ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

FRACTIONAL INTEGRALS OF IMAGINARY ORDER SUPPORTED ON CONVEX CURVES, AND THE DOUBLING PROPERTY

JAMES MARSHALL

1. Introduction and statement of results. Let $\Gamma : [0, \infty) \to \mathbf{R}^n$ be a curve in $\mathbf{R}^n$, $n \geq 2$, and define

$$H_\varepsilon f(x) = \int_0^\infty f(x - \Gamma(t)) \frac{dt}{(1 + t^2)^{1/2 + i\varepsilon}}$$

and

$$H_{\varepsilon,\delta} f(x) = \int_\delta^\infty f(x - \Gamma(t)) \frac{dt}{t^{1 + i\varepsilon}},$$

for $x \in \mathbf{R}^n$, $f \in C_0^\infty(\mathbf{R}^n)$, $\varepsilon > 0$ and $\delta > 0$.

We seek conditions on Γ so that H_ε is a bounded linear operator on $L^2(\mathbf{R}^n)$ and the family of operators $\{H_{\varepsilon,\delta}\}$, for a fixed ε, is uniformly bounded on $L^2(\mathbf{R}^n)$.

The motivation for examining these operators is the work done by a number of researchers over the last 20 years in studying the L^p-boundedness of the Hilbert transform $\mathbf{H}_\Gamma$ and the maximal operator $\mathbf{M}_\Gamma$, defined for $x \in \mathbf{R}^n$ and $f \in C_0^\infty(\mathbf{R}^n)$ as follows

$$\mathbf{H}_\Gamma f(x) = \text{p.v.} \int_{-\infty}^\infty f(x - \Gamma(t)) \frac{dt}{t}$$

(a principle value integral), and

$$\mathbf{M}_\Gamma f(x) = \sup_{h > 0} \frac{1}{h} \int_0^h |f(x - \Gamma(t))| \, dt.$$

Early inquiries into the L^p-boundedness of these operators, by Nagel, Rivière, Stein and Wainger, considered well-curved and two-sided homogeneous curves. A curve Γ in $\mathbf{R}^n$ is said to be well-curved if $\Gamma(0) = 0$

Received by the editors on August 21, 1995, and in revised form on February 24, 1996.

and a segment of the curve containing the origin lies in the subspace of $\mathbf{R}^n$ spanned by $\{\Gamma^{(j)}(0)\}_{j=1}^{\infty}$. We say Γ is a two-sided homogeneous curve if the following two conditions hold

$$\Gamma(t) = \begin{cases} \delta_t e & t > 0, \\ \delta_t f & t < 0, \\ 0 & t = 0, \end{cases}$$

(here δ_t is a one-parameter group of dilations and e and f are vectors in $\mathbf{R}^n$) and

$$\{\xi \mid \xi \cdot \Gamma(t) \equiv 0, t > 0\} = \{\xi \mid \xi \cdot \Gamma(t) \equiv 0, t < 0\}.$$

It was shown in [**5**] that for two-sided homogeneous curves $\mathbf{H}_\Gamma$ and $\mathbf{M}_\Gamma$ were bounded on $L^2(\mathbf{R}^n)$ and that, for well-curved curves there was L^2-boundedness for their local variants, which are given by

$$H_\Gamma f(x) = \text{p.v.} \int_{-1}^{1} f(x - \Gamma(t)) \frac{dt}{t}$$

and

$$M_\Gamma f(x) = \sup_{1 \geq h > 0} \frac{1}{h} \int_0^h |f(x - \Gamma(t))| \, dt.$$

We note that these results also hold for H_ε and $H_{\varepsilon,\delta}$, with the proofs proceeding exactly as for the Hilbert transform in [**5**], except that integration over dyadic intervals is replaced by integration over intervals of the form $[\delta_j, \delta_{j+1}]$, where $\delta_j = e^{2\pi j/\varepsilon}$.

In the 1980's attention was turned to convex curves Γ. We present theorems for H_ε and $H_{\varepsilon,\delta}$ analogous to those obtained for the Hilbert transform and maximal operator by Nagel, Vance, Wainger and Weinberg in [**2, 3, 4**]. Let us first restrict the setting to $\mathbf{R}^2$.

Theorem 1. *Suppose* $\Gamma : [0,\infty) \to \mathbf{R}^2$ *is of the form* $\Gamma(t) = (t, \gamma(t))$*, where*

$$\begin{gathered} \gamma : [0,\infty) \longrightarrow \mathbf{R} \quad \textit{is convex}, \\ \gamma \in C^2(0,\infty) \quad \textit{and} \quad \gamma(0) = \gamma'(0) = 0. \end{gathered} \tag{1.1}$$

Suppose also that the function h given by

$$h(t) = t\gamma'(t) - \gamma(t), \qquad t > 0, \tag{1.2}$$

satisfies the "doubling property"

(1.3) *there exists a constant C, $1 < C < \infty$, so that for each $t > 0$,*

$$h(Ct) \geq 2h(t).$$

Then

$$\|H_\varepsilon f\|_2 \leq A\|f\|_2$$

and

$$\|H_{\varepsilon,\delta}\|_2 \leq B\|f\|_2, \quad \textit{for } f \in C_0^\infty(\mathbf{R}^2),$$

where A and B are positive constants which depend only on ε. H_ε and $H_{\varepsilon,\delta}$ then extend by continuity to all of $L^2(\mathbf{R}^2)$.

Nagel, Vance, Wainger and Weinberg [**4**] have shown that, under the hypotheses of Theorem 1, the maximal operator $\mathbf{M}_\Gamma$ is bounded on $L^2(\mathbf{R})$ and [**2**] that, for the Hilbert transform, the doubling property (1.3), where γ is now an odd curve, is both a necessary and sufficient condition for the boundedness of $\mathbf{H}_\Gamma$ on $L^2(\mathbf{R}^2)$.

Next we extend the notion of convexity to $\mathbf{R}^n$ in the following manner, see [**3**, p. 486]. Suppose

$$\Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)), \tag{1.4}$$

where $\gamma_k \in C^n[0,\infty)$ and $\Gamma(0) = 0$. For $1 \leq j \leq n$ and $t > 0$, define

$$D_j(t) = \det \begin{pmatrix} 1 & \gamma_2'(t) & \cdots & \gamma_j'(t) \\ 0 & \gamma_2''(t) & \cdots & \gamma_j''(t) \\ \vdots & \vdots & & \vdots \\ 0 & \gamma_2^{(j)}(t) & \cdots & \gamma_j^{(j)}(t) \end{pmatrix}$$

and $D_0(t) = 1$.

Also set

$$N_j(t) = \det \begin{pmatrix} t & \gamma_2(t) & \cdots & \gamma_j(t) \\ 1 & \gamma_2'(t) & \cdots & \gamma_j'(t) \\ \vdots & \vdots & \vdots & \\ 0 & \gamma_2^{(j-1)}(t) & \cdots & \gamma_j^{(j-1)}(t) \end{pmatrix}.$$

Our convexity assumption is that $D_j(t) > 0$ if $t > 0$, for $2 \leq j \leq n$. We then define the auxiliary functions

$$h_j(t) = \frac{N_j(t)}{D_{j-1}(t)}, \quad 0 < t < \infty,\ 1 \leq j \leq n.$$

Given these definitions we can extend the result of Theorem 1 to $\mathbf{R}^n$.

Theorem 2. *Assume that* $\Gamma(t)$ *satisfies* (1.4), *and suppose that*

$$D_j(t) > 0, \qquad 1 \leq j \leq n, \qquad t > 0$$

and

(1.5) *there is a* $C > 1$ *so that for* $2 \leq j \leq n$,

$$h_j(Ct) \geq 2h_j(t) \quad \textit{for } t > 0.$$

Then

$$\|H_\varepsilon f\|_2 \leq A\|f\|_2$$

and

$$\|H_{\varepsilon,\delta} f\|_2 \leq B\|f\|_2$$

where A *and* B *depend only on* n *and* ε.

We note that, if $n = 2$, Theorem 2 yields the result only for strictly convex curves, i.e., curves with $\gamma''(t) > 0$ for $t > 0$, while Theorem 1 admits curves which may have $\gamma''(t) = 0$ for some values of t.

In this setting, Nagel, Vance, Wainger and Weinberg considered the local version of the Hilbert transform and showed [**3**] that, under the hypotheses of Theorem 2, with γ_k an odd function in $C^n[-1, 1]$, that H_Γ is bounded on $L^2(\mathbf{R}^n)$ if and only if condition (1.5) holds. This result, with $\gamma_k \in C^n(-\infty, \infty)$, is also valid for the global Hilbert transform $\mathbf{H}_\Gamma$. As for the maximal operator, in this case the question of L^2-boundedness remains unanswered.

Finally, define a piecewise linear curve γ by

$$\gamma(t) = 0 \quad \text{for } 0 \leq t \leq 1 \quad \text{and} \quad \gamma(t) = t - 1 \quad \text{for } 1 < t, \tag{1.6}$$

and then smooth out γ by replacing it with an appropriately chosen polynomial for $1/2 \le t \le 3/2$. Specifically, define γ^* by

$$(1.7)\qquad \gamma^*(t) = \begin{cases} 0 & 0 \le t \le 1/2, \\ -t^4/2 + 2t^3 - 9/4t^2 + t - 5/32 & 1/2 \le t \le 3/2, \\ t-1 & t > 3/2. \end{cases}$$

Then, for $t > 3/2$, $h^*(t) = 1$ where $h^*(t) = t(\gamma^*)'(t) - \gamma^*(t)$. Thus, h^* does not possess the doubling property, while γ^* satisfies all of the other hypotheses of Theorem 1. Thus, for $\Gamma(t) = (t, \gamma^*(t))$, $\mathbf{H}_\Gamma$ is not bounded on $L^2(\mathbf{R}^2)$. From this, it follows, by the method employed below in Section 5 for $H_{\varepsilon,\delta}$, that $\mathbf{H}_\Gamma$ is also not bounded on $L^2(\mathbf{R}^2)$ if $\Gamma(t) = (t, \gamma(t))$, with γ as in (1.6). However, for these two curves, both H_ε and $H_{\varepsilon,\delta}$ are bounded on $L^2(\mathbf{R}^2)$, as is the maximal operator, although this is not the first instance in which the results for the maximal operator are known to differ from those of the Hilbert transform; see [**1**]. Furthermore, for fixed ε the boundedness of $H_{\varepsilon,\delta}$ is uniform in δ.

2. The L^2-boundedness of H_ε follows from that of $H_{\varepsilon,\delta}$. Since $\widehat{H_\varepsilon f}(\xi) = m_\varepsilon(\xi)\hat{f}(\xi)$ where m_ε is given by

$$m_\varepsilon(\xi) = \int_0^\infty e^{i\xi\cdot\Gamma(t)} \frac{dt}{(1+t^2)^{1/2+i\varepsilon}}, \quad \xi \in \mathbf{R}^n,$$

and $\widehat{H_{\varepsilon,\delta} f}(\xi) = m_{\varepsilon,\delta}(\xi)\hat{f}(\xi)$, where $m_{\varepsilon,\delta}$ is given by

$$m_{\varepsilon,\delta}(\xi) = \int_\delta^\infty e^{i\xi\cdot\Gamma(t)} \frac{dt}{t^{1+i\varepsilon}}, \quad \xi \in \mathbf{R}^n,$$

the boundedness on $L^2(\mathbf{R}^n)$ of these operators can be established by showing that m_ε and $m_{\varepsilon,\delta}$ are bounded functions on $\mathbf{R}^n$.

In the proofs of Theorems 1 and 2, we will show that, for all $\delta > 0$, there exists a positive constant B, depending only on n and ε, such that

$$(2.1)\qquad \left| \int_\delta^\infty e^{i\varepsilon\cdot\Gamma(t)} \frac{dt}{t^{1+i\varepsilon}} \right| \le B.$$

Then the estimate

$$|(1+t^2)^{-1/2-i\varepsilon} - t^{1-2i\varepsilon}| \le (1+2\varepsilon)t^{-3}, \quad \text{for } t>0 \tag{2.2}$$

shows that

$$\begin{aligned}\left|\int_1^\infty e^{i\xi\cdot\Gamma(t)}\frac{dt}{(1+t^2)^{1/2+i\varepsilon}} - \int_1^\infty e^{i\xi\cdot\Gamma(t)}\frac{dt}{t^{1+2i\varepsilon}}\right| \\ \le \int_1^\infty |(1+t^2)^{-1/2-2i\varepsilon} - t^{-1-2i\varepsilon}|\,dt \le \frac{1+2\varepsilon}{2}.\end{aligned} \tag{2.3}$$

Thus, (2.1) and the triangle inequality yield

$$\left|\int_1^\infty e^{i\xi\cdot\Gamma(t)}\frac{dt}{(1+t^2)^{1/2+i\varepsilon}}\right| \le \frac{1+2\varepsilon}{2} + B. \tag{2.4}$$

Since $|\int_0^1 e^{i\xi\cdot\Gamma(t)}(1+t^2)^{-(1/2+i\varepsilon)}\,dt| \le \log(\sqrt{2}/2+1)$, combining this with (2.5) and the triangle inequality then shows that m_ε is a bounded function.

The estimate in (2.3) can be shown by splitting the function $g(s) = s^{-1/2-i\varepsilon}$ into its real and imaginary parts and applying the mean value theorem to each of them.

3. Proof of Theorem 1.

Case (i). Set $g(t) = \xi\cdot\Gamma(t) = xt + y\gamma(t)$, where $\xi = (x,y)$ with $x>0$ and $y>0$. Note that

$$g'(t) = x + y\gamma'(t) > 0$$

and

$$g''(t) = y\gamma''(t) \ge 0.$$

It follows that we can find $\eta > 0$ such that $\eta g'(\eta) = 1$. Thus, for any large N,

$$\int_\delta^N e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}} = \int_\delta^\eta e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}} + \int_\eta^N e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}} = A + B.$$

Integration by parts then shows

$$
\begin{aligned}
|A| &= \left| \frac{e^{ig(\eta)}}{-i\varepsilon\eta^{i\varepsilon}} - \frac{e^{ig(\delta)}}{-i\varepsilon\delta^{i\varepsilon}} + \frac{1}{\varepsilon}\int_\delta^\eta g'(t)e^{ig(t)}\frac{dt}{t^{i\varepsilon}} \right| \\
&\le \frac{2}{\varepsilon} + \frac{1}{\varepsilon}\int_0^\eta g'(t)\,dt \le \frac{3}{\varepsilon}.
\end{aligned} \tag{3.1}
$$

A different integration by parts gives

$$
\begin{aligned}
|B| &\le \left| \frac{e^{ig(N)}}{ig'(N)N^{1+i\varepsilon}} - \frac{e^{ig(\eta)}}{g'(\eta)\eta^{1+i\varepsilon}} \right| \\
&\quad + \left| \frac{1}{i}\int_\eta^N \frac{g''(t)e^{ig(t)}}{(g'(t))^2 t^{1+i\varepsilon}}\,dt \right| \\
&\quad + \left| \frac{1+i\varepsilon}{i}\int_\eta^N \frac{e^{ig(t)}}{g'(t)t^{2+i\varepsilon}}\,dt \right| \\
&\le 5+\varepsilon.
\end{aligned} \tag{3.2}
$$

We note that the same result holds if we replace $g(t)$ with $-g(t)$.

Case (ii). Set $g(t) = xt - y\gamma(t)$, with $x > 0$ and $y > 0$. Nagel, Vance, Wainger and Weinberg [**2**, Lemma 1] have shown that, for a curve satisfying (1.1) the doubling property is equivalent to the following condition:

(3.3) there exists $C > 1$ such that $h(t) \le Ct(\gamma'(s) - \gamma'(t))$ whenever $0 < t \le s/C$.

Using this constant C, write

$$
\int_\delta^\infty e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}} = \int_\delta^\eta + \int_\eta^{\xi_1/C} + \int_{\xi_1/C}^{C\xi_2} + \int_{C\xi_2}^\infty e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}},
$$

where $g(\eta) = 1$ and $g'(t) = 0$ for $\xi_1 \le t \le \xi_2$. (Following this proof we will demonstrate how to proceed if $g'(t) \ne 0$ for $t > 0$.)

To begin, observe that $g''(t) = -y\gamma''(t) \le 0$, so it follows that for $t > C\xi_2$, we can find $\Delta > 0$ with $\Delta \cdot |g'(C\xi_2 + \Delta)| = 1$. Next, write

$$
\int_{C\xi_2}^\infty e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}} = \int_{C\xi_2}^{C\xi_2+\Delta} e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}} + \int_{C\xi_2+\Delta}^\infty e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}.
$$

Then the integration by parts from (3.1) gives

$$\left| \int_{C\xi_2}^{C\xi_2+\Delta} e^{ig(t)} \frac{dt}{t^{1+i\varepsilon}} \right| \le \frac{2}{\varepsilon} + \frac{1}{\varepsilon} |g'(C\xi_2 + \Delta)| \int_{C\xi_2}^{C\xi_2+\Delta} dt \le \frac{3}{\varepsilon}.$$

For any $N > C\xi_2 + \Delta$, the integration by parts from (3.2) shows

$$\begin{aligned} \left| \int_{C\xi_2+\Delta}^{N} e^{ig(t)} \frac{dt}{t^{1+i\varepsilon}} \right| &\le \frac{5+\varepsilon}{(C\xi_2 + \Delta)|g'(C\xi_2 + \Delta)|} \\ &\le \frac{(5+\varepsilon)\Delta}{C\xi_2 + \Delta} \le 5 + \varepsilon. \end{aligned}$$

Next, we observe that $h'(t) = t\gamma''(t)$. Thus, if $\gamma''(t) = 0$ for $a \le t \le b$, then $h(t) = \alpha$ for $a \le t \le b$, where α is a positive constant. But $h(Ct) \ge 2h(t)$ for $t > 0$, so $h(Ca) \ge 2h(a)$. Thus, we must have $Ca > b$.

Since $g'(t) = 0$ for $\xi_1 \le t \le \xi_2$, it follows from the preceding comments that we must have $\xi_2/\xi_1 < C$. Then

$$\left| \int_{\xi_1/C}^{C\xi_2} e^{ig(t)} \frac{dt}{t^{1+i\varepsilon}} \right| \le \log\left(C^2 \frac{\xi_2}{\xi_1} \right) \le 3 \log C.$$

Recalling that $g(\eta) = 1$, we use the method of (3.1) to show

$$\left| \int_{\delta}^{n} e^{ig(t)} \frac{dt}{t^{1+i\varepsilon}} \right| \le \frac{2}{\varepsilon} + \frac{1}{\varepsilon} \int_0^{\eta} g'(t)\, dt = \frac{3}{\varepsilon}.$$

Finally, to bound $| \int_{\eta}^{\xi_1/C} e^{ig(t)} t^{-(1+i\varepsilon)}\, dt|$, we note that (3.3) shows that if $0 < t \le \xi_1/C$, then

$$C_1 = C + 1 \ge \frac{t\gamma'(\xi_1) - \gamma(t)}{t(\gamma'(\xi_1) - \gamma'(t))},$$

so that

$$\frac{1}{t(\gamma'(\xi_1) - \gamma'(t))} \le \frac{C_1}{t\gamma'(\xi_1) - \gamma(t)}, \quad 0 < t \le \xi_1/C. \tag{3.4}$$

But $g(t) = y(t\gamma'(\xi_1) - \gamma(t))$ and $g'(t) = y(\gamma'(\xi_1) - \gamma'(t))$, so (3.4) is equivalent to

$$\frac{1}{tg'(t)} \leq \frac{C_1}{g(t)}, \quad 0 < t \leq \xi_1/C. \tag{3.5}$$

Thus, the method of (3.2) shows

$$\begin{aligned} \left| \int_\eta^{\xi_1/C} e^{ig(t)} \frac{dt}{t^{1+i\varepsilon}} \right| &\leq \left| \frac{e^{ig(\xi_1/C)}}{g'(\xi_1/C)(\xi_1/C)^{1+i\varepsilon}} - \frac{e^{ig(\eta)}}{g'(\eta)\eta^{1+i\varepsilon}} \right| \\ &\quad + \int_\eta^{\xi_1/C} \frac{(-g''(t))}{(g'(t))^2 t}\, dt \\ &\quad + (1+\varepsilon) \int_\eta^{\xi_1/C} \frac{dt}{t^2 g'(t)} \\ &= I + II + III. \end{aligned} \tag{3.6}$$

But, by (3.5),

$$I \leq \frac{C_1}{g(\xi_1/C)} + \frac{C_1}{g(\eta)} \leq 2C_1$$

and

$$III \leq C_1^2(1+\varepsilon) \int_\eta^{\xi_1/C} \frac{g'(t)}{(g(t))^2}\, dt \leq C_1^2(1+\varepsilon).$$

As for II, another integration by parts gives

$$II = \frac{1}{(\xi_1/C)g'(\xi_1/C)} - \frac{1}{\eta g'(\eta)} + \int_\eta^{\xi_1/C} \frac{g'(t)}{(tg'(t))^2}\, dt \leq C_1 + C_1^2.$$

Now, if $0 < g'(t) = x - y\gamma'(t)$ for $t > 0$, then $x/y > \gamma'(t)$ for all $t > 0$. Observe also that $g(t) = y((x/y)t - \gamma(t))$ and $g'(t) = y(x/y - \gamma'(t))$. Then, for $N > \eta$, where $f(\eta) = 1$, (3.3) shows that, for $0 < t \leq N = CN/C$, we have

$$\frac{1}{tg'(t)} \leq \frac{C_1}{g(t)}, \quad 0 < t \leq N.$$

Thus, $\int_\eta^N e^{ig(t)} t^{-(1+i\varepsilon)}\, dt$ may be handled as $\int_\eta^{\xi_1/C} e^{ig(t)} t^{-(1+i\varepsilon)}\, dt$ was above, and $\int_\delta^\eta e^{ig(t)} t^{-(1+i\varepsilon)}\, dt$ is handled as before.

Finally, we note that, as in case (i), the proof of case (ii) goes through without difficulty if we replace $g(t)$ with $-g(t)$. This completes the proof of Theorem 1. $\square$

4. Proof of Theorem 2. Recall that we are now considering a curve $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$. For $x = (x_1, x_2, \dots, x_n)$, set $g(t) = \sum_{k=1}^n x_k \gamma_k(t)$, where $\gamma_1(t) = t$. Then, for a curve Γ satisfying the hypotheses of Theorem 2, Nagel, Vance, Wainger and Weinberg have proven the following result.

Lemma [**3**, p. 498]. *There exist constants C and D so that, for every $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ with $x_n \neq 0$ and associated function $g(t) = \sum_{k=1}^n x_k \gamma_k(t)$, we have*

$$|g(t)| \leq C \cdot t \cdot |g'(t)| \quad \textit{for every } t \in [0, \infty) \backslash \Omega, \tag{4.1}$$

where the exceptional set $\Omega = \Omega_x = \cup_{m=1}^M (c_m, d_m)$ satisfies $M \leq 2n-2$ and $d_m/c_m \leq D$ for all m. (Moreover, neither g' nor g'' is ever 0 on $[0, \infty) \backslash \Omega$.)

Note that, since g' and g'' are never zero on $[0, \infty) \backslash \Omega$, they must be of constant sign on each of the $2n-1$ nonoverlapping intervals which comprise this set. We further subdivide these intervals at the solutions of $|g(t)| = 1$, of which there can be at most $2n$, since g' has no more than $n-1$ zeros, see [**3**, p. 493]. Thus, $[0, \infty) \backslash \Omega$ can be written as the union of at most $4n-1$ intervals, on each of which we either have $|g(t)| \geq 1$ or $|g(t)| \leq 1$.

If $[a, b]$ is one of the intervals on which $|g(t)| \leq 1$, then the method of (3.1) leads to

$$\begin{aligned}
\left| \int_a^b e^{ig(t)} \frac{dt}{t^{1+i\varepsilon}} \right| &\leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} \left| \int_a^b g'(t)\, dt \right| \\
&\leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} |g(b) - g(a)| \\
&\leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} (|g(b)| + |g(a)|) \\
&\leq \frac{4}{\varepsilon}.
\end{aligned}$$

If we instead have $|g(t) \geq 1$ on $[a,b]$, then the method of (3.2) shows

$$\begin{aligned}\left|\int_a^b e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}\right| &\leq \left(\left|\frac{e^{ig(b)}}{ig'(b)b^{1+i\varepsilon}}\right| + \left|\frac{e^{ig(a)}}{ig'(a)a^{1+i\varepsilon}}\right|\right)\\ &\quad + \left|\frac{1}{i}\int_a^b \frac{g''(t)e^{ig(t)}}{(g'(t))^2 t^{1+i\varepsilon}}\right|\\ &\quad + \left|\frac{1+i\varepsilon}{i}\int_a^b \frac{e^{ig(t)}}{g'(t)t^{2+i\varepsilon}}\,dt\right|\\ &= I + II + III.\end{aligned}$$

Using (4.1), we then proceed as we did following (3.6) to conclude that

$$\left|\int_a^b e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}\right| \leq 2C + 2C^2$$

if $|g(t)| \geq 1$ for $a \leq t \leq b$. Furthermore,

$$\left|\int_\Omega e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}\right| \leq \sum_{m=1}^{M}\int_{c_m}^{d_m}\frac{dt}{t} \leq (2n-2)\log D.$$

Thus, for $x_n \neq 0$, $m_{\varepsilon,\delta}$ is bounded uniformly in δ for fixed ε. Since the set where $x_n = 0$ is of measure zero in $\mathbf{R}^n$, the proof of Theorem 2 is complete. □

5. The L^2-boundedness of $H_{\varepsilon,\delta}$ and the maximal operator supported on the curves $\Gamma(t) = (t, \gamma(t))$, with γ as in (1.6) and $\Gamma^*(t) = (t, \gamma^*(t))$ with γ^* as in (1.7). We first show the result for $H_{\varepsilon,\delta}$. With γ as in (1.6), set

$$g(t) = \begin{cases} xt & 0 \leq t \leq 1,\\ xt - y(t-1) & t > 1,\ y > x > 0.\end{cases}$$

Choose η such that $\eta(y-x) = 1$. First assume $\eta \geq 1$. Then, for $N > \eta$,

$$\begin{aligned}\left|\int_\eta^N e^{ig(t)}\frac{dt}{t^{1+\varepsilon}}\right| &= \left|\frac{e^{ig(N)}}{(x-y)N^{1+i\varepsilon}} - \frac{e^{ig(\eta)}}{(x-y)\eta^{1+i\varepsilon}}\right.\\ &\qquad \left. + \frac{1+i\varepsilon}{i}\int_\eta^N \frac{e^{ig(t)}}{(x-y)t^{2+i\varepsilon}}\,dt\right|\\ &\leq \frac{2}{\eta(y-x)} + \frac{1+\varepsilon}{\eta(y-x)} = 3+\varepsilon,\end{aligned} \tag{5.1}$$

and

$$
\left|\int_1^\eta e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}\right| = \left|\frac{-e^{ig(\eta)}}{i\varepsilon\eta^{i\varepsilon}} + \frac{e^{ig(1)}}{i\varepsilon} + \frac{1}{i\varepsilon}\int_1^\eta i(x-y)e^{ig(t)}\frac{dt}{t^{i\varepsilon}}\right| \le \frac{2}{\varepsilon} + \frac{(y-x)}{\varepsilon}\int_0^\eta dt = \frac{3}{\varepsilon}. \tag{5.2}
$$

Now, still with $\eta \ge 1$, suppose $x < 1$. Then, integration by parts as in (5.2) gives

$$
\left|\int_\delta^1 e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}\right| \le \frac{2}{\varepsilon} + \frac{x}{\varepsilon}\int_0^1 dt = \frac{3}{\varepsilon}. \tag{5.3}
$$

If, on the other hand, $x \ge 1$, choose β such that $\beta x = 1$ so that $\beta \le 1$. Then the method of (5.2) gives

$$
\left|\int_\delta^\beta e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}\right| \le \frac{3}{\varepsilon}, \tag{5.4}
$$

while the method of (5.1) gives

$$
\left|\int_\beta^1 e^{ig(t)}\frac{dt}{t^{1+i\varepsilon}}\right| \le 3 + \varepsilon. \tag{5.5}
$$

Now suppose $\eta < 1$. Then $\int_1^N e^{ig(t)}t^{-(1+i\varepsilon)}\,dt$ is dealt with by the method of (5.1). If $x \ge 1$, $\int_\delta^1 e^{ig(t)}t^{-(1+i\varepsilon)}\,dt$ is split as in (5.4) and (5.5), while if $x < 1$, the method of (5.3) is used.

We note that, in the case where

$$
g(t) = \begin{cases} xt & 0 \le t \le 1, \\ xt - y(t-1) & t > 1,\ x > y > 0, \end{cases}
$$

choosing η such that $\eta(x-y) = 1$ leads to essentially the same proof, as does the case with

$$
g(t) = \begin{cases} xt & 0 \le t \le 1, \\ xt + y(t-1) & t > 1,\ x > 0,\ y > 0, \end{cases}
$$

with $\eta(x+y) = 1$.

Now, with γ^* as in (1.7), we have

$$\begin{aligned}\left|\int_\delta^\infty e^{i(xt+y\gamma(t))}\frac{dt}{t^{1+i\varepsilon}}-\int_\delta^\infty e^{i(xt+y\gamma^*(t))}\frac{dt}{t^{1+\varepsilon}}\right| &= \left|\left(e^{i(xt+y\gamma(t))}-e^{i(xt+y\gamma^*(t))}\right)\frac{dt}{t^{1+i\varepsilon}}\right| \\ &\le \int_{1/2}^{3/2}\frac{dt}{t}=\log 3.\end{aligned}$$

It then follows that

$$\left|\int_\delta^\infty e^{i(xt+y\gamma^*(t))}\frac{dt}{t^{1+i\varepsilon}}\right|\le \log 3+B,$$

where

$$\left|\int_\delta^\infty e^{i(xt+y\gamma(t))}\frac{dt}{t^{1+i\varepsilon}}\right|\le B.$$

To see that $\mathbf{M}_{\Gamma^*}$ is bounded on $L^2(\mathbf{R}^2)$, first note that

$$\begin{aligned}\mathbf{M}_{\Gamma^*}f(x) &= \sup_{h>0}\frac{1}{h}\int_0^h |f(x-\Gamma^*(t))|\,dt \\ &\le \sup_{3/2\ge h>0}\frac{1}{h}\int_0^h |f(x-\Gamma^*(t))|\,dt \\ &\quad + \sup_{h>3/2}\frac{1}{h}\int_0^h |f(x-\Gamma^*(t))|\,dt.\end{aligned}$$

Now consider a curve $\tilde{\Gamma}(t)=(t,\tilde{\gamma}(t))$, where

$$\tilde{\gamma}(t)=\begin{cases}0 & 0\le t<1/2,\\ -t^4/2+2t^3-(9/4)t^2+t-5/2 & 1/2\le t<3/2,\\ t^3-(9/2)t^2+(31/4)t-23/8 & t\ge 3/2.\end{cases}$$

Then, by the result of Nagel, Vance, Wainger and Weinberg [**4**], $\mathbf{M}_{\tilde{\Gamma}}$ is bounded on $L^2(\mathbf{R}^2)$. Note that, for $0\le t<3/2$, $\tilde{\gamma}(t)=\gamma^*(t)$. Thus, for $0\le t<3/2$,

$$\begin{aligned}\sup_{3/2\ge h>0}\frac{1}{h}\int_0^h |f(x-\Gamma^*(t))|\,dt &= \sup_{3/2\ge h>0}\frac{1}{2}\int_0^h |f(x-\tilde{\Gamma}(t))|\,dt \\ &\le \sup_{h>0}\frac{1}{h}\int_0^h |f(x-\tilde{\Gamma}(t))|\,dt \\ &= \mathbf{M}_{\tilde{\Gamma}}f(x).\end{aligned}$$

For $h > 3/2$, with $x = (\xi, \eta)$,

$$\begin{aligned}
\frac{1}{h}\int_0^h |f(x-\Gamma^*(t))|\,dt &= \frac{1}{h}\int_0^{3/2} |f(x-\Gamma^*(t))|\,dt \\
&\quad + \frac{1}{h}\int_{3/2}^h |f(x-\Gamma^*(t))|\,dt \\
&\le \frac{1}{3/2}\int_0^{3/2} |f(x-\tilde{\Gamma}(t))|\,dt \\
&\quad + \frac{1}{3/2+\mu}\int_{3/2}^{3/2+\mu} |f(\xi - t, \eta-(t-1))|\,dt \\
&\le \sup_{\nu>0}\frac{1}{\nu}\int_0^{\nu} |f(x-\tilde{\Gamma}(t))|\,dt \\
&\quad + \frac{1}{\mu}\int_0^{\mu} \left|f\left(\left(\xi-\frac{3}{2}\right)-t, \left(\eta-\frac{1}{2}\right)-t\right)\right|\,dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup_{h>3/2}\frac{1}{h}\int_0^h |f(x-\gamma^*(t))|\,dt \\
&\qquad \le \sup_{\nu>0}\frac{1}{\nu}\int_0^{\nu} |f(x-\tilde{\gamma}(t))|\,dt \\
&\qquad\quad + \sup_{\mu>0}\frac{1}{\mu}\int_0^{\mu} \left|f\left(\left(\xi-\frac{3}{2}\right)-t, \left(\eta-\frac{1}{2}\right)-t\right)\right|\,dt \\
&\qquad = \mathbf{M}_{\tilde{\Gamma}}f(x) + \mathbf{N}f\left(\xi-\frac{3}{2}, \eta-\frac{1}{2}\right).
\end{aligned}$$

But, as noted above, $\mathbf{M}_{\tilde{\Gamma}}$ is bounded on $L^2(\mathbf{R}^2)$, while it follows from the theorem on the one-dimensional Hardy-Littlewood maximal function that

$$\mathbf{N}f(\xi,\eta) = \sup_{\mu>0}\frac{1}{\mu}\int_0^{\mu} |f(\xi-t,\eta-t)|\,dt$$

is a bounded operator on $L^p(\mathbf{R}^2)$ for $1 < p \le \infty$. Thus, $\mathbf{M}_{\Gamma^*}$ is bounded on $L^2(\mathbf{R}^2)$. The proof for $\mathbf{M}_{\Gamma}$, with γ as in (1.6), is similar.

6. An extension of Theorem 1. We can extend the result of Theorem 1 to certain piecewise linear convex curves. Suppose $\Gamma(t) = (t, \gamma(t))$, where γ is piecewise linear on intervals of the form $[t_{j+1}, t_j]$, where $t_j/t_{j+1} \leq C$ for some constant $C > 1$, $t_j \nearrow \infty$ as $j \to -\infty$ and $t_j \searrow 0$ as $j \to \infty$.

Set $\gamma(t_j) = a_j$. Then, on $[t_{j+1}, t_j]$, we have

$$\gamma(t) = \gamma_j(t) = m_j t + (a_j - m_j t_j) = m_j t + b_j, \tag{6.1}$$

where m_j is the slope of γ_j. Then, for $t_{j+1} < t < t_j$,

$$h(t) = h_j(t) = m_j t - (m_j t + b_j) = -b_j = m_j t_j - a_j.$$

If we then require that we choose the a_j's so that

$$\frac{a_j}{a_{j+1}} > C \quad \text{and} \quad m_{j-1} \geq 2m_j - \frac{a_j}{t_j}, \tag{6.2}$$

we guarantee that $h(Ct) \geq 2h(t)$ for all t for which h is defined.

For each integer j we can then choose δ_j such that $0 < \delta_j \leq t_j(1 - e^{-2^{-|j|}})$. Setting $\alpha_j = t_j - \delta_j$ and $\beta_j = t_j + \delta_j$, we then have

$$0 < \int_{t_j}^{\beta_j} \frac{dt}{t} < 2^{-|j|} \quad \text{and} \quad 0 < \int_{\alpha_j}^{t_j} \frac{dt}{t} \leq 2^{-|j|}. \tag{6.3}$$

Furthermore, for each j we may find a fourth degree polynomial $\varphi_j(t)$ with

$$\varphi_j(\alpha_j) = \gamma_j(\alpha_j), \varphi_j(\beta_j) = \gamma_{j-1}(\beta_j);$$
$$\varphi_j'(\alpha_j) = m_j, \varphi_j'(\beta_j) = m_{j-1};$$

and

$$\varphi_j''(\alpha_j) = 0 = \varphi_j''(\beta_j).$$

Then setting $\tilde{\gamma}(t) = \gamma_j(t)$, if $\beta_{j+1} < t < \alpha_j$ and $\tilde{\gamma}(t) = \varphi_j(t)$, if $\alpha_j \leq t \leq \beta_{j-1}$, we have a curve $\tilde{\gamma}$ which satisfies the hypotheses of Theorem 1.

Setting $\Gamma(t) = (t, \gamma_j(t))$ for $t_{j+1} \le t < t_j$, and $\tilde{\Gamma}(t) = (t, \tilde{\gamma}(t))$, we note that the multiplier function for $H_{\varepsilon,\delta}$ supported on Γ is given by

$$(6.4) \qquad \int_\delta^{t_J} e^{i(xt+y\gamma_J(t))} \frac{dt}{t^{1+i\varepsilon}} + \sum_{j=-\infty}^{J-1} \int_{t_{j+1}}^{t_j} e^{i(xt+y\gamma_j(t))} \frac{dt}{t^{1+i\varepsilon}}, \quad \text{for some } J \in \mathbf{Z}.$$

Then (6.3) and the triangle inequality show

$$(6.5) \quad \left| \int_\delta^{t_J} e^{i(xt+y\gamma_J(t))} \frac{dt}{t^{1+i\varepsilon}} - \int_\delta^{t_J} e^{i(xt+g\tilde{\gamma}(t))} \frac{dt}{t^{1+i\varepsilon}} \right| + \sum_{j=-\infty}^{J-1} \left| \int_{t_j+1}^{t_j} e^{i(xt+y\gamma_j(t))} \frac{dt}{t^{1+i\varepsilon}} - \int_{t_j+1}^{t_j} e^{i(xt+y\tilde{\gamma}(t))} \frac{dt}{t^{1+i\varepsilon}} \right| \le 6.$$

Since $|\int_\delta^\infty e^{i(xt+y\tilde{\gamma}(t))} t^{-(1+i\varepsilon)}\, dt| \le B$, where B depends only on ε, it follows from this and (6.5) that the multiplier function given by (6.4) has a bound which depends only on ε.

As a final comment, we note that M_Γ and $H_{\varepsilon,\delta}$ have both been shown to be bounded when supported on a curve for which the Hilbert transform is unbounded, but that there are no known examples of curves for which the results for these two operators differ.

REFERENCES

1. H. Carlsson and S. Wainger, *Maximal functions related to convex polygonal lines*, Indiana Univ. Math. J. **34** (1985), 815–823.

2. A. Nagel, J. Vance, S. Wainger and D. Wainger, *Hilbert transforms for convex curves*, Duke Math. J. **50** (1983), 735–744.

3. ———, *The Hilbert transform for convex curves in* $\mathbf{R}^n$, Amer. J. Math. **108** (1986), 485–504.

4. ———, *Maximal functions for convex curves*, Duke Math. J. **52** (1985), 715–722.

5. E.M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

Illinois College, Jacksonville, IL 62650