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SOME SUBCLASSES OF BMOA AND THEIR CHARACTERIZATION IN TERMS OF CARLESON MEASURES

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ABSTRACT. We study a collection of subclasses of BMOA defined by means of a modified Garcia norm and show that these classes are equivalently defined by means of a modified Carleson measure. We extend a result of C. Fefferman on series with nonnegative coefficients to these classes and also compare them with the classes of mean Lipschitz functions. Finally, we show some clear differences between the analytic and meromorphic cases of these classes.

1. Introduction. Let $\Delta = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$ and denote by $dxdy$ the usual area measure on Δ. The boundary of Δ will be denoted by $\partial\Delta$. For $z, w \in \Delta$ we let $g(z,w)$ be the Green's function of Δ with pole at w. The class of holomorphic functions on Δ will be denoted by A.

We are interested in the classical space BMOA of functions of bounded mean oscillation on Δ. There are several well-known equivalent definitions which can be found, for example, in [**7**] and [**11**]. The following is a variant of one of these definitions:

Definition. For $0 < p < \infty$, we say that $f \in Q_p$ if $f \in A$ and

$$||f||_{Q_p}^2 = \sup_{w\in\Delta} \iint_\Delta |f'(z)|^2 g^p(z,w)\,dx\,dy < \infty.$$

Moreover, if the above integrals tend to zero as $|w| \to 1$, then we say $f \in Q_{p,0}$.

These spaces, in their analytic and meromorphic forms, were introduced by the first author and his collaborators and have been studies in [**2**–**6**] and elsewhere. The key points (in the analytic case) are that

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for $1 < p < \infty$ the spaces $(Q_p,\ Q_{p,0})$ are all the same and equal to the Bloch space $\mathcal{B}$ (little Bloch space $\mathcal{B}_0$), see [**1, 17**]. For $p = 1$, we have $Q_1 = \mathrm{BMOA}$, $Q_{1,0} = \mathrm{VMOA}$, and, for $0 < p_1 < p_2 \le 1$, we have $Q_{p_1} \subsetneq Q_{p_2} \subset \mathrm{BMOA}$ and similarly for the lower order spaces. The motivation behind this paper is to investigate the differences between the Q_p-spaces for $0 < p \le 1$. To this end, we remind the reader of two well-known spaces of functions, the fractional Dirichlet spaces and the mean Lipschitz spaces.

Definition. (a) For $0 \le p \le 1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A$, we say that $f \in \mathcal{D}_p$ if

$$||f|_{\mathcal{D}_p}^2 = \sum_{n=1}^{\infty} n^p |a_n|^2 < \infty.$$

(b) For $1 \le p \le \infty$ and $0 < \alpha \le 1$, we say that $f \in A$ is in $\Lambda(p, \alpha)$ provided there is a finite constant C such that, for all $0 \le r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta \le \frac{C}{(1-r)^{(1-\alpha)p}}.$$

More simply we say $f \in \Lambda(p, \alpha)$ if and only if $M_p(f', r) = O(1-r)^{\alpha - 1}$.

Observe that the Dirichlet spaces $\mathcal{D}_p$ decrease with p and that

$$||f||_{\mathcal{D}_p}^2 \simeq \iint_{\Delta} |f'(z)|^2 (1-|z|)^{1-p} \, dx \, dy, \tag{1}$$

see, for example, [**20**].

In the above we use the notation $a \simeq b$ to denote comparability of the quantities, i.e., there are absolute positive constants c_1 and c_2 satisfying $c_1 b \le a \le c_2 b$. Similarly, we say that $a \lesssim b$ if only the second inequality holds. Of course, $\mathcal{D}_0$ above is just the usual Hardy space H^2 and $||f||_{\mathcal{D}_0} = ||f - f(0)||_2$, where $||f||_p$ denotes the usual Hardy space norm.

The spaces $\Lambda(p, \alpha)$ are discussed in [**9**] where it is proved that the spaces $\Lambda(p, 1/p)$ increase with p and are all contained in BMOA. This inclusion suggests a comparison with the Q_p-spaces.

In order to state our main results, we define the Möbius transform of a function $f \in A$ to be

$$f_w(z) = f\left(\frac{w-z}{1-\bar{w}z}\right) - f(w), \qquad z \in \Delta$$

for all $w \in \Delta$. Finally, for $0 \le p < \infty$ we say that a positive measure μ defined on Δ is a bounded p-Carleson measure provided

$$\mu(S(I)) = O(|I|^p) \tag{2}$$

for all subarc I of $\partial\Delta$. As usual, $|I|$ denotes the arc length of I and $S(I)$ denotes the Carleson box based on I. When $p = 1$, we have the standard definition of a Carleson measure, see for example [**7**]. If the righthand side of (2) is $o(|I|^p)$ then we say that μ is a compact p-Carleson measure.

Theorem 1.1. *Suppose that $0 < p \le 1$ and that $f \in A$. The following are equivalent*:

(a) $f \in Q_p$.

(b) *The function f is Möbius bounded in $\mathcal{D}_{1-p}$, i.e.,* $\sup_{w\in\Delta} ||f_w||_{\mathcal{D}_{1-p}} < \infty$.

(c) *The measure $d\mu = |f'|^2(1-|z|)^p\, dx\, dy$ is a bounded p-Carleson measure.*

Remark. The equivalence of (a) and (b), which we include for completeness, is in [**4**].

When $p = 1$, the above theorem is very well known and is contained in works of C. Fefferman, A. Garcia and Ch. Pommerenke, see [**7**] for an exposition on these works. Curiously, the above theorem does not make it completely transparent that the spaces Q_p actually vary with p. In fact, the quintessential example $f(z) = \log(1+z)$ is in Q_p for all $0 < p \le 1$. In [**5**], examples were constructed by using series with Hadarmard gaps. Constructions of this sort raise the question of characterizing those $f = \sum a_n z^n \in Q_p$ for which $a_n \ge 0$. For BMOA, this is a well-known unpublished result of C. Fefferman, see for example [**8, 14**] and [**18**].

Theorem 1.2. *Suppose that* $0 < p \le 1$ *and that* $f(z) = \sum a_n z^n \in A$.

(a) *The condition*

$$\sup_k \frac{1}{k^p} \sum_{n=0}^{\infty} (n+1)^{1-p} \left[\sum_{m=0}^{\min(n,k)} \frac{|a_{2n-m+1}|}{(m+1)^{1-p}} \right]^2 < \infty \tag{3}$$

implies that $f \in Q_p$.

(b) *If* $a_n \ge 0$ *for all* $n = 0, 1, \ldots$ *and* $f \in Q_p$, *then condition* (3) *holds.*

Corollary 1.3. *If* $f(z) = \sum b_n z^n$, $g(z) = \sum a_n z^n$ *and* $|b_n| \le a_n$ *for all* $n = 0, 1, 2, \ldots$, *then*

$$||f||_{Q_p} \lesssim ||g||_{Q_p}.$$

Our last main result compares the Q_p spaces with the mean Lipschitz spaces defined above. We will say that $f \in HG$ if $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in A$ and there is some $\lambda > 1$ where

$$n_{k+1}/n_k \ge \lambda > 1 \tag{4}$$

for all $k = 0, 1, \ldots$.

Theorem 1.4. *Suppose that* $2 < p < \infty$ *and that* $0 < q < 1$ *is defined by the equation*

$$\frac{1}{p} = \frac{1-q}{2}. \tag{5}$$

Then

(a) $\Lambda(p, 1/p) \subsetneq \cap_{\varepsilon > 0} Q_{q+\varepsilon}$.

(b) $Q_q \cap HG \subsetneq \Lambda(p, 1/p)$.

(c) *There exists a function* $f \in A$ *satisfying*:

$$f \in \bigcap_{q>0} Q_q \Big\backslash \bigcup_{p<\infty} \Lambda(p, 1/p).$$

Corollary 1.5. *For* $0 < p \le 1$,

$$Q_p \bigcap HG = \mathcal{D}_{1-p} \bigcap HG.$$

The paper is organized as follows. Section 2 contains the proof of Theorem 1.1, Section 3 contains the proof of Theorem 1.4, Section 4 contains the proof of Theorem 1.2 and Section 5 examines the meromorphic case.

2. p-Carleson measures. We begin with a characterization for the p-Carleson measure on Δ. All values of p, $0 < p < \infty$, are dealt with simultaneously.

Lemma 2.1. *Let μ be a positive measure on Δ. Then, for $0 < p < \infty$,*

(i) *μ is a bounded p-Carleson measure if and only if*

$$\sup_{w\in\Delta} \iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) < \infty, \tag{1}$$

(ii) *μ is a compact p-Carleson measure if and only if*

$$\lim_{|w|\to 1} \iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) = 0. \tag{2}$$

Proof. (i) Suppose that (1) is true. Then, for every Carleson box $S(I) = \{z \in \Delta : 1-h \le |z| < 1, |\theta - \arg z| \le h\}$ and $w = (1-h)e^{i(\theta+h/2)}$, we have

$$\begin{aligned}\iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) &\ge \inf_{z\in S(I)} \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p \mu(S(I)) \\ &\gtrsim \frac{\mu(S(I))}{|I|^p}.\end{aligned} \tag{3}$$

So

$$C_\mu = \sup_I \frac{\mu(S(I))}{|I|^p} \lesssim \sup_{w\in\Delta} \iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) < \infty.$$

Conversely, we assume that μ is a bounded p-Carleson measure on Δ, that is, $C_\mu < \infty$. Further,

$$\mu(\Delta) \lesssim C_\mu. \tag{4}$$

If $|w| \le 3/4$, we have the following trivial estimate

$$\iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) \lesssim \mu(\Delta) \lesssim C_\mu.$$

If $|w| > 3/4$, we put $E_n = \{z \in \Delta : |z - w/|w|| < 2^n(1-|w|)\}$ and get $\mu(E_n) \lesssim C_\mu 2^{np}(1-|w|)^p$ for all $n = 1, 2, 3, \ldots$. We also have

$$\frac{1-|w|^2}{|1-\bar{w}z|^2} \lesssim \frac{1}{1-|w|}, \qquad z \in E_1,$$

and, hence for $n \ge 1$ and $E_0 = \varnothing$,

$$\frac{1-|w|^2}{|1-\bar{w}z|^2} \lesssim \frac{1}{2^{2n}(1-|w|)}, \qquad z \in E_n \backslash E_{n-1}.$$

Consequently,

$$\begin{aligned} \iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) &\le \sum_{n=1}^\infty \iint_{E_n\backslash E_{n-1}} \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) \\ &\lesssim \sum_{n=1}^\infty \frac{\mu(E_n)}{2^{2np}(1-|w|)^p} \\ &\lesssim C_\mu \sum_{n=1}^\infty \frac{1}{2^{np}}, \end{aligned} \tag{5}$$

that is to say, (1) holds.

(ii) First we suppose that (2) holds. By applying (3), we immediately see that μ is a compact p-Carleson measure.

On the other hand, if μ is a compact p-Carleson measure, then μ must be bounded. Now, for $t \in (0,1)$, let $\chi_{\Delta\setminus\Delta_t}(z)$ be the characteristic function of $\Delta\setminus\Delta_t$, where $\Delta_t = \{z : |z| < t\}$. Further, let $d\mu_t(z) = \chi_{\Delta\setminus\Delta_t}(z)\, d\mu(z)$. Then, from (4) and (5), we get

$$\begin{aligned}
\iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) &= \left(\iint_{\Delta_t} + \iint_{\Delta\setminus\Delta_t}\right)\left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu(z) \\
&\le \left(\frac{1-|w|^2}{(1-t)^2}\right)^p \mu(\Delta) \\
&\quad + \iint_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^p d\mu_t(z) \\
&\lesssim \left(\frac{1-|w|^2}{(1-t)^2}\right)^p \mu(\Delta) + C_{\mu_t} \\
&\lesssim \left(\frac{1-|w|^2}{(1-t)^2}\right)^p C_\mu \\
&\quad + \sup_I \frac{\mu((\Delta\setminus\Delta_t)\cap S(I))}{|I|^p}.
\end{aligned}$$

This implies that (2) is true. Hence, the lemma is proved. □

The following two results are well known: $f \in \mathrm{BMOA}$, $f \in \mathrm{VMOA}$, if and only if $|f'|^2(1-|z|)\, dx\, dy$ is a bounded (compact) 1-Carleson measure ([**12, 13**] or [**11, 19, 27**]) and $f \in \mathcal{B}$, $f \in \mathcal{B}_0$, if and only if $|f'|^2(1-|z|)^2\, dx\, dy$ is a bounded (compact) 2-Carleson measure [**22, 23**].

Now we characterize Q_p and $Q_{p,0}$ by means of p-Carleson measures for $0 < p < \infty$. We will use this result in proving Theorem 1.1.

Theorem 2.2. *Let $f \in A$. Then, for $0 < p < \infty$, we have:*

(i) *$f \in Q_p$ if and only if $d\mu = |f'|^2(1-|z|)^p\, dx\, dy$ is a bounded p-Carleson measure.*

(ii) *$f \in Q_{p,0}$ if and only if $d\mu = |f'|^2(1-|z|)^p\, dx\, dy$ is a compact p-Carleson measure.*

Proof. By Lemma 2.1 it suffices to show that

$$\iint_\Delta |f'(z)|^2 \frac{(1-|z|^2)^p(1-|w|^2)^p}{|1-\bar{w}z|^{2p}}\, dx\, dy \simeq \iint_\Delta |f'(z)|^2 g^p(z,w)\, dx\, dy \tag{6}$$

for all $w \in \Delta$. By a change of variables argument, it suffices to prove (6) for $w = 0$. This is equivalent to

$$\int_0^1 M_2^2(f',r)(1-r^2)^p r\, dr \simeq \int_0^1 M_2^2(f',r)(\log(1/r))^p r\, dr$$

which follows from elementary estimates for the logarithm function combined with the fact that $M_p(g,r)$ increases with r. □

Remark. For $1 < p < \infty$, Theorem 2.2 was proved in [**21**] and [**26**].

Proof of Theorem 1.1. The equivalence of (a) and (c) follows from Theorem 2.2. Denoting $\varphi_w(z) = (w-z)/(1-\bar{w}z)$ we get by changing variables that

$$\begin{aligned}\sup_{w\in\Delta}\iint_\Delta |f'(z)|^2(1-|\varphi_w(z)|^2)^p\, dx\, dy &= \sup_{w\in\Delta}\iint_\Delta |f'(\varphi_w(z))|^2(1-|z|^2)^p|\varphi_w'(z)|^2\, dx\, dy\\ &= \sup_{w\in\Delta}\iint_\Delta |f_w'(z)|^2(1-|z|^2)^p\, dx\, dy\end{aligned}$$

and thus (a) is equivalent to (b), see [**5**, Theorem 7]. Thus, the proof is completed. □

The proof of Theorem 1.1 implies the following lower order version:

Corollary 2.3. *Suppose that* $0 < p \le 1$ *and that* $f \in A$. *The following are equivalent*:

(a) $f \in Q_{p,0}$.

(b) $\lim_{|w|\to 1} ||f_w||_{\mathcal{D}_{1-p}} = 0$.

(c) *The measure* $d\mu = |f'|^2(1-|z|)^p\,dx\,dy$ *is a compact* p*-Carleson measure.*

3. Mean Lipschitz spaces. In order to prove strict inclusions in Theorem 1.4, we need a criterion for a Hadamard gap series to belong to $\Lambda(p, 1/p)$. We first observe that the binomial theorem combined with Stirling's formula yields

$$\frac{1}{(1-\bar{w}z)^p} = \sum_{n=0}^{\infty} b_n \bar{w}^n z^n \tag{1}$$

where $b_n \simeq (1+n)^{p-1}$ for $n = 0, 1, \dots$.

Lemma 3.1. *Let* $1 \le p < \infty$. *A function* $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ *belonging to* HG *satisfies* $f \in \Lambda(p, 1/p)$ *if and only if* $|a_k| = O(n_k^{-1/p})$.

Proof. First suppose that $|a_k| = O(n_k^{-1/p})$ and that (1.4) holds for some $\lambda > 1$. Since the number of Taylor coefficients a_k when $n_k \in I_n = \{k \in \mathbf{N} : 2^n \le k < 2^{n+1}\}$ is at most $[\log_\lambda 2] + 1$, we get

$$\begin{aligned} M_p(f', r) \le M_\infty(f', r) &\le \sum_{k=0}^{\infty} n_k |a_k| r^{n_k - 1} \\ &\lesssim \frac{1}{r} \sum_{k=0}^{\infty} n_k^{1-1/p} r^{n_k} \lesssim \frac{1}{r} \sum_{n=1}^{\infty} \frac{r^n}{n^{1/p}} \\ &\simeq \frac{1}{r} \frac{1}{(1-r)^{1-1/p}}. \end{aligned}$$

Thus, since $M_p(f', r)$ increases with r, $f \in \Lambda(p, 1/p)$.

Conversely, suppose $f \in \Lambda(p, 1/p)$. By Khinchin's inequality [**10**, Appendix A], we have

$$M_2(f', r) \simeq M_p(f', r) \lesssim \left(\frac{1}{1-r}\right)^{1-1/p}$$

and so $n_k |a_k| r^{n_k - 1} \lesssim (1-r)^{1/p-1}$. Taking $r = 1 - n_k^{-1}$, we get

$$n_k |a_k| \simeq n_k |a_k| \left(1 - \frac{1}{n_k}\right)^{n_k - 1} \lesssim n_k^{1-1/p}$$

and so $|a_k| = O(n_k^{-1/p})$. Thus, the proof is completed. $\square$

Proof of Theorem 1.4. (a) We suppose first that $f \in \Lambda(p, 1/p)$ so that $M_p(f', s) = O(1-s)^{1/p-1}$. Then, for the Carleson box $S(I) = \{z \in \Delta : 1-h \le |z| < 1, |\theta - \arg z| \le h\}$, we get by Hölder's inequality and the assumptions that $p > 2$ and (1.5) that

$$\begin{aligned}\mu(S(I)) &= \int_0^h \left(\int_{\theta-h}^{\theta+h} |f'((1-s)e^{i\varphi})|^2 \, d\varphi \right) s^{q+\varepsilon}(1-s) \, ds \\ &\le \int_0^h \left(\int_{\theta-h}^{\theta+h} |f'((1-s)e^{i\varphi})|^p \, d\varphi \right)^{2/p} (2h)^{1-2/p} s^{q+\varepsilon} \, ds \\ &\lesssim h^{1-2/p} \int_0^h (M_p(f', 1-s))^2 s^{q+\varepsilon} \, ds \\ &\lesssim h^{1-2/p} \int_0^h (s^{1/p-1})^2 s^{q+\varepsilon} \, ds \simeq |I|^{q+\varepsilon}.\end{aligned}$$

By Theorem 1.1, $f \in Q_{q+\varepsilon}$ for all $\varepsilon > 0$ and thus the inclusion is proved.

In order to prove the strict inclusion, we consider a function $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} = \sum_{n=1}^{\infty} \sqrt{n} 2^{-n/p} z^{2^n}$. Then $|a_k| n_k^{1/p} = \sqrt{n}$, and by Lemma 3.1, $f \notin \Lambda(p, 1/p)$. On the other hand,

$$\sum_{n=0}^{\infty} 2^{n(1-(q+\varepsilon))} \left(\sum_{n_k \in I_n} |a_k|^2 \right) = \sum_{n=0}^{\infty} 2^{-n\varepsilon} n < \infty,$$

and, by [**5**, Theorem 6], $f \in Q_{q+\varepsilon}$ for all $0 < \varepsilon \le 2/p$.

(b) Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ belongs to $HG \cap Q_{1-2/p}$. By Lemma 3.1, it suffices to show

$$n_k^{2/p} |a_k|^2 \lesssim 1.$$

But this is obvious since $f \in Q_{1-2/p} \subset \mathcal{D}_{2/p}$.

In Corollary 3.2, we construct a Hadamard gap series f_p such that $f_p \in \Lambda(p, 1/p)$ but $f_p \notin Q_{1-2/p}$. This shows the inclusion to be strict.

(c) Suppose we can construct a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfying:

(i) $||\Delta_n f||_2 \leq 2^{-n}$, where $\Delta_n f(e^{i\theta}) = \sum_{k\in I_n} a_k e^{ik\theta}$ for $n = 1, 2, \dots$, and

(ii) there exists $n = n(p, m)$, for all $p = 3, 4, \dots; m = 1, 2, \dots$, such that

$$||\Delta_n f||_p \geq m 2^{-n/p}.$$

Then, for every $q > 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{n(1-q)} \Big(\sum_{k\in I_n} |a_k|\Big)^2 &\leq \sum_{n=0}^{\infty} 2^{n(1-q)} 2^n \Big(\sum_{k\in I_n} |a_k|^2\Big) \\ &= \sum_{n=0}^{\infty} 2^{n(1-q)} (2^n ||\Delta_n f||_2^2) \\ &\leq \sum_{n=0}^{\infty} 2^{-nq} < \infty,\end{aligned}$$

and so $f \in Q_q$ (cf. [**5**, Theorem 5]).

Since the spaces $\Lambda(p, 1/p)$ are monotonically increasing, see [**9**, Corollary 2.3], it suffices to show that $f \notin \Lambda(p, 1/p)$ for $p = 3, 4, \dots$. Fix such a p. By (ii) there exists $\{n_m\}$ such that $||\Delta_{n_m} f||_p 2^{n_m/p} \geq m$ for $m = 1, 2, \dots$. Thus, $\sup_n ||\Delta_n f||_p 2^{n/p} = \infty$ and hence $f \notin \Lambda(p, 1/p)$, see [**9**, Theorem 3.1].

The construction. Let $r_1, r_2, \dots,$ be an enumeration of the pairs $\{(p, m) : p = 3, 4, \dots; m = 1, 2, \dots\}$. We need to find integers n_j,

$$n_1 < n_2 < \dots,$$

and polynomials f_j satisfying:

(iii) f_j polynomials of degree $\leq 2^{n_j}$

(iv) $||f_j||_2 \leq 2^{-n_j}$,

(v) $||f_j||_{\pi_1(r_j)} \geq \pi_2(r_j) 2^{-n_j/\pi_1(r_j)}$.

(Here π_1, π_2 are projections on first and second coordinates of the pairs r_j.)

Assume $\{f_j\}$ have been constructed, then define

$$f(z) = \sum_{j=1}^{\infty} z^{2^{n_j}} f_j(z).$$

It is then easily seen that f satisfies (i) and (ii) so we're done once we construct $\{f_j\}$.

Construction of the sequence $\{f_j\}$: Given n_{j-1}, $p = 3, 4, \dots,$ and $m = 1, 2, \dots,$ we must find $n_j > n_{j-1}$ and polynomials f_j of degree 2^{n_j} such that

(iv)′ $||f_j||_2 \leq 2^{-n_j}$

(v)′ $||f_j||_p \geq m2^{-n_j/p}$.

But the existence of f_j follows immediately from the density of polynomials in H^p and $H^p \subsetneq H^2$. The proof of the theorem is completed. □

We next show in Theorem 1.4(a) the lower bound $1 - 2/p$ is sharp in the sense that $\Lambda(p, 1/p) \not\subset Q_{1-2/p}$ for $2 < p < \infty$.

Corollary 3.2. *There exists $f_p \in \Lambda(p, 1/p)$ with $f_p \notin Q_{1-2/p}$ for all $2 < p < \infty$.*

Proof. Let

$$f_p(z) = \sum_{k=0}^{\infty} a_k z^{n_k} = \sum_{n=0}^{\infty} 2^{-n/p} z^{2^n}.$$

Since

$$n_k^{1/p}|a_k| = (2n)^{1/p} 2^{-n/p} = 1 < \infty,$$

$f_p \in \Lambda(p, 1/p)$ by Lemma 3.1.

On the other hand,

$$\begin{aligned}\sum_{k=0}^{\infty} n_k^{1-(1-2/p)} \left(\sum_{n_k \in I_n} |a_k|^2 \right) &= \sum_{n=0}^{\infty} (2n)^{2/p} 2^{-2n/p} \\ &= \sum_{n=0}^{\infty} 1 = \infty.\end{aligned}$$

By [**5**, Theorem 6], see also [**2**, Theorem 5], $f \notin Q_{1-2/p}$ and the corollary is proved. □

Proof of Corollary 1.5. We already know that $Q_p \subset \mathcal{D}_{1-p}$. Let $f \in \mathcal{D}_{1-p} \cap HG$. Then, by definition,

$$\infty > \sum_{n=0}^{\infty} n^{1-p}|a_n|^2 \geq \sum_{n=0}^{\infty} 2^{n(1-p)}\bigg(\sum_{k\in I_n}|a_k|^2\bigg).$$

By [**5**, Theorem 6], $f \in Q_p \cap HG$, and the proof is completed. $\square$

4. Functions with nonnegative coefficients. In this section we prove Theorem 1.2 in the introduction. For $p = 1$, the published proofs of this result involve some aspect of the duality between H^1 and BMOA, due to C. Fefferman. In the absence of an analogue to these theories, we must proceed directly using the definition of Q_p.

We assume that $f(z) = \sum a_n z^n$ and that the sequence $\{a_n\}$ satisfies condition (1.3) in the introduction. As proved in Section 2, it suffices to prove:

$$\sup_{w\in\Delta}\iint_{\Delta}|f'(z)|^2\frac{(1-|z|^2)^p(1-|w|^2)^p}{|1-\bar{w}z|^{2p}}\,dx\,dy < \infty. \tag{1}$$

Let b_n be the binomial coefficients in (3.1); then

$$\begin{aligned}\frac{f'(z)}{(1-\bar{w}z)^p} &= \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n\sum_{n=0}^{\infty}b_n\bar{w}^n z^n\\ &= \sum_{n=0}^{\infty}\bigg(\sum_{m=0}^{n}(m+1)a_{m+1}b_{n-m}\bar{w}^{n-m}\bigg)z^n\\ &= \sum_{n=0}^{\infty}(n+1)c_{n+1}z^n = \bigg(\sum_{1}^{\infty}c_n z^n\bigg)'.\end{aligned}$$

Now invoking (1.1) and $b_n \simeq (n+1)^{p-1}$, we get

$$\begin{aligned}\iint_\Delta |f'(z)|^2 \frac{(1-|z|^2)^p}{|1-\bar{w}z|^{2p}}\,dx\,dy &\simeq \Big\| \sum c_n z^n \Big\|^2_{\mathcal{D}_{1-p}} \\ &= \sum_{n=0}^{\infty} (n+1)^{1-p} |c_{n+1}|^2 \\ &\lesssim \sum_{n=0}^{\infty} (n+1)^{1-p} \\ &\quad \bigg(\sum_{m=0}^{n} \frac{(m+1)|a_{m+1}||w|^{n-m}}{(n+1)(n-m+1)^{1-p}} \bigg)^2.\end{aligned}$$

Fix $w \in \Delta$, and let k be the positive integer satisfying: $(k+1)^{-1} < 1-|w| \le k^{-1}$. Using the above inequality, we see that it suffices to prove

$$\sup_k (I_k/k^p) < \infty, \tag{2}$$

where, for $k = 1, 2, \ldots$, we let

(3)

$$I_k = \sum_{n=0}^{\infty} (n+1)^{1-p} \bigg(\sum_{m=0}^{n} \frac{(m+1)|a_{m+1}|}{(n+1)(n-m+1)^{1-p}} \Big(1 - \frac{1}{k+1} \Big)^{n-m} \bigg)^2.$$

Lemma 4.1. *If* $\{a_n\}$ *satisfies* (1.3), *then there is a finite constant* c *satisfying*

$$\sum_{m=1}^{n} m a_m \le cn \tag{4}$$

for $n = 1, 2, \ldots$.

Proof. It clearly suffices to prove that there is a constant c satisfying:

$$\sum_{m=n}^{2n} a_m \le c \tag{5}$$

for $n = 1, 2, \dots$. But (1.3) implies that there is a $c < \infty$ with

$$\sum_{n=k}^{2k} (n+1)^{1-p} \left[\sum_{m=0}^{n} \frac{|a_{2n-m}|}{(m+1)^{1-p}} \right]^2 \le ck^p,$$

for $k = 1, 2, \dots$, and hence

$$\sum_{n=k}^{2k} \left[\sum_{m=n}^{2n} |a_m| \right]^2 \lesssim ck.$$

Condition (5) now follows, and the proof is complete. □

Using the lemma we can simplify (3) by observing that

$$\sum_{n=0}^{\infty} (n+1)^{1-p} \left(\sum_{0 \le m \le n/2} \frac{(m+1)|a_{m+1}|}{(n+1)(n-m+1)^{1-p}} \left(1 - \frac{1}{k+1}\right)^{n-m} \right)^2$$
$$\lesssim \sum_{n=0}^{\infty} (n+1)^{p-1} \left(1 - \frac{1}{k+1}\right)^n \simeq k^p$$

by the binomial theorem (see (3.1)). It now follows that we need only prove: Condition (1.3) implies

$$\sup_k \frac{1}{k^p} \sum_{n=0}^{\infty} (n+1)^{1-p} \left(\sum_{0 \le m \le n/2} \frac{|a_{n-m+1}|}{(m+1)^{1-p}} \left(1 - \frac{1}{k}\right)^m \right)^2$$

or, more simply, $\sup I'_k / k^p < \infty$ where

$$I'_k = \sum_{n=0}^{\infty} (n+1)^{1-p} \left(\sum_{m=0}^{n} \frac{|a_{2n-m+1}|}{(m+1)^{1-p}} \left(1 - \frac{1}{k}\right)^m \right)^2. \tag{6}$$

As our last reduction we first observe that for $0 \le m \le k$ we have $(1 - 1/k)^m \simeq 1$. Fixing a large integer M, then splitting the sum in (6) into two parts and applying (1.3), we get

$$\frac{1}{2} I'_k \le \sum_{n=0}^{\infty} (n+1)^{1-p} \left(\sum_{m=0}^{\min(n, Mk)} \frac{|a_{2n-m+1}|}{(m+1)^{1-p}} \right)^2$$
$$+ \sum_{n=Mk}^{\infty} (n+1)^{1-p} \left(\sum_{m=Mk}^{n} \frac{|a_{2n-m+1}|}{(m+1)^{1-p}} \left(1 - \frac{1}{k}\right)^m \right)^2$$
$$\le c(Mk)^p + \left(\frac{1 - 1/k}{1 - 1/(Mk)} \right)^{2Mk} \cdot I'_{Mk}.$$

Thus,

$$\frac{1}{2}\sup_k \frac{I'_k}{k^p} \le cM^p + \frac{1}{4}\sup_k \frac{I'_{Mk}}{(Mk)^p}$$

provided M is sufficiently large. This yields

$$\sup_k (I'_k/k^p) \le 4cM^p$$

provided f is a polynomial. A limit argument completes the proof of part (a) of Theorem 1.2.

The proof of part (b) is easy. By reversing the first step in the above proof, we obtain

$$\begin{aligned}(1-|w|)^p \sum_{n=0}^{\infty}(n+1)^{1-p}&\bigg(\sum_{m=0}^{n}\frac{(m+1)|a_{m+1}|\,|w|^{n-m}}{(n+1)(n-m+1)^{1-p}}\bigg)^2 \\ &\lesssim \iint_{\Delta} |f'(z)|^2 \frac{(1-|w|^2)^p(1-|z|^2)^p}{|1-\bar{w}z|^{2p}}\,dx\,dy \\ &\le C < \infty\end{aligned}$$

for all $w \in \Delta$. Now (1.3) is obtained by replacing $1-|w|$ with k^{-1} and $|w|^{n-m}$ with 1 provided $n-m \le k$. The remaining terms can be ignored. This completes the proof of Theorem 1.2. □

5. Merormorphic case. Since analytic functions f in the unit disk Δ are also meromorphic in Δ, we would expect that some of the characcterizations of Section 2 would carry over to the class of meromorphic functions, provided that the ordinary derivative of f is replaced by the spherical derivative $f^{\#}$, where

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}, \qquad z \in \Delta.$$

The purpose of this section is not only to generalize known results but also to point out here some of the difference between the results for the classes of analytic functions and the corresponding classes of meromorphic functions. Indeed, the analogue of Theorem 2.2 (ii) remains true in the meromorphic case, but the analogue of Theorem 2.2 (i) is not valid when $p > 1$.

We first recall the notation and terminology associated with the classes of meromorphic functions which correspond to the classes of analytic functions, $\mathcal{B}$, $\mathcal{B}_0$, BMOA, VMOA, Q_p and $Q_{p,0}$ introduced in Section 1. The set of functions meromorphic in Δ is denoted by M. The family of normal meromorphic functions in Δ is denoted by N and is defined

$$N = \{f \in M : \sup_{z\in\Delta}(1-|z|^2)f^{\#}(z) < \infty\}$$

[**15**]. The family of little normal (meromorphic) functions will be denoted by N_0 and is defined as

$$N_0 = \{f \in M : \lim_{|z|\to 1}(1-|z|^2)f^{\#}(z) = 0\}.$$

For $0 < p < \infty$ and $f \in M$, we say that $f \in Q_p^{\#}$ if

$$\sup_{w\in\Delta}\iint_{\Delta} f^{\#}(z)^2 g^p(z,w)\,dx\,dy < \infty$$

and that $f \in Q_{p,0}^{\#}$ if

$$\lim_{|w|\to 1}\iint_{\Delta} f^{\#}(z)^2 g^p(z,w)\,dx\,dy = 0.$$

In [**3**, Theorem 2 and Corollary 3] it was proved that $Q_p^{\#} = N$ and $Q_{p,0}^{\#} = N_0$ for all $p > 1$. For $p = 1$ it is well known that $Q_1^{\#} = \mathrm{UBC}$ (the functions of uniformly bounded characteristic) and $Q_{1,0}^{\#} = \mathrm{UBC}_0$ (the functions of uniformly vanishing characteristic) [**24**]. For $0 < p < 1$ we obtain, also in the meromorphic case, new subclasses of UBC, UBC_0, (or N, N_0) which, as far as we know, have not earlier appeared in the literature. In [**5**, Theorem 8] the nesting property of these classes was proved, that is, $Q_p^{\#} \subsetneq Q_q^{\#}$, $Q_{p,0}^{\#} \subsetneq Q_{q,0}^{\#}$ for $0 < p < q \leq 1$. In special case $q = 1$ we have the strict inclusions $Q_p^{\#} \subsetneq \mathrm{UBC}$, $Q_{p,0}^{\#} \subsetneq \mathrm{UBC}_0$ for $0 < p < 1$. The strict inclusions were shown by bounded Hadamard gap series. Thus, some bounded functions exist which do not belong to $Q_p^{\#}$ or $Q_{p,0}^{\#}$ for $0 < p < 1$.

Using the definition of bounded p-Carleson measures, Yamashita proved the following theorem:

Theorem Y [**25**, Theorem 2]. *If $f \in \mathrm{UBC}$, then $(f^{\#})^2(1-|z|)\,dx\,dy$ is a bounded 1-Carleson measure.*

Pavizevic showed that, in fact, the equivalence is true.

Theorem P [**16**, Theorem 3]. *Let $f \in M$. Then $f \in \mathrm{UBC}$ if and only if $(f^{\#})^2(1-|z|)\,dx\,dy$ is a bounded 1-Carleson measure.*

In [**5**] we showed the vanishing characteristic version of Pavizevic's result.

Theorem AXZ1 [**5**, Remark 4]. *Let $f \in M$. Then $f \in \mathrm{UBC}_0$ if and only if $(f^{\#})^2(1-|z|)\,dx\,dy$ is a compact 1-Carleson measure.*

Now we are going to generalize Theorems P and AXZ1 for any $Q_p^{\#}$, $Q_{p,0}^{\#}$, and any p, $0<p<1$. This combined with the above mentioned theorems will partially, for $0<p\le 1$, correspond to Theorem 2.2. Some clear differences appear as compared with the analytic case when we consider the $Q_p^{\#}$ classes for $1<p<\infty$.

Theorem 5.1. *Let $f \in M$. Then, for $0<p\le 1$, $f \in Q_p^{\#}$ if and only if $d\nu = (f^{\#})^2(1-|z|)^p\,dx\,dy$ is a bounded p-Carleson measure on Δ.*

Proof. From Lemma 2.1 it follows that $d\nu = (f^{\#})^2(1-|z|)^p\,dx\,dy$ is a bounded p-Carleson measure if and only if

$$\sup_{w\in\Delta}\iint_{\Delta} f^{\#}(z)^2\left(\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2}\right)^p dx\,dy < \infty.$$

Because the above integral equals $\iint_{\Delta} f^{\#}(z)^2(1-|\varphi_w(z)|^2)^p\,dx\,dy$, the assertion follows from [**5**, Proposition 2]. $\square$

For $1<p<\infty$ the meromorphic case differs from the analytic case, see Theorem 2.2 (i), as the following theorem shows.

Theorem 5.2. *Let $f \in M$. Then, for $1<p<\infty$, the following* *statement is not equivalent.* $f \in Q_p^{\#}$ *if and only if* $d\nu = (f^{\#})^2(1-|z|)^p\,dx\,dy$ *is a bounded* p*-Carleson measure on* Δ*.*

Proof. Because $Q_p^{\#} = N$ for all $1 < p < \infty$, see [**3**, Theorem 2], $d\nu$ being a bounded p-Carleson measure does not imply that $f \in Q_p^{\#}$ by [**6**, Example 2]. Thus, the theorem is proved. $\square$

In case of compact p-Carleson measures we have a more complete description which is similar to the analytic case.

Theorem 5.3. *Let* $f \in M$*. Then, for* $0 < p < \infty$*,* $f \in Q_{p,0}^{\#}$ *if and only if* $d\nu = (f^{\#})^2(1-|z|)^p\,dx\,dy$ *is a compact* p*-Carleson measure on* Δ*.*

Proof. If $f \in Q_{p,0}^{\#}$, then, by the inequality

$$1 - x^2 \le 2\log(1/x), \qquad 0 < x \le 1,$$

and Lemma 2.1, $d\nu$ is a compact p-Carleson measure on Δ.

Suppose now that $d\nu$ is a compact p-Carleson measure on Δ. Then, for any $w \in \Delta$ and any r, $0 < r < 1$, we have (we denote the pseudohyperbolic disk $\{z \in \Delta : |(w-z)/(1-\bar{w}z)| < r\}$ by $D(w,r)$)

$$\lim_{|w|\to 1} \iint_{D(w,r)} f^{\#}(z)^2\,dx\,dy = 0. \tag{1}$$

By [**24**, Lemma 3.2 (II)] (1) implies that $f \in N_0$. Then, for $w \in \Delta$, we have

$$\begin{aligned}
&\iint_{\Delta} f^{\#}(z)^2 g^p(z,w)\,dx\,dy \\
&\qquad = \left(\iint_{D(w,1/4)} + \iint_{\Delta\setminus D(w,1/4)}\right) f^{\#}(z)^2 g^p(z,w)\,dx\,dy \\
&\qquad \lesssim \left(\sup_{z\in D(w,1/4)} (1-|z|^2) f^{\#}(z)\right)^2 \\
&\qquad\quad \iint_{D(w,1/4)} (1-|z|^2)^{-2}\left(\log\frac{1}{|z|}\right)^p dx\,dy \\
&\qquad\quad + \iint_{\Delta} f^{\#}(z)^2\left(\frac{1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2}\right)^p dx\,dy,
\end{aligned}$$

which shows that $f \in Q^{\#}_{p,0}$. Hence, the proof is completed. $\square$

Remark. The spherical Dirichlet space $A\mathcal{D}_s(\Delta)$ is the set of functions $f \in M$ which satisfy

$$\iint_{\Delta} f^{\#}(z)^2 dxdy < \infty.$$

Yamashita has proved $A\mathcal{D}_s(\Delta) \subset \mathrm{UBC}_0$, cf., [**25**, Theorem 1]. However, there is a large gap between the classes $A\mathcal{D}_s(\Delta)$ and UBC_0 as the following theorem shows.

Theorem AXZ2 [**5**, Corollary 5].

$$A\mathcal{D}_s(\Delta) \subsetneq \bigcap_{0<p\leq 1} Q^{\#}_{p,0}.$$

Proof. We give a short proof of the inclusion relation applying Theorem 5.3. Let $f \in A\mathcal{D}_s(\Delta)$. Then, for $0 < p \leq 1$ and the Carleson box $S(I)$,

$$\begin{aligned}\nu(S(I)) &= \iint_{S(I)} f^{\#}(z)^2(1-|z|)^p\,dx\,dy \\ &\leq |I|^p \iint_{S(I)} f^{\#}(z)^2\,dx\,dy.\end{aligned}$$

By the assumption that $\iint_{S(I)} f^{\#}(z)^2\,dx\,dy \to 0$ as $|I| \to 0$. Hence, $d\nu$ is a compact p-Carleson measure, and by Theorem 5.3, $f \in Q^{\#}_{p,0}$. To prove the strictness of the inclusion, the same Hadamard gap series as in [**5**, Corollary 4] is used. Thus, the proof is completed. $\square$

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