ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 4, Fall 1997

SOME REMARKS ON THE DUNFORD-PETTIS PROPERTY

NARCISSE RANDRIANANTOANINA

ABSTRACT. Let A be the disk algebra, Ω be a compact Hausdorff space and μ be a Borel measure on Ω. It is shown that the dual of $C(\Omega, A)$ has the Dunford-Pettis property. This proved in particular that the spaces $L^1(\mu, L^1/H_0^1)$ and $C(\Omega, A)$ have the Dunford-Pettis property.

1. Introduction. Let E be a Banach space, Ω be a compact Hausdorff space and μ be a finite Borel measure on Ω. We denote by $C(\Omega, E)$ the space of all E-valued continuous functions from Ω and for $1 \leq p < \infty$, $L^p(\mu, E)$ stands for the space of all (class of) E-valued p-Bochner integrable functions with its usual norm. A Banach space E is said to have the Dunford-Pettis property if every weakly compact operator with domain E is completely continuous, i.e., takes weakly compact sets into norm compact subsets of the range space. There are several equivalent definitions. The basic result proved by Dunford and Pettis in [**11**] is that the space $L^1(\mu)$ has the Dunford-Pettis property. A. Grothendieck [**12**] initiated the study of Dunford-Pettis property in Banach spaces and showed that $C(K)$-spaces have this property. The Dunford-Pettis property has a rich history; the survey articles by J. Diestel [**8**] and A. Pełczyński [**15**] are excellent sources of information. In [**8**] it was asked if the Dunford-Pettis property can be lifted from a Banach E to $C(\Omega, E)$ or $L^1(\mu, E)$. M. Talagrand [**18**] constructed counterexamples for these questions so the answer is negative in general. There are, however, some positive results. For instance, J. Bourgain showed (among other things) in [**2**] that $C(\Omega, L^1)$ and $L^1(\mu, C(\Omega))$ both have the Dunford-Pettis property; K. Andrews [**1**] proved that if E^* has the Schur property then $L^1(\mu, E)$ has the Dunford-Pettis property. F. Delbaen [**7**] showed that if A is the disc algebra, then $L^1(\mu, A)$ has the Dunford-Pettis property. In [**17**], E. Saab and P. Saab observed that if $\mathcal{A}$ is a C^*-algebra with the Dunford-Pettis property then $C(\Omega, \mathcal{A})$

Received by the editors on December 31, 1994.
1991 AMS *Mathematics Subject Classification.* 46E40, 46E25.
Key words and phrases. Bochner spaces, Dunford-Pettis property, weakly compact sets.

has the Dunford-Pettis property and they asked, see [**17**, Question 14], if a similar result holds if one considers the disk algebra A. In this note we provide a positive answer to the above question by showing that the dual of $C(\Omega, A)$ has the Dunford-Pettis property. This implies in particular that both $L^1(\mu, L^1/H_0^1)$ and $C(\Omega, A)$ have the Dunford-Pettis property. Our approach is to study a "Random version" of the minimum norm lifting from L^1/H_0^1 into L^1.

The notation and terminology used and not defined in this note can be found in [**9**] and [**10**].

2. Minimum norm lifting. Let us begin by fixing some notations. Throughout, m denotes the normalized Haar measure on the circle $\mathbf{T}$. The space H_0^1 stands for the space of integrable functions on $\mathbf{T}$ such that $\hat{f}(n) = \int_{\mathbf{T}} f(\theta)e^{-in\theta}dm(\theta) = 0$ for $n \leq 0$.

It is a well-known fact that $A^* = L^1/H_0^1 \oplus_1 M_S(\mathbf{T})$ where $M_S(\mathbf{T})$ is the space of singular measures on $\mathbf{T}$ (see, for instance, [**15**]). Consider the quotient map $q : L^1 \to L^1/H_0^1$. This map has the following important property: for each $x \in L^1/H_0^1$, there exists a unique $f \in L^1$ so that $q(f) = x$ and $\|f\| = \|x\|$. This fact provides a well-defined map called the minimum norm lifting

$$\sigma : L^1/H_0^1 \rightsquigarrow L^1 \quad \text{s.t.} \quad q(\sigma(x)) = x \quad \text{and} \quad \|\sigma(x)\| = \|x\|.$$

One of the many important features of σ is that it preserves weakly compact subsets, namely, the following was proved in [**15**].

Proposition 1. *If K is a relatively weakly compact subset of L^1/H_0^1, then $\sigma(K)$ is relatively weakly compact in L^1.*

Our goal in this section is to extend the minimum norm lifting to certain classes of spaces that contains L^1/H_0^1. In particular, we will introduce a random-version of the minimum norm lifting.

First we will extend the minimum norm lifting to A^*.

We define a map $\gamma : L^1/H_0^1 \oplus_1 M_s(\mathbf{T}) \rightsquigarrow L^1 \oplus_1 M_s(\mathbf{T})$ as follows:

$$\gamma(\{x, s\}) = \{\sigma(x), s\}.$$

Clearly γ defines a minimum norm lifting from A^* into $M(\mathbf{T})$.

In order to proceed to the next extension, we need the following proposition.

Proposition 2. *Let σ and γ be as above. Then*

a) $\sigma : L^1/H^1_0 \rightsquigarrow L^1$ *is norm-universally measurable, i.e., the inverse image of every norm Borel subset of L^1 is norm universally measurable in L^1/H^1_0;*

b) $\gamma : A^* \rightsquigarrow M(\mathbf{T})$ *is weak*-universally measurable, i.e., the inverse image of every weak*-Borel subset of $M(\mathbf{T})$ is weak*-universally measurable in A^*.*

Proof. For a), notice that L^1/H^1_0 and L^1 are Polish spaces (with the norm topologies) and so is the product $L^1 \times L^1/H^1_0$. Consider the following subset of $L^1 \times L^1/H^1_0$:

$$\mathcal{A} = \{(f, x); q(f) = x, \|f\| = \|x\|\}.$$

The set $\mathcal{A}$ is a Borel subset of $L^1 \times L^1/H^1_0$. In fact, $\mathcal{A}$ is the intersection of the graph of q, which is closed, and the subset $\mathcal{A}_1 = \{(f, x), \|f\| = \|x\|\}$ which is also closed. Let π be the restriction on $\mathcal{A}$ of the second projection of $L^1 \times L^1/H^1_0$ onto L^1/H^1_0. The operator π is of course continuous and hence $\pi(\mathcal{A})$ is analytic. By Theorem 8.5.3 of [**6**], there exists a universally measurable map $\phi : \pi(\mathcal{A}) \to L^1$ whose graph belongs to $\mathcal{A}$. The existence and the uniqueness of the minimum norm lifting imply that $\pi(\mathcal{A}) = L^1/H^1_0$ and ϕ must be σ.

The proof of b) is done with a similar argument using the fact that A^* and $M(\mathbf{T})$ with the weak* topologies are countable reunions of Polish spaces, and their norms are weak*-Borel measurable. The proposition is proved. $\square$

Let (Ω, Σ, μ) be a probability space. For a measurable function $f : \Omega \to L^1/H^1_0$, the function $\omega \mapsto \sigma(f(\omega))$ $(\Omega \to L^1)$ is μ-measurable by Proposition 2. We define an extension of σ on $L^1(\mu, L^1/H^1_0)$ as follows:

$$\tilde{\sigma} : L^1(\mu, L^1/H^1_0) \rightsquigarrow L^1(\mu, L^1) \quad \text{with} \quad \tilde{\sigma}(f)(\omega) = \sigma(f(\omega)) \quad \text{for } \omega \in \Omega.$$

The map $\tilde{\sigma}$ is well defined and $\|\tilde{\sigma}(f)\| = \|f\|$ for each $f \in L^1(\mu, L^1/H^1_0)$. Also, if we denote by $\tilde{q} : L^1(\mu, L^1) \to L^1(\mu, L^1/H^1_0)$, the map $\tilde{q}(f)(\omega) = q(f(\omega))$, we get that $\tilde{q}(\tilde{\sigma}(f)) = f$.

Similarly, if $f:\Omega\to A^*$ is weak*-scalarly measurable, the function $\omega\mapsto\gamma(f(\omega))$, $\Omega\to M(\mathbf{T})$, is weak*-scalarly measurable. As above, we define $\tilde{\gamma}$ as follows. For each measure $G\in M(\Omega,A^*)$, fix $g:\Omega\to A^*$ its weak*-density with respect to its variation $|G|$. We define

$$\tilde{\gamma}(G)(A)=\text{weak}^*-\int_A\gamma(g(\omega))\,d|G|(\omega)\quad\text{for all } A\in\Sigma.$$

Clearly $\tilde{\gamma}(G)$ is a measure and it is easy to check that $\|\tilde{\gamma}(G)\|=\|G\|$, in fact $|\tilde{\gamma}(G)|=|G|$.

The rest of this section is devoted to the proof of the following result that extends the property of σ stated in Proposition 1 to $\tilde{\sigma}$.

Theorem 1. *Let K be a relatively weakly compact subset of $L^1(\mu,L^1/H^1_0)$. The set $\tilde{\sigma}(K)$ is relatively weakly compact in $L^1(\mu,L^1)$.*

We will need a few general facts for the proof. In the sequel, we will identify, for a given Banach space F, the dual of $L^1(\mu,F)$ with the space $L^\infty(\mu,F^*_\sigma)$ of all maps h from Ω to F^* that are weak*-scalarly measurable and essentially bounded with the uniform norm, see [**14**].

Definition 1. Let E be a Banach space. A series $\sum_{n=1}^\infty x_n$ in E is said to be *weakly unconditionally Cauchy* (WUC) if, for every $x^*\in E^*$, the series $\sum_{n=1}^\infty|x^*(x_n)|$ is convergent.

The following lemma is well known.

Lemma 1. *If S is a relatively weakly compact subset of a Banach space E, then for every* WUC *series $\sum_{n=1}^\infty x^*_n$ in E^*, $\lim_{m\to\infty}x^*_n(x)=0$ uniformly on S.*

The following proposition which was essentially proved in [**16**] is the main ingredient for the proof of Theorem 1. For what follows $(e_n)_n$ denote the unit vector basis of c_0 and (Ω,Σ,μ) is a probability space.

Proposition 3 [**16**]. *Let Z be a subspace of a real Banach space E and $(f_n)_n$ be a sequence of maps from Ω to E that are measurable and $\sup_n\|f_n\|_\infty\le 1$. Let $a<b$ (real numbers), then:*

There exist a sequence $g_n \in \operatorname{conv}\{f_n, f_{n+1}, \dots\}$ *measurable subsets* C *and* L *of* Ω *with* $\mu(C \cup L) = 1$ *such that*

(i) *If* $\omega \in C$ *and* $T \in \mathcal{L}(E/Z, \ell^1)$, $\|T\| \leq 1$; *then, for each* $h_n \in \operatorname{conv}\{g_n, g_{n+1}, \dots\}$, *either* $\limsup_{n\to\infty}\langle h_n(\omega), T^* e_n\rangle \leq b$ *or* $\liminf_{n\to\infty}\langle h_n(\omega), T^* e_n\rangle \geq a$;

(ii) $\omega \in L$, *there exists* $k \in \mathbf{N}$ *so that for each infinite sequence of zeros and ones* Γ, *there exists* $T \in \mathcal{L}(E/Z, \ell^1)$, $\|T\| \leq 1$ *such that, for* $n \geq k$,

$$\Gamma_n = 1 \Longrightarrow \langle g_n(\omega), T^* e_n\rangle \geq b$$
$$\Gamma_n = 0 \Longrightarrow \langle g_n(\omega), T^* e_n\rangle \leq a.$$

Proof. Let $\pi : E \to E/Z$ be the quotient map. Let $K_0 := \{T \circ \pi; T \in \mathcal{L}(E/Z, \ell^1)_1\}$. The set K_0 is clearly a weak*-closed subset of $\mathcal{L}(E, \ell^1)_1$. The proposition is obtained by applying to the sequence $(f_n)_n$ the construction used in the proof of Theorem 1 of [**16**] starting from $K_0(\omega) = K_0$ defined above. $\square$

We will also make use of the following fact:

Lemma 2 [**15**, p. 45]. *Let* $(U_n)_n$ *be a bounded sequence of positive elements of* $L^1(\mathbf{T})$. *If* $(U_n)_n$ *is not uniformly integrable, then there exists a* WUC *series* $\sum_{l=1}^{\infty} a_l$ *in the disk algebra* A *such that* $\limsup_{l\to\infty}\sup_n |\langle a_l, U_n\rangle| > 0$.

Proof of Theorem 1. Assume without loss of generality that K is a bounded subset of $L^\infty(\mu, L^1/H_0^1)$. The set $\tilde{\sigma}(K)$ is a bounded subset of $L^\infty(\mu, L^1(\mathbf{T}))$. Let $|\tilde{\sigma}(K)| = \{|\tilde{\sigma}(f)|; f \in K\}$. Notice that for each $f \in L^1(\mu, L^1/H_0^1)$, there exists $h \in L^\infty(\mu, H^\infty_\sigma) = L^1(\mu, L^1/H_0^1)^*$ with $\|h\| = 1$ and $|\tilde{\sigma}(f)(\omega)| = \tilde{\sigma}(f)(\omega).h(\omega)$ (the multiplication of the function $\tilde{\sigma}(f)(\omega) \in L^1(\mathbf{T})$ with the function $h(\omega) \in H^\infty(\mathbf{T})$) for almost every $\omega \in \Omega$.

Consider $\varphi_n = |\tilde{\sigma}(f_n)|$ to be a sequence of $L^1(\mu, L^1(\mathbf{T}))$ with $(f_n)_n \subset K$, and choose $(h_n)_n \in L^\infty(\mu, H^\infty_\sigma)$ so that $\varphi_n(\omega) = \tilde{\sigma}(f_n)(\omega).h_n(\omega)$ for all $n \in \mathbf{N}$.

Lemma 3. *There exists* $\psi_n \in \operatorname{conv}\{\varphi_n, \varphi_{n+1}, \dots\}$ *so that for almost* *every* $\omega \in \Omega$,

$$\lim_{n\to\infty} \langle \psi_n(\omega), Te_n\rangle \quad \textit{exists for each } T \in \mathcal{L}(c_0, A).$$

To prove the lemma, let $(a(k), b(k))_{k\in\mathbf{N}}$ be an enumeration of all pairs of rationals with $a(k) < b(k)$. We will apply Proposition 3 successively starting from $(\varphi_n)_n$ for $E = L^1(\mathbf{T})$ and $Z = H_0^1(\mathbf{T})$. Note that Proposition 3 is valid only for real Banach spaces so we will separate the real part and the imaginary part.

Inductively, we construct sequences $(\varphi_n^{(k)})_{n\geq 1}$ and measurable subsets C_k, L_k of Ω satisfying:

(i) $C_{k+1} \subseteq C_k$, $L_k \subseteq L_{k+1}$, $\mu(C_k \cup L_k) = 1$,

(ii) for all $\omega \in C_k$ and $T \in \mathcal{L}(L^1/H_0^1, \ell^1)$, $\|T\| \leq 1$ and $j \geq k$, either

$$\limsup_{n\to\infty} \mathrm{Re}\,\langle \varphi_n^{(j)}(\omega), T^*e_n\rangle \leq b(k),$$

or

$$\liminf_{n\to\infty} \mathrm{Re}\,\langle \varphi_n^{(j)}(\omega), T^*e_n\rangle \geq a(k),$$

(iii) for all $\omega \in L_k$, there exists $l \in \mathbf{N}$ so that for each Γ infinite sequences of zeros and ones, there exists $T \in \mathcal{L}(L^1/H_0^1, \ell^1)$, $\|T\| \leq 1$ such that if $n \geq l$,

$$\Gamma_n = 1 \Longrightarrow \mathrm{Re}\,\langle \varphi_n^{(k)}(\omega), T^*e_n\rangle \geq b(k)$$

$$\Gamma_n = 0 \Longrightarrow \mathrm{Re}\,\langle \varphi_n^{(k)}(\omega), T^*e_n\rangle \leq a(k);$$

(iv) $\varphi_n^{(k+1)} \in \mathrm{conv}\,\{\varphi_n^{(k)}, \varphi_{n+1}^{(k)}, \dots\}$.

Again this is just an application of Proposition 3 starting from the sequence $\Omega \to C(\mathbf{T})^*$ ($\omega \mapsto \mathrm{Re}\,(\varphi_n(\omega))$) where $\langle \mathrm{Re}\,(\varphi_n(\omega)), f\rangle = \mathrm{Re}\,\langle \varphi_n(\omega), f\rangle$ for all $f \in C(\mathbf{T})$. Let $C = \cap_k C_k$ and $L = \cup_k L_k$.

Claim. $\mu(L) = 0$.

Proof. To see the claim, assume that $\mu(L) > 0$. Since $L = \cup_k L_k$, there exists $k \in \mathbf{N}$ so that $\mu(L_k) > 0$. Consider $\varphi_n^k \in \operatorname{conv}\{\varphi_n, \varphi_{n+1}, \dots\}$, and let $\mathcal{P} = \{k \in \mathbf{N}, b(k) > 0\}$ and $\mathcal{N} = \{k \in \mathbf{N}, a(k) < 0\}$. Clearly $\mathbf{N} = \mathcal{P} \cup \mathcal{N}$.

Let us assume first that $k \in \mathcal{P}$. Using (iii) with $\Gamma = (1, 1, 1, \dots)$, for each $\omega \in L_k$, there exists $T \in \mathcal{L}(c_0, H^\infty)$, $\|T\| \le 1$ so that $\operatorname{Re}\langle \varphi_n^{(k)}(\omega), Te_n\rangle \ge b(k)$. Using a similar argument as in [**16**, Proposition 5], one can construct a map $T : \Omega \to \mathcal{L}(c_0, H^\infty)$ with

a) $\omega \mapsto T(\omega)e$ is weak*-scalarly measurable for every $e \in c_0$;

b) $\|T(\omega)\| \le 1$ for all $\omega \in \Omega$ and $T(\omega) = 0$ for $\omega \in \Omega \backslash L_k$.

c) $\operatorname{Re}\langle \varphi_n^{(k)}(\omega), T(\omega)e_n\rangle \ge b(k)$ for all $\omega \in L_k$.

So we get that

$$\liminf_{n\to\infty} \int_{L_k} \operatorname{Re}\langle \varphi_n^{(k)}(\omega), T(\omega)e_n\rangle \, d\mu(\omega) \ge b(k)\mu(L_k)$$

which implies that

$$\liminf_{n\to\infty} \left| \int_{L_k} \langle \varphi_n^{(k)}(\omega), T(\omega)e_n\rangle \, d\mu(\omega)\right| \ge b(k)\mu(L_k).$$

If $k \in \mathcal{N}$, we repeat the same argument with $\Gamma = (0, 0, 0, \dots)$ to get that

$$\liminf_{n\to\infty} \left| \int_{L_k} \langle \varphi_n^{(k)}(\omega), T(\omega)e_n\rangle \, d\mu(\omega)\right| \ge |a(k)|\mu(L_k).$$

So in both cases, if $\delta = \max(b(k)\mu(L_k), |a(k)|\mu(L_k))$, there exists a map $T : \Omega \to \mathcal{L}(c_0, H^\infty)$ (measurable for the weak* topology) so that

$$\liminf_{n\to\infty} \left| \int_{L_k} \langle \varphi_n^k(\omega), T(\omega)e_n\rangle \, d\mu(\omega)\right| \ge \delta. \tag{1}$$

To get the contradiction, let

$$\varphi_n^{(k)} = \sum_{i=p_n}^{q_n} \lambda_i^n |\tilde{\sigma}(f_i)(\omega)| = \sum_{i=p_n}^{q_n} \lambda_i^n \tilde{\sigma}(f_i)(\omega).h_i(\omega)$$

with $\sum_{i=p_n}^{q_n} \lambda_i^n = 1$, $p_1 < q_1 < p_2 < q_2 < \cdots$ and $h_i \in L^\infty(\mu, H_\sigma^\infty)$.

Condition (1) is equivalent to:

$$\liminf_{n\to\infty}\left|\sum_{i=p_n}^{q_n}\lambda_i^n\int_{L_k}\langle\tilde{\sigma}(f_i)(\omega).h_i(\omega),T(\omega)e_n\rangle\,d\mu(\omega)\right|\geq\delta.$$

Therefore there exists $N\in\mathbf{N}$ so that, for each $n\geq N$,

$$\sum_{i=p_n}^{q_n}\lambda_i^n\left|\int_{L_k}\langle\tilde{\sigma}(f_i)(\omega).h_i(\omega),T(\omega)e_n\rangle\,d\mu(\omega)\right|\geq\delta/2;$$

for each $n\geq N$, choose $i(n)\in[p_n,q_n]$ so that

$$\left|\int_{L_k}\langle\tilde{\sigma}(f_{i(n)})(\omega).h_{i(n)}(\omega),T(\omega)e_n\rangle\,d\mu(\omega)\right|\geq\delta/2,$$

and we obtain that, for each $n\geq N$,

$$\left|\int_{L_k}\langle\sigma(f_{i(n)}(\omega)),T(\omega)e_n.h_{i(n)}(\omega)\rangle\,d\mu(\omega)\right|\geq\delta/2. \tag{2}$$

Notice that, for every $\omega\in\Omega$, $T(\omega)e_n\in H^\infty(\mathbf{T})$ and $h_{i(n)}(\omega)\in H^\infty(\mathbf{T})$ so the product $T(\omega)e_n.h_{i(n)}(\omega)\in H^\infty(\mathbf{T})$ and therefore

$$\langle\sigma(f_{i(n)}(\omega)),T(\omega)e_n.h_{i(n)}(\omega)\rangle=\langle f_{i(n)}(\omega),T(\omega)e_n.h_{i(n)}(\omega)\rangle.$$

For $n\geq N$, fix

$$\phi_n(\omega)=\begin{cases}T(\omega)e_n.h_{i(n)}(\omega) & \omega\in L_k\\ 0 & \omega\notin L_k.\end{cases}$$

If we set $\phi_n=0$ for $n<N$ then the series $\sum_{i=1}^\infty\phi_i$ is a WUC series in $L^\infty(\mu,H^\infty_\sigma)$; to see this, notice that for each $\omega\in\Omega$, $\sum_{n=1}^\infty T(\omega)e_n$ is a WUC series in H^∞ (hence in $L^\infty(\mathbf{T})$) so $\sum_{n=1}^\infty|T(\omega)e_n|$ is a WUC series in $L^\infty(\mathbf{T})$. Now let $x\in L^1(\mu,L^1/H^1_0)$, the predual of $L^\infty(\mu,H^\infty_\sigma)$, and fix $v\in L^1(\mu,L^1)$ with $\tilde{q}(v)=x$. We have

$$\begin{aligned}\sum_{n=1}^\infty|\langle\phi_n,x\rangle|&=\sum_{n=1}^\infty|\langle\phi_n,v\rangle|\\&=\sum_{n=N}^\infty|\langle T(\cdot)e_n.h_{i(n)}(\cdot).\chi_{L_k}(\cdot),v\rangle|\\&\leq\sum_{n=N}^\infty\|h_{i(n)}\|\langle|T(\cdot)e_n|,|v|\rangle\\&\leq\sum_{n=1}^\infty\langle|T(\cdot)e_n|,|v|\rangle<\infty.\end{aligned}$$

Now (2) is equivalent to: for each $n \geq N$,

$$|\langle \phi_n, f_{i(n)} \rangle| \geq \delta/2$$

which is a contradiction since $\{f_i, i \in \mathbf{N}\} \subseteq K$ is relatively weakly compact and $\sum_{n=1}^{\infty} \phi_n$ is a WUC series. The claim is proved. $\square$

To complete the proof of the lemma, let us fix a sequence $(\xi_n)_n$ so that $\xi_n \in \operatorname{conv}\{\varphi_n^{(k)}, \varphi_{n+1}^{(k)}, \dots\}$ for every $k \in \mathbf{N}$, we get by (ii) that $\lim_{n\to\infty} \operatorname{Re}\langle \xi_n(\omega), T^* e_n \rangle$ exists for every $T \in \mathcal{L}(L^1/H_0^1, \ell^1)$. Fix $T \in \mathcal{L}(c_0, A)$. Since $(\xi_n(\omega)) \in L^1(\mathbf{T})$, it is clear that $\langle \xi_n(\omega), Te_n \rangle = \langle \xi_n(\omega), S^* e_n \rangle$ where S is the restriction of T^* on L^1/H_0^1. We repeat the same argument as above for the imaginary part (starting from $(\xi_n)_n$) to get a sequence $(\psi_n)_n$ with $\psi_n \in \operatorname{conv}\{\xi_n, \xi_{n+1}, \dots\}$ so that $\lim_{n\to\infty} \operatorname{Im}\langle \psi_n(\omega), Te_n \rangle$ exists for every $T \in \mathcal{L}(c_0, A)$. The lemma is proved. $\square$

To finish the proof of the theorem, we will show that for almost every ω, the sequence $(\psi_n(\omega))_{n\geq 1}$ is uniformly integrable. If not, there would be a measurable subset Ω' of Ω with $\mu(\Omega') > 0$ and $(\psi_n(\omega))_{n\geq 1}$ not uniformly integrable for each $\omega \in \Omega'$. Hence, by Lemma 2, for each $\omega \in \Omega'$, there exists $T \in \mathcal{L}(c_0, A)$ so that

$$\limsup_{m\to\infty} \sup_n |\langle \psi_n(\omega), Te_m \rangle| > 0.$$

So there would be increasing sequences (n_j) and (m_j) of integers, $\delta > 0$, so that $|\langle \psi_{n_j}(\omega), Te_{m_j} \rangle| > \delta$ for all $j \in \mathbf{N}$; choose an operator $S : c_0 \to c_0$ so that $Se_{n_j} = e_{m_j}$; we have $|\langle \psi_{n_j}(\omega), TSe_{n_j} \rangle| > \delta$. But, by Lemma 3, $\lim_{n\to\infty} |\langle \psi_n(\omega), TSe_n \rangle|$ exists so $\lim_{n\to\infty} |\langle \psi_n(\omega), TSe_n \rangle| > \delta$. We have just shown that for each $\omega \in \Omega'$ there exists an operator $T \in \mathcal{L}(c_0, A)$ so that $\lim_{n\to\infty} |\langle \psi_n(\omega), Te_n \rangle| > 0$ and, as before, we can choose the operator T measurably, i.e., there exists $T : \Omega \to \mathcal{L}(c_0, A)$, measurable for the strong operator topology so that:

a) $\|T(\omega)\| \leq 1$ for every $\omega \in \Omega$;

b) $\lim_{n\to\infty} |\langle \psi_n(\omega), T(\omega)e_n \rangle| = \delta(\omega) > 0$ for $\omega \in \Omega'$;

c) $T(\omega) = 0$ for $\omega \notin \Omega'$.

These conditions imply that

$$\lim_{n\to\infty}\int |\langle\psi_n(\omega),T(\omega)e_n\rangle|\,d\mu(\omega)=\int_{\Omega'}\delta(\omega)=\delta>0,$$

and we can find measurable subsets $(B_n)_n$ so that

$$\liminf_{n\to\infty}\left|\int_{B_n}\langle\psi_n(\omega),T(\omega)e_n\rangle\,d\mu(\omega)\right|>\delta/4$$

and one can get a contradiction using a similar construction as in the proof of Lemma 3.

We have just shown that, for each sequence $(f_n)_n$ in K, there exists a sequence $\psi_n \in \operatorname{conv}(|\tilde{\sigma}(f_n)|, |\tilde{\sigma}(f_{n+1})|, \dots)$ so that for almost every $\omega\in\Omega$, the set $\{\psi_n(\omega), n\geq 1\}$ is relatively weakly compact in $L^1(\mathbf{T})$. By Ulger's criteria of weak compactness for Bochner space [**19**], the set $|\tilde{\sigma}(K)|$ is relatively weakly compact in $L^1(\mu, L^1(\mathbf{T})) = L^1(\Omega\times\mathbf{T}, \mu\otimes m)$. Hence $\tilde{\sigma}(K)$ is uniformly integrable in $L^1(\Omega\times\mathbf{T},\mu\otimes m)$ which is equivalent to $\tilde{\sigma}(K)$ relatively weakly compact in $L^1(\mu, L^1(\mathbf{T}))$. This completes the proof. □

Theorem 1 can be extended to the case of spaces of measures.

Corollary 1. *Let K be a relatively weakly compact subset of $M(\Omega, A^*)$. The set $\tilde{\gamma}(K)$ is relatively weakly compact in $M(\Omega, M(\mathbf{T}))$.*

The following lemma will be used for the proof.

Lemma 4. *Let $\Pi : M(\mathbf{T})\to L^1$ be the usual projection. The map Π is weak* to norm universally measurable.*

Proof. For each $n\in\mathbf{N}$ and $1\leq k<2^n$, let $D_{n,k}=\{e^{it};(k-1)\pi/2^{n-1}\leq t<k\pi/2^{n-1}\}$. Define, for each measure λ in $M(\mathbf{T})$, $R_n(\lambda)=g_n\in L^1$ to be the function $\sum_{k=1}^{2^n}2^n\lambda(D_{n,k})\chi_{D_{n,k}}$. It is not difficult to see that the map $\lambda\mapsto\lambda(D_{n,k})$ is weak*-Borel, so the map R_n is weak* Borel measurable as a map from $M(\mathbf{T})$ into L^0. But $R_n(\lambda)$ converges almost everywhere to the derivative of λ with respect to m. If

$R(\lambda)$ is such a limit, the map R is weak* Borel measurable and therefore $M_s(\mathbf{T}) = R^{-1}(\{0\})$ is weak* Borel measurable. Now fix B a Borel measurable subset of L^1. Since L^1 is a Polish space and the inclusion map of L^1 into $M(\mathbf{T})$ is norm to weak* continuous, B is a weak* analytic subset of $M(\mathbf{T})$ which implies that $\Pi^{-1}(B) = B + M_s(\mathbf{T})$ is a weak* analytic (and hence weak* universally measurable) subset of $M(\mathbf{T})$. Thus the proof of the lemma is complete. $\square$

To prove the corollary, let K be a relatively weakly compact subset of $M(\Omega, A^*)$. There exists a measure μ in (Ω, Σ) so that K is uniformly continuous with respect to μ. For each $G \in K$, choose $\omega \mapsto g(\omega)(\Omega \to A^*)$ a weak*-density of G with respect to μ. Let $g(\omega) = \{g_1(\omega), g_2(\omega)\}$ be the unique decomposition of $g(\omega)$ in $L^1/H_0^1 \oplus_1 M_s(\mathbf{T})$. We claim that the function $\omega \mapsto g_1(\omega)$ belongs to $L^1(\mu, L^1/H_0^1)$. To see this, notice that the function $\omega \to \gamma(g(\omega)) = \{\sigma(g_1(\omega)), g_2(\omega)\}$ is a weak*-density of $\tilde{\gamma}(G)$ with respect to μ. By the above lemma, $\omega \mapsto \Pi(\gamma(g(\omega))) = \sigma(g_1(\omega))$ $(\Omega \to L^1)$ is norm measurable and hence $\omega \mapsto g_1(\omega)$ $(\Omega \to L^1/H_0^1)$ is norm measurable and the claim is proved.

We get that $g(\omega) = \{g_1(\omega), g_2(\omega)\}$ where $g_1(.) \in L^1(\mu, L^1/H_0^1)$ and $g_2(.)$ defines a measure in $M(\Omega, M(\mathbf{T}))$. So $K = K_1 + K_2$ where K_1 is a relatively weakly compact subset of $L^1(\mu, L^1/H_0^1)$ and K_2 is a relatively weakly compact subset of $M(\Omega, M(\mathbf{T}))$. It is now easy to check $\tilde{\gamma}(K) = \tilde{\sigma}(K_1) + K_2$ and an appeal to Theorem 2 completes the proof. $\square$

Remark 1. Hensgen initiated the study of possible existence and uniqueness of minimum norm lifting σ from $L^1(X)/H_0^1(X)$ to $L^1(X)$ in [**13**]. He proved, see [**13**, Theorem 3.6] that if X is reflexive then $\sigma(K)$ is relatively weakly compact in $L^1(X)$ if and only if K is relatively weakly compact in $L^1(X)/H_0^1(X)$.

3. The Dunford-Pettis property. In this section we prove our main results concerning the spaces $L^1(\mu, L^1/H_0^1)$ and $C(\Omega, A)$. Let us first recall some characterizations of the Dunford-Pettis property that are useful for our purpose.

Proposition 4 [**8**]. *Each of the following conditions is equivalent to* *the Dunford-Pettis property for a Banach space* X

(i) *If* $(x_n)_n$ *is a weakly Cauchy sequence in* X *and* $(x_n^*)_n$ *is a weakly null sequence in* X^**, then* $\lim_{n\to\infty} x_n^*(x_n) =$;

(ii) *If* $(x_n)_n$ *is a weakly null sequence in* X *and* $(x_n^*)_n$ *is a weakly Cauchy sequence in* X^**, then* $\lim_{n\to\infty} x_n^*(x_n) = 0$.

It is immediate from the above proposition that if X^* has the Dunford-Pettis property then so does X.

We are now ready to present our main theorem.

Theorem 2. *Let* Ω *be a compact Hausdorff space, the dual of* $C(\Omega, A)$ *has the Dunford-Pettis property.*

Proof. Let $(G_n)_n$ and $(\xi_n)_n$ be weakly null sequences of $M(\Omega, A^*)$ and $M(\Omega, A^*)^*$ respectively, and consider the inclusion map $J : C(\Omega, A) \to C(\Omega, C(\mathbf{T}))$. By Corollary 1, the set $\{\tilde{\gamma}(G_n); n \in \mathbf{N}\}$ is relatively weakly compact in $M(\Omega, M(\mathbf{T}))$.

Claim. *For each* $G \in M(\Omega, A^*)$ *and* $\xi \in M(\Omega, A^*)^*$*,* $\langle G, \xi\rangle = \langle \tilde{\gamma}(G), J^{**}(\xi)\rangle$.

Proof. Notice that the claim is trivially true for $G \in M(\Omega, A^*)$ and $f \in C(\Omega, A)$. For $\xi \in M(\Omega, A^*)^*$, fix a net $(f_\alpha)_\alpha$ of elements of $C(\Omega, A)$ that converges to ξ for the weak*-topology. We have

$$\begin{aligned}\langle G, \xi\rangle &= \lim_\alpha \langle G, f_\alpha\rangle \\ &= \lim_\alpha \langle \tilde{\gamma}(G), J(f_\alpha)\rangle \\ &= \langle \tilde{\gamma}(G), J^{**}(\xi)\rangle,\end{aligned}$$

and the claim is proved. $\square$

To complete the proof of the theorem, we use the claim to get that, for each $n \in \mathbf{N}$,

$$\langle G_n, \xi_n\rangle = \langle \tilde{\gamma}(G_n), J^{**}(\xi_n)\rangle.$$

Since $(J^{**}(\xi_n))_n$ is a weakly null sequence in $M(\Omega, M(\mathbf{T}))^*$ and $\{\tilde{\gamma}(G_n); n \in \mathbf{N}\}$ is relatively weakly compact, we apply the fact that

$M(\Omega, M(\mathbf{T}))$ has the Dunford-Pettis property (it is an L^1-space) to conclude that the sequence $(\langle\tilde{\gamma}(G_n), J^{**}(\xi_n)\rangle)_n$ converges to zero and so does the sequence $(\langle G_n, \xi_n\rangle)_n$. This completes the proof. □

Corollary 2. *Let Ω be a compact Hausdorff space and μ a finite Borel measure on Ω. The following spaces have the Dunford-Pettis property*: $L^1(\mu, L^1/H_0^1)$, $L^1(\mu, A^*)$ *and* $C(\Omega, A)$.

Proof. For the space $L^1(\mu, L^1/H_0^1)$, it is enough to notice that the space $L^1(\mu, L^1/H_0^1)$ is complemented in $M(\Omega, L^1/H_0^1)$ which in turn is a complemented subspace of $M(\Omega, A^*)$.

For $L^1(\mu, A^*)$, we use the fact that $A^* = L^1/H_0^1 \oplus_1 M_S(\mathbf{T})$. It is clear that $L^1(\mu, A^*) = L^1(\mu, L^1/H_0^1) \oplus_1 L^1(\mu, M_S(\mathbf{T}))$ and, since $L^1(\mu, M_S(\mathbf{T}))$ is an L^1-space, the space $L^1(\mu, A^*)$ has the Dunford-Pettis property. □

Remark 2. F. Delbaen obtained in [**7**] a result closely related to the results presented here. He showed that the space $L^1(\mu, A)$ has the Dunford-Pettis property.

The use of the minimum norm lifting to prove that some spaces have the Dunford-Pettis property was initiated by J. Chaumat in [**4**], see also I. Cnop and F. Delbaen [**5**] independently, where it was shown that the dual of the disc algebra A has the Dunford-Pettis property. Although we did not refer directly to the fact that A^* has the Dunford-Pettis property, the proof presented here is an extension of the approach used in [**4**] and [**5**].

It should be noted that J. Bourgain [**3**] also used a different type of extension of the minimum norm lifting to show that the Hardy space H^∞ has the Dunford-Pettis property.

Acknowledgments. The author would like to thank Nigel Kalton for showing him the argument used in Lemma 4, Wolfgang Hensgen for some fruitful comments and the referee for bringing [**7**] to his attention.

Addendum. After this paper was submitted, we learned that

Manuel D. Contreras and Santiago Díaz have proved with completely different techniques that $C(\Omega, A)$ and $C(\Omega, H^\infty)$ have the Dunford Pettis property (see Proc. Amer. Math. Soc. **124** (1996), 3413–3416).

REFERENCES

1. K. Andrews, *Dunford-Pettis set in the space of Bochner integrable functions*, Math. Ann. **241** (1979), 35–42.

2. J. Bourgain, *New classes of $\mathcal{L}^P$-spaces*, Lecture Notes in Math. **889** (1981).

3. ———, *New Banach space properties of the disc algebra and H^∞*, Acta Math. **152** (1984), 1–48.

4. J. Chaumat, *Une généralisation d'un théorème de Dunford-Pettis*, Université de Paris XI, Orsay, 1974.

5. I. Cnop and F. Delbaen, *A Dunford-Pettis theorem for $L^1/H^{\infty\perp}$*, J. Funct. Anal. **24** (1977), 364–378.

6. D.L. Cohn, *Measure theory*, Birkhäuser, Basel, 1980.

7. F. Delbaen, *Dunford-Pettis property*, Unpublished note.

8. J. Diestel, *The Dunford-Pettis property*, Contemp. Math. **2** (1980), 15–60.

9. ———, *Sequences and series in Banach spaces*, Graduate Texts in Math. **92** (1984).

10. J. Diestel and J.J. Uhl, *Vector measures*, Math. Surveys **15** (1977).

11. N. Dunford and B.J. Pettis, *Linear operators on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392.

12. A. Grothendieck, *Sur les applications lineaires faiblements compactes d'espaces de type $C(K)$*, Canad. J. Math. **5** (1953), 129–173.

13. W. Hensgen, *Contributions to the geometry of vector-valued H^∞ and L^1/H^1_0 spaces*, Habilitationsschrift, Universitat Regensburg, 1992.

14. A. Ionescu-Tulsea and C. Ionescu-Tulsea, *Topics in the theory of lifting*, Springer-Verlag, Berlin, 1969.

15. A. Pełczynski, *Banach spaces of analytic functions and absolutely summing operators*, American Math. Soc., Providence, 1977.

16. N. Randrianantoanina, *Complemented copies of ℓ^1 and Pełczyński's property* (V*) *in Bochner spaces*, Canad. J. Math. **48** (1996), 625–640.

17. E. Saab and P. Saab, *Stability problems in Banach spaces*, Lecture Notes in Pure Appl. Math. **136** (1992), 367–394.

18. M. Talagrand, *La propriété de Dunford-Pettis dans $C(K, E)$ et $L_1(E)$*, Israel J. Math. **44** (1983), 317–321.

19. A. Ulger, *Weak compactness in $L^1(\mu, X)$*, Proc. Amer. Math. Soc. **113** (1991), 143–149.

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712-1082

Current address: Department of Mathematics, Miami University, Oxford, OH 45056-1641

E-mail address: randrin@muohio.edu