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CONVOLUTION AND FOURIER-FEYNMAN TRANSFORMS

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ABSTRACT. In this paper, for a class of functionals on Wiener space of the form $F(x) = \exp\{\int_0^T f(t, x(t))\, dt\}$, we show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms. This allows us to compute the transform of the convolution product without computing the convolution product.

1. Introduction and preliminaries. The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [**1**]. In [**3**] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [**7**] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which extended the results in [**1, 3**] and gave various relationships between the L_1 and L_2 theories. In [**5**], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space and for a class of functionals of the type

$$F(x) = f\bigg(\int_0^T \alpha_1(t)\, dx(t), \dots, \int_0^T \alpha_n(t)\, dx(t)\bigg)$$

showed that the Fourier-Feynman transform of the convolution product was a product of Fourier-Feynman transforms.

In this paper we consider a class of functionals which play an important role in quantum mechanics, namely functionals of the form

$$F(x) = \exp\bigg\{\int_0^T f(t, x(t))\, dt\bigg\}$$

for appropriate $f : [0, T] \times \mathfrak{R} \to \mathbf{C}$. For any two functionals in the class, we first establish the existence of their convolution product. We then show that the Fourier-Feynman transform of their convolution is

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a product of their transforms. For specific functionals, this allows us to compute the transform of their convolution product without actually computing their convolution product. Finally, in Section 5 we discuss the associativity of the convolution product.

In [**3, 7**] all of the functionals F on Wiener space and all of the complex-valued functions f on $\mathfrak{R}^n$ were assumed to be Borel measurable. But, as was pointed out in [**8**, p. 170], the concept of scale-invariant measurability in Wiener space and Lebesgue measurability in $\mathfrak{R}^n$ is precisely correct for the analytic Fourier-Feynman theory.

Let $C_0[0,T]$ denote Wiener space; that is, the space of real-valued continuous functions x on $[0,T]$ such that $x(0)=0$. Let M denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote Wiener measure. $(C_0[0,T],M,m)$ is a complete measure space and we denote the Wiener integral of a functional F by

$$\int_{C_0[0,T]} F(x)m(dx).$$

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable [**4, 8**] provided $\rho E\in M$ for each $\rho>0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N)=0$ for each $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F\approx G$.

Let $\mathbf{C},\mathbf{C}_+$ and $\mathbf{C}_+^\sim$ denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let F be a $\mathbf{C}$-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$J(\lambda)=\int_{C_0[0,T]} F(\lambda^{-1/2}x)m(dx)$$

exists as a finite number for all $\lambda>0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}_+$ such that $J^*(\lambda)=J(\lambda)$ for all $\lambda>0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda\in\mathbf{C}_+$ we write

$$\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)=J^*(\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that $\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$ exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q, and we write

$$\int_{C_0[0,T]}^{anf_q} F(x)m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T)}^{anw_\lambda} F(x)m(dx)$$

where $\lambda \to -iq$ through $\mathbf{C}_+$.

Notation. i) For $\lambda \in \mathbf{C}_+$ and $y \in C_0[0,T]$, let

$$(T_\lambda(F))(y) = \int_{C_0[0,T]}^{anw_\lambda} F(x+y)m(dx). \tag{1.1}$$

ii) Given a number p with $1 \leq p \leq +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

iii) Let $1 < p \leq 2$, and let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C_0[0,T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0. \tag{1.2}$$

Then we write

$$\lim_{n\to\infty}(w_s^{p'})(H_n) \approx H \tag{1.3}$$

and we call H the scale invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ.

We are finally ready to state the definition of the L_p analytic Fourier-Feynman transform [**7**] and our definition of the convolution product.

Definition. Let $q \neq 0$ be a real number. For $1 < p \leq 2$ we define the *L_p analytic Fourier-Feynman transform* $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbf{C}_+$),

$$(T_q^{(p)}(F))(y) = \lim_{\lambda\to -iq}(w_s^{p'})(T_\lambda(F))(y) \tag{1.4}$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$(T_q^{(1)}(F))(y) = \lim_{\lambda \to -iq} (T_\lambda(F))(y) \tag{1.5}$$

for s-a.e. y. We note that, for $1 \le p \le 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

Definition. Let F and G be functionals on $C_0[0,T]$. For $\lambda \in \mathbf{C}_+^{\sim}$ we define their convolution product (if it exists) by

$$(1.6)\quad (F * G)_\lambda(y) \\ = \begin{cases} \displaystyle\int_{C_0[0,T]}^{anw_\lambda} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda \in \mathbf{C}_+ \\ \displaystyle\int_{C_0[0,T]}^{anf_q} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda = -iq,\ q \in \mathfrak{R} - \{0\}. \end{cases}$$

Remarks. i) When $\lambda = -iq$, we will denote $(F * G)_\lambda$ by $(F * G)_q$.

ii) Our definition of convolution is different than the definition given by Yeh in [**10**] and used by Yoo in [**11**]. For one thing, our convolution product is commutative, that is to say

$$(F * G)_\lambda(y) = (G * F)_\lambda(y).$$

In [**10**] and [**11**] Yeh and Yoo studied relationships between their convolution product and Fourier-Wiener transforms.

2. Transforms. First we describe the class of functionals $A = A_{pr}$ that we will be working with in this paper. For $1 \le p \le 2$ and $r \in (2p/(2p-1), \infty]$, let $L_{pr}([0,T] \times \mathfrak{R})$ be the space of all $\mathbf{C}$-valued Lebesgue measurable functions f on $[0,T] \times \mathfrak{R}$ such that $f(t,\cdot)$ is in $L_p(\mathfrak{R})$ for almost all $t \in [0,T]$ and as a function of t, $\|f(t,\cdot)\|_p$ is in $L_r([0,T])$. We define $A = A_{pr}$ to be the class of functionals F, such that for some $f \in L_{pr}([0,T] \times \mathfrak{R})$,

$$F(x) = \exp\left\{ \int_0^T f(t, x(t))\, dx \right\}. \tag{2.1}$$

Then, see [**8**, pp. 170–171], $F(x)$ is defined s-a.e. and is scale-invariant measurable.

Theorem 2.1. *Let $F \in A$ be given by (2.1) with $f \in L_{pr}([0,T]\times\mathfrak{R})$. Then $T_\lambda(F)$ exists for all $\lambda \in \mathbf{C}_+$ and is given by*

(2.2)
$$\begin{aligned}(T_\lambda(F))(y) &= 1+\sum_{n=1}^{\infty}\int_{\Delta_n*(T)}\int_{\mathfrak{R}^n}\prod_{j=1}^{n}\bigg[\bigg(\frac{\lambda}{2\pi(t_j-t_{j-1})}\bigg)^{1/2} \\ &\qquad \cdot f(t_j,u_j+y(t_j))\exp\bigg\{-\frac{\lambda(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\bigg\}\bigg]\,d\vec{u}\,d\vec{t} \\ &= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathfrak{R}^n}\prod_{j=1}^{n}\bigg[\bigg(\frac{\lambda}{2\pi(t_j-t_{j-1})}\bigg)^{1/2} \\ &\quad \cdot f(t_j,u_j)\exp\bigg\{-\frac{\lambda[(u_j-u_{j-1})-(y(t_j)-y(t_{j-1}))]^2}{2(t_j-t_{j-1})}\bigg\}\bigg]\,d\vec{u}\,d\vec{t}\end{aligned}$$

where $\Delta_n(T)=\{\vec{t}=(t_1,\dots,t_n)\in[0,T]^n : 0<t_1<t_2<\cdots<t_n\le T\}$, $\vec{u}=(u_1,\dots,u_n)$ and $t_0\equiv 0\equiv u_0$.

Proof. First note that

$$\begin{aligned}F(x) &= \exp\bigg\{\int_0^T f(t,x(t))\,dt\bigg\} \\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\bigg[\int_0^T f(t,x(t))\,dt\bigg]^n \\ &= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\prod_{j=1}^{n}[f(t_j,x(t_j))]\,d\vec{t}.\end{aligned}\tag{2.3}$$

Hence, for $\lambda>0$, using a well-known Wiener integration formula, we obtain

$$\begin{aligned}(T_\lambda(F))(y) &= \int_{C_0[0,T]} F(\lambda^{-1/2}x+y)m(dx) \\ &= 1+\int_{C_0[0,T]}\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\prod_{j=1}^{n}\bigg[\bigg(\frac{\lambda}{2\pi(t_j-t_{j-1})}\bigg)^{1/2} \\ &\qquad \cdot f(t_j,u_j+y(t_j))\exp\bigg\{-\frac{\lambda(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\bigg\}\bigg]\,d\vec{u}\,d\vec{t}.\end{aligned}\tag{2.4}$$

Finally, by analytic continuation in λ, we obtain that equation (2.2) holds throughout $\mathbf{C}_+$. □

Theorem 2.2. *Let $F \in A$ be as in Theorem 2.1. Then for all $p \in [1,2]$, the Fourier-Feynmann transform $T_q^{(p)}(F)$ exists for all real $q \neq 0$ and is given by the formula*

(2.5)
$$\begin{aligned}(T_q^{(p)}(F))(y) &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^{n} \bigg[\bigg(\frac{-qi}{2\pi(t_j - t_{j-1})} \bigg)^{1/2} \\ &\qquad \cdot f(t_j, u_j + y(t_j)) \exp\bigg\{ \frac{qi(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} \bigg\} \bigg] \, d\vec{u} \, d\vec{t} \\ &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^{n} \bigg[\bigg(\frac{-iq}{2\pi(t_j - t_{j-1})} \bigg)^{1/2} f(t_j, u_j) \\ &\qquad \cdot \exp\bigg\{ \frac{iq[(u_j - u_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{2(t_j - t_{j-1})} \bigg\} \bigg] \, d\vec{u} \, d\vec{t}\end{aligned}$$

Proof. By [**7**, Section 4] and [**8**, pp. 170–171], $T_q^{(p)}(F)$ exists for all real $q \neq 0$ and is scale-invariant measurable. Also note that because $\Phi(z) = e^z$ is an entire function of order 1 and because $f \in L_{pr}([0,T] \times \mathfrak{R})$, it was shown in [**7**] that, for any $q \in \mathfrak{R}$,

$$\sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} |q|^{n/2} \prod_{j=1}^{n} \bigg[\bigg(\frac{1}{2\pi(t_j - t_{j-1})} \bigg)^{1/2} |f(t_j, u_j)| \bigg] \, d\vec{u} \, d\vec{t} < \infty.$$

Thus, for all $y \in C_0[0,T]$, the series on the righthand side of equation (2.5) converges absolutely (and uniformly in q on compact subsets of $\mathfrak{R} - \{0\}$). Furthermore the series converges in the $L_p(C_0[0,T])$ mean. Thus, our representation (2.5) for $T_q^{(p)}(F)$ follows from equation (2.2). □

3. Convolutions. In our first theorem, with F and G in $A = A_{pr}$, we obtain a series representation for the convolution product of F and G.

Theorem 3.1. *Let $F \in A$ be given by (2.1), and let $G \in A$ be given by*

$$G(x) = \exp\left\{ \int_0^T g(t, x(t))\, dt \right\} \tag{3.1}$$

with $g \in L_{pr}([0,T] \times \mathfrak{R})$. Then their convolution product $(F*G)_\lambda$ exists for all $\lambda \in \mathbf{C}_+$ and is given by

$$\begin{aligned}(F*G)_\lambda(y) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^{n} \bigg[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{1/2} \\ \cdot \exp\left\{ - \frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} \right\} \bigg\{ f\left(t_j, \frac{y(t_j) + u_j}{\sqrt{2}} \right) \\ + g\left(t_j, \frac{y(t_j) - u_j}{\sqrt{2}} \right) \bigg\} \bigg] \, d\vec{u}\, d\vec{t}.\end{aligned} \tag{3.2}$$

Proof. For $\lambda > 0$, using a well-known Wiener integration formula, we obtain

$$\begin{aligned}(F*G)_\lambda(y) &= \int_{C_0[0,T]} F\left(\frac{y + \lambda^{-1/2} x}{\sqrt{2}} \right) G\left(\frac{y - \lambda^{-1/2} x}{\sqrt{2}} \right) m(dx) \\ &= \int_{C_0[0,T]} \exp\left\{ \int_0^T f\left(t, \frac{y(t) + \lambda^{-1/2} x(t)}{\sqrt{2}} \right) dt \right\} \\ &\qquad \cdot \exp\left\{ \int_0^T g\left(t, \frac{y(t) - \lambda^{-1/2/} x(t)}{\sqrt{2}} \right) dt \right\} m(dx) \\ &= \int_{C_0[0,T]} \exp\left\{ \int_0^T \left[f\left(t, \frac{y(t) + \lambda^{-1/2} x(t)}{\sqrt{2}} \right) \right.\right. \\ &\qquad \left.\left. + g\left(t, \frac{y(t) - \lambda^{-1/2} x(t)}{\sqrt{2}} \right) \right] dt \right\} m(dx) \\ &= \int_{C_0[0,T]} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_0^T \left[f\left(t, \frac{y(t) + \lambda^{-1/2} x(t)}{\sqrt{2}} \right) \right.\right.\right. \\ &\qquad \left.\left.\left. + g(t, \frac{y(t) - \lambda^{-1/2} x(t)}{\sqrt{2}}) \right] dt \right]^n \right\} m(dx)\end{aligned}$$

$$= 1 + \sum_{n=1}^{\infty} \int_{C_0[0,T]} \int_{\Delta_n(T)} \prod_{j=1}^{n} \bigg[f\bigg(t_j, \frac{y(t_j) + \lambda^{-1/2}x(t_j)}{\sqrt{2}}\bigg) + g\bigg(t_j, \frac{y(t_j) - \lambda^{-1/2}x(t_j)}{\sqrt{2}}\bigg)\bigg] d\vec{t} m(dx)$$

$$= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\Re^n} \prod_{j=1}^{n} \bigg[\bigg(\frac{1}{2\pi(t_j - t_{j-1})}\bigg)^{1/2} \cdot \exp\bigg\{ - \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\bigg\} \cdot \bigg\{ f\bigg(t_j, \frac{y(t_j) + \lambda^{-1/2}u_j}{\sqrt{2}}\bigg) + g\bigg(t_j, \frac{y(t_j) - \lambda^{-1/2}u_j}{\sqrt{2}}\bigg)\bigg\}\bigg] d\vec{u}\, d\vec{t}$$

$$= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\Re^n} \prod_{j=1}^{n} \bigg[\bigg(\frac{\lambda}{2\pi(t_j - t_{j-1})}\bigg)^{1/2} \cdot \exp\bigg\{ - \frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\bigg\} \cdot \bigg\{ f\bigg(t_j, \frac{y(t_j) + u_j}{\sqrt{2}}\bigg) + g\bigg(t_j, \frac{y(t_j) - u_j}{\sqrt{2}}\bigg)\bigg\}\bigg] d\vec{u}\, d\vec{t}.$$

Now, by analytic continuation in λ, we obtain that equation (3.2) is valid throughout $\mathbf{C}_+$. □

In our next theorem we obtain a series representation for $T_\lambda(F * G)_\lambda$.

Theorem 3.2. *Let f, g, F and G be as in Theorem* 3.1. *Then, for all $\lambda \in \mathbf{C}_+, T_\lambda(F * G)_\lambda$ exists and is given by the formula*

(3.3)

$$(T_\lambda(F * G)_\lambda)(z) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\Re^{2n}} \prod_{j=1}^{n} \bigg[\{ f(t_j, v_j + z(t_j)/\sqrt{2}) + g(t_j, r_j + z(t_j)/\sqrt{2})\} \bigg(\frac{\lambda}{2\pi(t_j - _{j-1})}\bigg) \cdot \exp\bigg\{ - \frac{\lambda(v_j - v_{j-1})^2}{2(t_j - t_{j-1})} - \frac{\lambda(r_j - r_{j-1})^2}{2(t_j - t_{j-1})}\bigg\}\bigg] d\vec{r}\, d\vec{v}\, d\vec{t}.$$

Proof. For $\lambda > 0$, using (3.2) we see that

$$
\begin{aligned}
(T_\lambda(F*G)_\lambda)(z) &= \int_{C_0[0,T]} (F*G)_\lambda(\lambda^{-1/2}x+z)m(dx) \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathfrak{R}^n}\int_{C_0[0,T]}\prod_{j=1}^{n}\bigg[\bigg(\frac{\lambda}{2\pi(t_j-t_{j-1})}\bigg)^{1/2} \\
&\quad\cdot\exp\bigg\{-\frac{\lambda(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\bigg\} \\
&\quad\cdot\bigg\{f\bigg(t_j,\frac{\lambda^{-1/2}x(t_j)+z(t_j)+u_j}{\sqrt{2}}\bigg) \\
&\qquad+g\bigg(t_j,\frac{\lambda^{-1/2}x(t_j)+z(t_j)-u_j}{\sqrt{2}}\bigg)\bigg\}m(dx)\bigg]\,d\vec{u}\,d\vec{t} \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathfrak{R}^{2n}}\prod_{j=1}^{n}\bigg[\bigg(\frac{\lambda}{2\pi(t_j-t_{j-1})}\bigg) \\
&\quad\cdot\exp\bigg\{-\frac{\lambda(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}-\frac{\lambda(w_j-w_{j-1})^2}{2(t_j-t_{j-1})}\bigg\} \\
&\quad\cdot\bigg\{f\bigg(t_j,\frac{z(t_j)+w_j+u_j}{\sqrt{2}}\bigg) \\
&\qquad\qquad+g\bigg(t_j,\frac{z(t_j)+w_j-u_j}{\sqrt{2}}\bigg)\bigg\}\bigg]\,d\vec{w}\,d\vec{u}\,d\vec{t}.
\end{aligned}
$$

Next, in the above expression we make the substitutions

$$v_j=\frac{w_j+u_j}{\sqrt{2}} \quad\text{and}\quad r_j=\frac{w_j-u_j}{\sqrt{2}}$$

for $j=1,2,\dots,n$. The Jacobian of this transformation is one and for $j=1,\dots,n$, we have that

$$(u_j-u_{j-1})^2+(w_j-w_{j-1})^2=(v_j-v_{j-1})^2+(r_j-r_{j-1})^2.$$

Thus, for $\lambda > 0$,

$$
\begin{aligned}
(T_\lambda(F*G)_\lambda)(z) &= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathfrak{R}^{2n}}\prod_{j=1}^{n}\bigg[\bigg(\frac{\lambda}{2\pi(t_j-t_{j-1})}\bigg) \\
&\quad\cdot\exp\bigg\{-\frac{\lambda(v_j-v_{j-1})^2}{(2(t_j-t_{j-1})}-\frac{\lambda(r_j-r_{j-1})^2}{2(t_j-t_{j-1})}\bigg\} \\
&\quad\cdot\bigg\{f\bigg(t_j,v_j+\frac{z(t_j)}{\sqrt{2}}\bigg)+g\bigg(t_j,r_j+\frac{z(t_j)}{\sqrt{2}}\bigg)\bigg\}\bigg]\,d\vec{r}\,d\vec{v}\,d\vec{t}.
\end{aligned}
$$

Finally, by analytic continuation in λ, we obtain that equation (3.3) is valid through $\mathbf{C}_+$. $\square$

4. Transforms of convolutions.

Lemma 4.1. *If $T_\lambda(F)$, $T_\lambda(G)$ and $T_\lambda(F*G)_\lambda$ exist for $\lambda>0$, then*

$$(T_\lambda(F*G)_\lambda)(z) = (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}). \tag{4.1}$$

Proof. For $\lambda>0$, using (1.1) and (1.6) we see that

$$\begin{aligned}(T_\lambda(F*G)_\lambda)(z) &= \int_{C_0[0,T]}(F*G)_\lambda(z+\lambda^{-1/2}y)m(dy)\\ &= \int_{C_0^2[0,T]} F(z/\sqrt{2}+\lambda^{-1/2}(y+x)/\sqrt{2})\\ &\qquad\cdot G(z/\sqrt{2}+\lambda^{-1/2}(y-x)/\sqrt{2})m(dy)m(dx).\end{aligned}$$

But $w_1=(y+x)/\sqrt{2}$ and $w_2=(y-x)/\sqrt{2}$ are independent standard Wiener processes and $m\times m$ is rotation invariant in $C_0^2[0,T]$, and hence,

$$\begin{aligned}(T_\lambda(F*G)_\lambda)(z) &= \int_{C_0^2[0,T]} F(z/\sqrt{2}+\lambda^{-1/2}w_1)\\ &\qquad\cdot G(z/\sqrt{2}+\lambda^{-1/2}w_2)m(dw_1)m(dw_2)\\ &= \int_{C_0[0,T]} F(z/\sqrt{2}+\lambda^{-1/2}w_1)m(dw_1)\\ &\qquad\cdot\int_{C_0[0,T]} G(z/\sqrt{2}+\lambda^{-1/2}w_2)m(dw_2)\\ &= (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}).\end{aligned}$$

As a corollary to Lemma 4.1, we obtain an interesting relationship involving convolutions and analytic Wiener integrals. $\square$

Theorem 4.1. *Let F and G be as in Theorem 3.1. Then for all $\lambda\in\mathbf{C}_+$,*

$$(T_\lambda(F*G)_\lambda)(z) = (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}). \tag{4.2}$$

Proof. By Lemma 4.1, equation (4.2) holds for all $\lambda > 0$. The result now follows because $T_\lambda(F), T_\lambda(G)$ and $T_\lambda(F * G)_\lambda$ all have analytic extensions throughout $\mathbf{C}_+$. □

Our next theorem shows that the Fourier-Feynman transform of the convolution product is the product of their transforms.

Theorem 4.2. *Let f, g, F and G be as in Theorem 3.1. Then, for all real $q \neq 0$,*

$$(T_q^{(p)}(F * G)_q)(z) = (T_q^{(p)}(F))(z/\sqrt{2})(T_q^{(p)}(G))(z/\sqrt{2}) \tag{4.3}$$

for $1 \leq p \leq 2$.

Proof. As was noted in the proof of Theorem 2.2, the series expansions for $(T_q^{(p)}(F))(z/\sqrt{2})$ and $(T_q^{(p)}(G))(z/\sqrt{2})$ both converge absolutely for all $z \in C_0[0,T]$. Hence the righthand side of (4.3) is a bounded continuous function of λ on $\mathbf{C}_+^-$ for all $z \in C_0[0,T]$. Hence, using equation (4.1), $T_q^{(p)}(F * G)_q$ exists and is given by equation (4.3) for all desired values of p and q. □

Corollary. *Let f and F be as in Theorem 2.1. Then for all real $q \neq 0$,*

$$(T_q^{(p)}(F * F)_q)(z) = [(T_q^{(p)}(F))(z/\sqrt{2})]^2$$

for $1 \leq p \leq 2$.

Remark. Formula (4.3) is useful in that it permits one to calculate $T_q^{(p)}(F * G)_q$ without actually calculating $(F * G)_q$. In practice, $T_\lambda(F)$ and $T_\lambda(G)$ are usually easier to calculate than are $(F * G)_\lambda$ and $T_\lambda(F * G)_\lambda$.

Examples. We finish this section by finding $T_q^{(p)}(F_j * F_k)_q$ for various functionals on Wiener space $C_0[0,T]$. In view of equation (4.3), we need only compute the transforms of the various functionals F_j. The results are summarized below where the expressions for $T_\lambda(F_j)$ are valid for all $\lambda \in \mathbf{C}_+^\sim$ unless otherwise indicated.

Let α be any real function in $L_2[0,T]$, and let $c \in \mathbf{C}$. Then

TABLE 1.

$F_j(x)$	$T_\lambda(F_j)(z)$
$F_1(x) = 1$	1
$F_2(x) = \int_0^T \alpha(t)\,dx(t)$	$\int_0^T \alpha(t)\,dz(t)$
$F_3(x) = \int_0^T x^2(t)\,dt$	$\int_0^T z^2(t)\,dt + \frac{T^2}{(2\lambda)}$
$F_4(x) = [\int_0^T \alpha(t)\,dx(t)]^2$	$\int_0^T \alpha(t)\,dz(t)]^2 + \frac{\\|\alpha\\|^2}{\lambda}$
$F_5(x) = \exp\{c\int_0^T \alpha(t)\,dx(t)\}$	$\exp\{c\int_0^T \alpha(t)\,dz(t) + \frac{c^2\\|\alpha\\|^2}{(2\lambda)}\}$
$F_6(x) = \exp\{c[\int_0^T \alpha(t)\,dx(T)]^2\}$	$\sqrt{\frac{\lambda}{(\lambda-2c\\|\alpha\\|^2)}}\exp\{(\frac{c\lambda}{(\lambda-2c\\|\alpha\\|^2)})(\int_0^T \alpha(t)\,dz(t))^2\}$
	provided $\mathrm{Re}\,(c/\lambda) < (2\\|\alpha\\|^2)^{-1}$
$F_7(x) = \exp\{c\int_0^T x(t)\,dt\}$	$\exp\{c\int_0^T z(t)\,dt + \frac{c^2T^3}{(6\lambda)}\}$

Now, using Table 1, together with equation (4.3), one can immediately compute $T_q^{(p)}(F_j * F_k)_q$ for $j,k \in \{1,2,\dots,7\}$. For example,

$$(T_q^{(p)}(F_3 * F_7)_q)(z) = \left[\int_0^T \frac{z^2(t)dt}{2} + \frac{iT^2}{2q}\right]\exp\left\{c\int_0^T \frac{z(t)dt}{\sqrt{2}} + \frac{ic^2T^3}{6q}\right\}.$$

5. Associativity of the convolution product. We finish this paper with a brief discussion of the associativity of the convolution product. Recall that the convolution product is commutative; that is to say, $(F * G)_\lambda = (G * F)_\lambda$ for all F, G and λ.

In the following we will restrict our attention to the case $\lambda > 0$; extensions to $\mathbf{C}_+$ follow by analytic continuation in λ. Furthermore, we will assume that the scale-invariant measurable functionals F, G, H and K are nice enough so that their various transforms and convolution products all exist throughout $\mathbf{C}_+^{\sim}$. Then $T_\lambda(F) \approx T_\lambda(G)$ implies that $F \approx G$.

i) In general the convolution product defined by equation (1.6) is not associative; that is to say,

$$((F * G)_\lambda * H)_\lambda \neq (F * (G * H)_\lambda)_\lambda$$

since, by Lemma 4.1,

$$\begin{aligned}(T_\lambda((F * (G * H)_\lambda)_\lambda))(z) &= (T_\lambda(F))(z/\sqrt{2}) \\ &\quad \cdot (T_\lambda(G))(z/2)(T_\lambda(H))(z/2) \\ &\neq (T_\lambda(F))(z/2) \\ &\quad \cdot (T_\lambda(G))(z/2)(T_\lambda(H))(z/\sqrt{2}) \\ &= (T_\lambda(((F * G)_\lambda * H)_\lambda))(z).\end{aligned}$$

In particular, for $G(z) \equiv H(z) \equiv 1$ on $C_0[0,T]$,

$$\begin{aligned}(T_\lambda((F * (1 * 1)_\lambda)_\lambda))(z) &= (T_\lambda(F))(z/\sqrt{2}) \\ &\neq (T_\lambda(F))(z/2) \\ &= (T_\lambda(((F * 1)_\lambda * 1)_\lambda))(z).\end{aligned}$$

ii) However, if for each functional F on $C_0[0,T]$, we define the functional F_λ^* on $C_0[0,T]$ by

$$F_\lambda^*(z) = (1 * F)_\lambda(z), \tag{5.1}$$

then we have the associativity result

$$(F_\lambda^* * (G * H)_\lambda)_\lambda = ((F * G)_\lambda * H_\lambda^*)_\lambda \tag{5.2}$$

since, by equation (4.1),

$$\begin{aligned}(T_\lambda((F_\lambda^* * (G * H)_\lambda)_\lambda))(z) &= (T_\lambda(F))(z/2) \\ &\quad \cdot (T_\lambda(G))(z/2)(T_\lambda(H))(z/2) \\ &= (T_\lambda(((F * G)_\lambda * H_\lambda^*)_\lambda))(z).\end{aligned}$$

iii) Next, if we define F_λ^{**} on $C_0[0,T]$ by

$$F_\lambda^{**}(z) = (1 * F_\lambda^*)_\lambda(z), \tag{5.3}$$

then we have the associativity result

$$(((F * G)_\lambda * H_\lambda^*)_\lambda * K_\lambda^{**})_\lambda = (F_\lambda^{**} * ((G * H)_\lambda * K_\lambda^*)_\lambda)_\lambda \tag{5.4}$$

since, by equation (4.1),

$$\begin{aligned}(T_\lambda((((F * G)_\lambda * H_\lambda^*)_\lambda * K_\lambda^{**})_\lambda))(z) &\\ &= (T_\lambda(F))(z/(2\sqrt{2}))(T_\lambda(G))(z/(2\sqrt{2}))\\ &\quad \cdot (T_\lambda(H))(z/(2\sqrt{2}))(T_\lambda(K))(z/(2\sqrt{2}))\\ &= (T_\lambda((F_\lambda^{**} * ((G * H)_\lambda * K_\lambda^*)_\lambda)_\lambda))(z).\end{aligned}$$

Remark. iv) Both sides of (5.2) are, of course, also equal to $(G_\lambda^* * (F * H)_\lambda)_\lambda$, while both sides of (5.4) are also equal to

$$(G_\lambda^{**} * ((F * K)_\lambda * H_\lambda^*)_\lambda)_\lambda$$

as well as nine other similar expressions.

v) The procedures used in ii) and iii) above can easily be extended to more factors by defining $F_\lambda^{***}(z) = (1 * F_\lambda^{**})_\lambda(z)$, $F_\lambda^{****}(z) = (1 * F_\lambda^{***})_\lambda(z)$, etc.

REFERENCES

1. M.D. Brue, *A functional transform for Feynman integrals similar to the Fourier transform*, Thesis, University of Minnesota, 1972.

2. R.H. Cameron and D.A. Storvick, *An operator valued function space integral and a related integral equation*, J. Math. Mech. **18** (1968), 517–552.

3. ———, *An L_2 analytic Fourier-Feynman transform*, Michigan Math. J. **23** (1976), 1–30.

4. K.S. Chang, *Scale-invariant measurability in Yeh-Wiener space*, J. Korean Math. Soc. **19** (1982), 61–67.

5. T. Huffman, C. Park and D. Skoug, *Analytic Fourier-Feynman transforms and convolution*, Trans. Amer. Math. Soc. **347** (1995), 661–673.

6. G.W. Johnson and D.L. Skoug, *The Cameron-Storvick function space integral: An $L(L_p, L_{p'})$ theory*, Nagoya Math. J. **60** (1976), 93–137.

7. ———, *An L_p analytic Fourier-Feynman transform*, Michigan Math. J. **26** (1979), 103–127.

8. ———, *Scale-invariant measurability in Wiener space*, Pacific J. Math. **83** (1979), 157–176.

9. E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, NJ, 1971.

10. J. Yeh, *Convolution in Fourier-Wiener transform*, Pacific J. Math. **15** (1965), 731–738.

11. I. Yoo, *Convolution in Fourier-Wiener transform on abstract Wiener space*, to appear in Rocky Mountain Math. J.

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