the reverse Hölder classes.

Theorem 8.1. *For a non-negative function ϕ, the following are equivalent*:

(1) *ϕ is a multiplier of (A_∞);*

(2) *ϕ is in (RH_s) for all $s > 1$;*

(3) *ϕ^n is in (A_∞) for all $n > 0$.*

For multipliers of monotonic weights, condition (2) can be replaced by the apparently weaker condition that ϕ satisfies the reverse Hölder inequality for every $s > 1$ on intervals adjacent to the origin.

Corollary 8.2. *A function ϕ on $[0,\infty)$ is a multiplier of the decreasing (A_∞) weights if and only if for all $s > 1$ there exists a constant C_s such that for all t*

$$\left(\frac{1}{t}\int_0^t \phi^s\,dx\right)^{1/s} \le \frac{C_s}{t}\int_0^t \phi\,dx. \tag{15}$$

The same is true for multipliers of increasing weights.

Proof. We will prove this for multipliers of decreasing weights; the proof for the increasing case is identical.

Let ϕ be a multiplier of the deceasing (A_∞) weights. Since $w(t) \equiv 1$ is in (A_∞), ϕ^n is in (A_∞) for every $n > 0$. By Theorem 8.1, ϕ is in (RH_s) for every $s > 1$ and so inequality (15) holds.

To prove the converse: if inequality (15) holds, then by Hölder's inequality (15) holds for ϕ^n for every $n > 0$. Therefore, by Lemma 7.1 ϕ^n is in (A_∞), so by Theorem 8.1 ϕ is a multiplier of all of (A_∞) and so in particular of the decreasing (A_∞) weights. □

The proof of Corollary 8.2 actually shows more.

Corollary 8.3. *The multipliers of the decreasing (A_∞) weights on $[0,\infty)$ are exactly those multipliers of (A_∞) weights in general which are themselves decreasing. The same is true in the increasing case.*

Condition (2) of Theorem 8.1 also lets us prove a somewhat surprising description of the multipliers of the increasing (A_∞) weights. Recall that by Theorem 4.3 each increasing (A_∞) weight is in every reverse Hölder class. The next result is an immediate consequence of this fact.

Corollary 8.4. *A function ϕ on $[0,\infty)$ is a multiplier of the increasing (A_∞) weights if and only if it is an increasing (A_∞) weight.*

We can also characterize the multipliers of the decreasing (A_∞) weights using condition (2) and Corollary 4.6. However, this proves a great deal more, so a discussion of this is postponed until after Theorem 8.10 below.

We now consider the multipliers of the monotonic (A_p) weights. Johnson and Neugebauer [**7**] characterized the multipliers of all of the (A_p) weights in terms of the geometry of BMO. Here we give a new proof of their result which depends on a weak version of the Helson-Szegö theorem for all (A_p) classes.

Lemma 8.5. *Fix $p>1$ and let $w=e^{\phi}$ be an (A_p) weight. Then there exists a constant C_p, depending only on p, such that $\inf\{\|\phi-f\|_{\mathrm{BMO}} : f\in L^\infty\}\le C_p$.*

Proof. This lemma is an immediate consequence of a theorem proved in Garcia-Cuerva [**5**, p. 436], a result which in turn is a corollary of the Jones factorization theorem. This result shows that ϕ can be written in the form $\phi(t)=f(t)+\gamma\log Mg(t)-\delta\log Mh(t)$, where f is in L^∞, g,h are in L^1, $0<\gamma<1$ and $0<\delta<p-1$. Further, there exists a constant C independent of ϕ such that $\|\phi-f\|_{BMO}\le C(\gamma+\delta)$. The lemma follows immediately. □

Theorem 8.6. *Given $p>1$, a function ϕ is a multiplier of the (A_p) weights if and only if $\psi=\log\phi$ is in the* BMO *closure of L^∞.*

Proof. Fix $p > 1$ and suppose that ϕ is a multiplier of the (A_p) weights. Then ϕ^n lies in (A_p) for all $n > 0$, so by Lemma 8.5 there exists a constant C_p and a bounded function u_n such that $\|n\psi - u_n\|_{BMO} \leq C_p$, or equivalently, $\|\psi - u_n/n\|_{BMO} \leq C_p/n$. It follows immediately that ψ is in the BMO closure of L^∞.

To prove the converse: suppose ψ is in the BMO closure of L^∞ and w is in (A_p). Since $\log(A_p)$ is an open subset of BMO (see Journé [**10**, pp. 32–33]), there exists an $\varepsilon > 0$ such that if $\|v\|_{BMO} < \varepsilon$ then $e^v w$ is also in (A_p). But given ε, there exists a bounded function u such that $\|\psi - u\|_{BMO} < \varepsilon$. Let $v = \psi - u$. To complete the proof, note that e^u is bounded and bounded away from zero, and is thus a multiplier of (A_p). Therefore $e^u e^v w = \phi w$ is in (A_p) and so ϕ is a multiplier. □

Theorem 8.6 has an immediate corollary.

Corollary 8.7. *A function ϕ is a multiplier of every (A_p) class, $p > 1$, if and only if there exists $p_0 > 1$ such that ϕ is a multiplier of the (A_{p_0}) weights.*

Together, Lemma 8.5 and Theorem 8.6 give a characterization of the multipliers of the (A_p) weights which is the analogue of Theorem 8.1.

Corollary 8.8. *For each $p > 1$, ϕ is a multiplier of the (A_p) weights if and only if ϕ^n is in (A_p) for all $n > 0$.*

We can now characterize the multipliers of the monotonic (A_p) weights. First we give the analogue of Corollary 8.3. The proof follows at once from Corollary 8.8, from the fact that the only increasing (A_1) weights are bounded and bounded away from zero (see the remarks before Theorem 4.5) and from the fact that decreasing (A_∞) weights, by Theorem 4.1, are in (A_1).

Corollary 8.9. *For each $p \geq 1$, the multipliers of the decreasing (A_p) weights on $[0, \infty)$ are exactly those multipliers of (A_p) weights in general which are themselves decreasing. The same is true in the increasing case.*

Corollary 8.8 also lets us extend the characterization of monotonic (A_p) weights in Theorem 4.5 to a characterization of monotonic multipliers of (A_p) weights. The proof is immediate.

Theorem 8.10. *A monotonic function ϕ on $[0,\infty)$ is a multiplier of each of the (A_p) classes if and only if*

$$\lim_{n\to\infty}\left(\frac{\phi(2^{k-n})}{\phi(2^k)}\right)^{1/n}=1,$$

and the limit is uniform in k.

Since every decreasing (A_∞) weight on $[0,\infty)$ is in (A_1), any decreasing multiplier of (A_∞) is actually a multiplier of the each of the (A_p) classes. Thus for decreasing weights, both classes of multipliers are characterized by Theorem 8.10. As alluded to above, this characterization of the decreasing multipliers of (A_∞) can also be proved directly using Theorem 8.1 and Corollary 4.6.

We got Theorem 8.10 from Theorem 4.5 by replacing a constant bound with a limiting value of 1. It is reasonable, therefore, to ask if we obtain other characterizations of monotonic multipliers when we replace the bounds in the characterizations of monotonic doubling weights in Section 3 with limits. Doing so yields several equivalent conditions which are sufficient but are not necessary.

Theorem 8.11. *Given a monotonic function $\phi=e^\psi$ on $[0,\infty)$, the following conditions are equivalent and imply that ϕ is a multiplier of each of the (A_p) classes*:

(1)
$$\lim_{\substack{t\to 0\\ t\to\infty}}\frac{\phi(2t)}{\phi(t)}=1;$$

(2)
$$\lim_{\substack{t\to 0\\ t\to\infty}}\frac{1}{t\phi(t)}\int_0^t\phi\,dx=1;$$

(3)
$$\lim_{\substack{t\to 0\\ t\to\infty}}\frac{1}{t}\int_0^t\psi\,dx-\psi(t)=0.$$

Proof. We will prove this for ϕ decreasing; the proof for ϕ increasing is essentially the same. We will first show that (1), (2) and (3) are equivalent, and then show that (3) implies that ϕ is a multiplier.

To see that (1) implies (2) as t tends to zero: fix t and partition $[0,t]$ into the intervals $[2^{-(n+1)}t, 2^{-n}t]$, $n \geq 0$. Then

$$\phi(t) \leq \frac{1}{t}\int_0^t \phi\,dx \leq \sum_{n=1}^{\infty} \phi(2^{-n}t)2^{-n}. \tag{16}$$

Fix $\varepsilon > 0$. Then for t sufficiently small, $\phi(t)/\phi(t/2) > 1-\varepsilon$, so by induction, $\phi(t/2^n) \leq \phi(t)(1-\varepsilon)^{-n}$. Therefore the righthand side of (16) is bounded by

$$\phi(t)\sum_{n=1}^{\infty}(2-2\varepsilon)^{-n} = \frac{\phi(t)}{1-2\varepsilon},$$

and so (16) becomes

$$1 \leq \frac{1}{t\phi(t)}\int_0^t \phi\,dx \leq \frac{1}{1-2\varepsilon}.$$

Since ε was arbitrary, (2) holds as t tends to zero.

The proof that (1) implies (2) as t tends to infinity is similar. Given $\varepsilon > 0$, there exists $T > 0$ such that if $t \geq T$ then $\phi(2t)/\phi(t) > 1-\varepsilon$. Fix $k > 0$ and partition $[0, 2^kT]$ into the intervals $[0,T]$ and $[2^nT, 2^{n+1}T]$, $0 \leq n < k$. Then

$$\phi(2^kT) \leq \frac{1}{2^kT}\int_0^T \phi\,dx + \sum_{n=1}^{k} \phi(2^nT)2^{n-k}. \tag{17}$$

Arguing as above, we see that the second term on the righthand side is bounded by $\phi(2^kT)/(1-2\varepsilon)$. Therefore, inequality (17) implies that (2) holds as t tends to infinity if

$$\frac{1}{2^kT\phi(2^kT)}\int_0^T \phi\,dx$$

tends to zero as k tends to infinity; but this is the case since if $\varepsilon < 1/2$ then $2^kT\phi(2^kT) > (2-2\varepsilon)^kT\phi(T)$.

That (2) implies (3) follows immediately from Jensen's inequality.

To see that (3) implies (1): since ψ is decreasing, (3) is equivalent to the existence of a positive function $C(t)$ such that $C(t)$ tends to zero as t tends to zero or infinity, and such that

$$\psi(t) \leq \frac{1}{t}\int_0^t \psi\, dx \leq \psi(t) + C(t).$$

But then

$$\begin{aligned} 2\psi(2t) + 2C(2t) &\geq \frac{1}{t}\int_0^t \psi\, dx + \frac{1}{t}\int_t^{2t} \psi\, dx \\ &\geq \psi(t) + \psi(2t), \end{aligned}$$

which implies that

$$\psi(2t) + 2C(2t) \geq \psi(t) \geq \psi(2t).$$

This is equivalent to condition (1).

To show that (3) implies that ϕ is a multiplier, we will apply Theorem 8.6. Define a sequence of bounded functions, ψ_n, $n > 0$, by

$$\psi_n(t) = \begin{cases} \psi(1/n) & \text{if } t < 1/n, \\ \psi(t) & \text{if } 1/n \leq t \leq n, \\ \psi(n) & \text{if } t > n. \end{cases}$$

If w is a decreasing function then for every interval $I = [s,t]$, $I(|w - I(w)|) \leq 2(I(w) - w(t))$. Therefore, to show that the ψ_n's converge to ψ in BMO, it will suffice to show that the supremum of

$$J_n(I) = I(\psi - \psi_n) - (\psi(t) - \psi_n(t))$$

over all intervals $I = [s,t]$ tends to 0 as n tends to infinity. Fix $n > 1$. There are three cases which correspond to the location of I relative to the interval $[1/n, n]$. If $1/n \leq s$ and $t \leq n$ then $J_n(I)$ is zero. If $t < 1/n$ or if $s > n$ then

$$\begin{aligned} J_n(I) &= \frac{1}{t-s}\int_s^t \psi\, dx - \psi(t) \\ &\leq \frac{1}{t}\int_0^t \psi\, dx - \psi(t). \end{aligned} \tag{18}$$

If $s < 1/n$ and $t > n$ then by applying inequality (18) twice we see that

$$J_n(I) \le \frac{1}{s}\int_0^s \psi\,dx - \psi(s) + \frac{1}{t}\int_0^t \psi\,dx - \psi(t). \tag{19}$$

By (3), the righthand sides of (18) and (19) are arbitrarily small for n large. Hence $J_n(I)$ tends to 0 uniformly for all I. □

The following example shows that the conditions in Theorem 8.11 are not necessary for a function to be a multiplier.

Example 8.12. There exists a decreasing multiplier of the (A_p) weights on $[0,\infty)$ which does not satisfy the conditions of Theorem 8.11. An example also exists for increasing multipliers.

Proof. We will construct a decreasing multiplier ϕ which does not satisfy condition (1) of Theorem 8.11. By Theorem 8.6, $1/\phi$ will be the desired increasing multiplier.

We will construct ϕ on $[0,1]$ and then extend it to $[0,\infty)$ by making it constant on $[1,\infty)$. For $n \ge 0$, define

$$a_n = \frac{1}{e}\sum_{k=n}^{\infty}\frac{1}{k!},$$

and partition $[0,1]$ into the intervals $I_n = (a_{n+1}, a_n]$. Fix $R > 1$ and define

$$\phi(t) = \sum_{n=0}^{\infty} R^n \chi_{I_n}(t).$$

Clearly there exists a sequence $\{t_n\}$ tending to zero such that $\phi(t_n)/\phi(2t_n) \ge R$, so ϕ does not satisfy condition (1) of Theorem 8.11. To prove that ϕ is nevertheless a multiplier, by Corollary 8.8 it will suffice to show that ϕ is a doubling weight for all choices of R. Fix t in I_n. Then

$$\begin{aligned}\frac{1}{t}\int_0^t \phi\,dx &= \frac{1}{et}\sum_{k=n+1}^{\infty}\frac{R^k}{k!} + \frac{R^n}{t}(t - a_{n+1}) \\ &= \frac{1}{et}\sum_{k=n+1}^{\infty}\frac{R^k - R^n}{k!} + R^n.\end{aligned}$$

By the ratio test this series converges more quickly than a geometric series. Therefore it is bounded by a constant (depending only on R) times its first term. Hence

$$\frac{1}{t}\int_0^t \phi\,dx \le R^n\left(\frac{C(R-1)}{et(n+1)!}+1\right). \tag{20}$$

Since $t \ge a_{n+1}$, $t(n+1)! \ge 1/e$, so the righthand side of (20) is bounded by $CR^n = C\phi(t)$, where C depends only on R. Therefore, by Theorem 3.7 ϕ is a doubling weight for all R. □

While Theorem 8.11 is not a complete characterization of the multipliers of monotonic (A_p) weights, it does yield, in Corollary 8.13, a large class of examples. Further, this corollary and the examples which follow give additional information about the size of the set of decreasing multipliers of (A_p).

Corollary 8.13. *If a decreasing function ϕ on $[0,\infty)$ is in* BMO *and is bounded away from zero, then ϕ is a multiplier of the (A_p) weights. The same is true for ϕ increasing.*

Proof. I will prove this for ϕ decreasing; the proof for ϕ increasing is essentially the same.

Since ϕ is bounded away from zero, condition (1) of Theorem 8.11 holds as t tends to infinity; similarly if ϕ is bounded it holds as t tends to zero. Therefore it will suffice to show that it holds for unbounded ϕ as t tends to zero.

Since ϕ is in BMO, there exists $\lambda > 0$ such that $w = e^{\lambda\phi}$ is in (A_2). Therefore, by Theorem 3.4 there exists $\alpha < 1$ such that $\alpha w(t) \le w(2t)$ for all t. If we take the logarithm we see that

$$\phi(t) \le \phi(2t) + \frac{1}{\lambda}\log\frac{1}{\alpha}.$$

Since ϕ is unbounded, for any $\delta > 1$ there exists $T > 0$ such that if $t < T$, then $\phi(t) \le \delta\phi(2t)$. Hence condition (1) holds. □

The condition that ϕ be bounded away from zero is natural since BMO functions can have arbitrary behavior close to zero. A more general condition is that both ϕ and $1/\phi$ are in BMO. The proof of Theorem 8.13 extends to show that monotonic functions ϕ with this property are multipliers. Johnson and Neugebauer examined the class of functions with this property [**9**]. They showed that if ϕ and $1/\phi$ are in BMO then ϕ^n is in (A_2) for all $n > 0$. Hence, by Corollary 8.8 such functions are multipliers.

The converse of Corollary 8.13 is not true: there exist monotonic multipliers of the (A_p) weights which satisfy the conditions of Theorem 8.11 which are not in BMO. An increasing example is easy to construct: for example, $\phi(t) = \max(\log(t)^2, 1)$. The following is an example for decreasing multipliers.

Example 8.14. There exists a decreasing function ϕ on $[0, \infty)$ which is bounded away from zero and satisfies condition (1) of Theorem 8.11, but which is not in BMO.

Proof. We will construct ϕ on $[0, 1]$ and extend it to $[0, \infty)$ by making it constant on $[1, \infty)$. Let

$$\phi(t) = \sum_{n=0}^{\infty} (n+1)^2 \chi_{I_n}(t), \qquad I_n = (2^{-(n+1)}, 2^{-n}].$$

Then

$$\lim_{t \to 0} \frac{\phi(2t)}{\phi(t)} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1.$$

If ϕ is in BMO, then there exists $\lambda > 0$ such that $e^{\lambda\phi}$ is an (A_2) weight. But

$$\lim_{t \to 0} \frac{e^{\lambda\phi(2t)}}{e^{\lambda\phi(t)}} = \lim_{n \to \infty} e^{-2\lambda n - \lambda} = 0,$$

which contradicts Theorem 3.4. Hence ϕ is not in BMO. $\square$

Example 8.14 and Corollary 8.8 show that the set of multipliers of decreasing (A_p) weights contains decreasing BMO functions which are bounded away from zero as a proper subset and is contained in the set of all doubling weights which, locally, are in every L^p space. Our last example shows that this second inclusion is also proper.

Example 8.15. There exists a decreasing doubling weight w on $[0,\infty)$ such that w^p is locally integrable for all $p>1$ but which is not a multiplier of any (A_p) class.

Proof. Suppose v is a decreasing function such that v^p is locally integrable for all p but v is not a doubling weight. Since the maximal operator is a bounded operator on all L^p spaces, $p>1$, $(Mv)^p$ is also locally integrable for all p. By Theorem 7.3 and the remarks preceding Lemma 7.1, Mv is not a doubling weight but $(Mv)^{1/2}$ is. Hence by Corollary 8.8, $w=(Mv)^{1/2}$ is the desired function.

It remains to produce v with the desired properties. We will construct v on $[0,1/e]$ and extend it to $[0,\infty)$ by making it constant on $[1/e,\infty)$. Let $a_0=1/e$, and $a_n=a_{n-1}^n$ for $n>0$. Define

$$v(t)=\sum_{n=0}^{\infty}\log(1/a_n)\chi_{I_n}(t), \qquad I_n=(a_{n+1},a_n].$$

Since v is dominated by $\log(1/t)$, v^p is locally integrable for all p. However, v is not a doubling weight. For if I and J are two adjacent intervals of equal length such that their common point is a_{n+1}, $I\subset I_{n+1}$ and $J\subset I_n$, then

$$\frac{v(I)}{v(J)}=\frac{\log a_{n+1}}{\log a_n}=n+1.$$

Hence by Lemma 3.1 v is not a doubling weight. □

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